MFPS XXVII

Exact and automatic reduction of rule-based models

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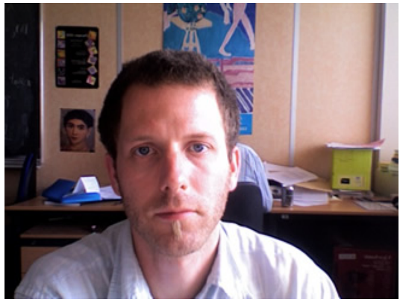

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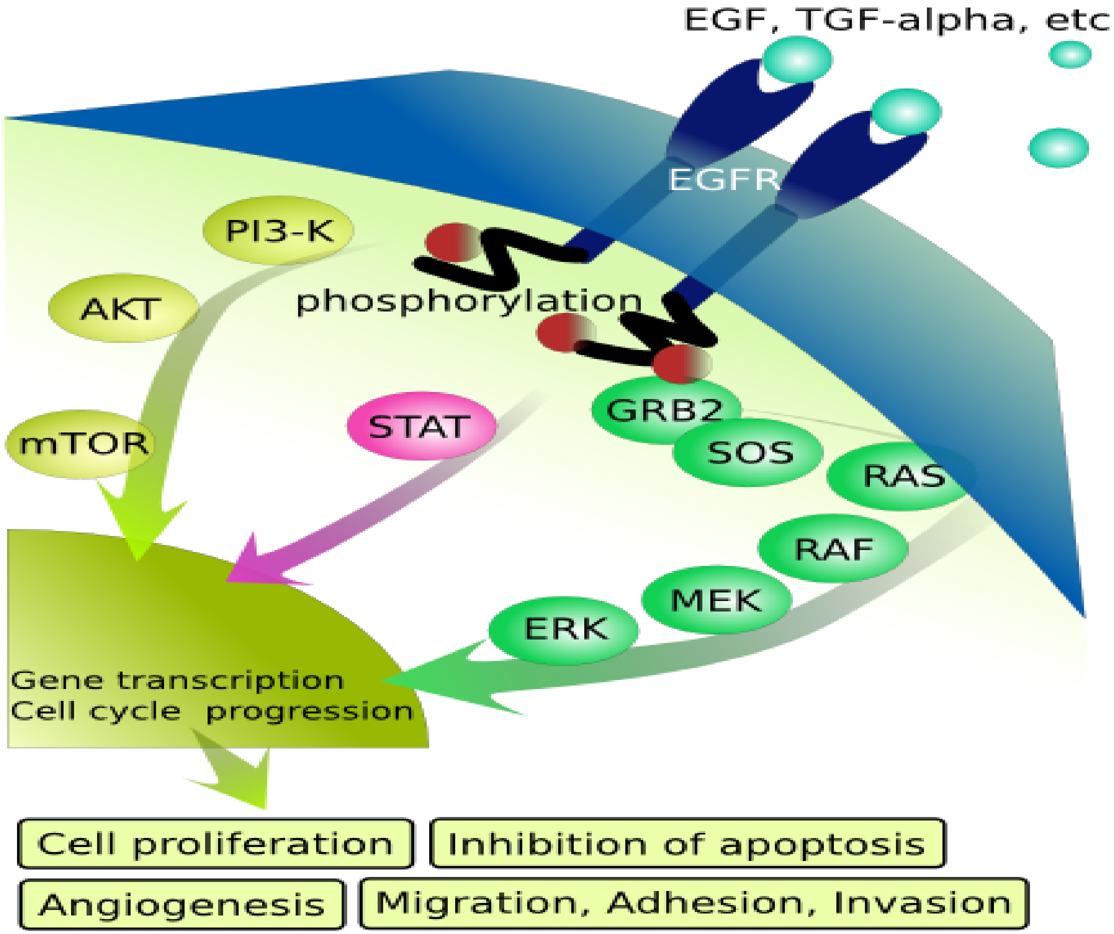
Overview

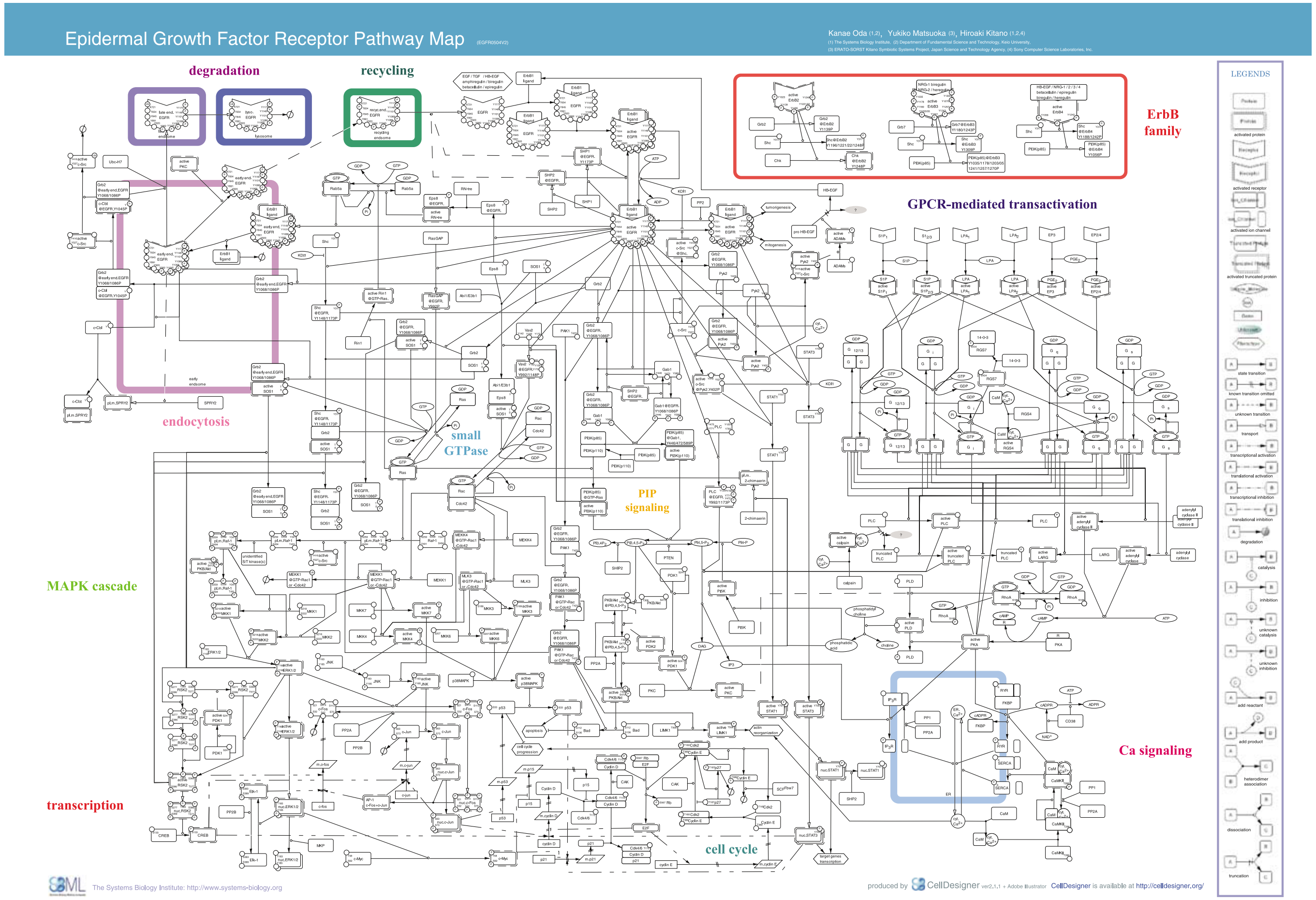
1. Context and motivations
2. Handmade ODEs
3. Abstract interpretation framework
4. Kappa
5. Concrete semantics
6. Abstract semantics
7. Conclusion

Signalling Pathways



Eikuch, 2007

Pathway maps



Oda, Matsuoka, Funahashi, Kitano, Molecular Systems Biology, 2005

Differential models

$$\begin{cases}
\frac{dx_1}{dt} = -k_1 \cdot x_1 \cdot x_2 + k_{-1} \cdot x_3 \\
\frac{dx_2}{dt} = -k_1 \cdot x_1 \cdot x_2 + k_{-1} \cdot x_3 \\
\frac{dx_3}{dt} = k_1 \cdot x_1 \cdot x_2 - k_{-1} \cdot x_3 + 2 \cdot k_2 \cdot x_3 \cdot x_3 - k_{-2} \cdot x_4 \\
\frac{dx_4}{dt} = k_2 \cdot x_3^2 - k_2 \cdot x_4 + \frac{v_4 \cdot x_5}{p_4 + x_5} - (k_3 \cdot x_4 - k_{-3} \cdot x_5) \\
\frac{dx_5}{dt} = \cdots \\
\quad \vdots \\
\frac{dx_n}{dt} = -k_1 \cdot x_1 \cdot c_2 + k_{-1} \cdot x_3
\end{cases}$$

– do not describe the structure of molecules;
– combinatorial explosion: forces choices that are not principled;
– a nightmare to modify.

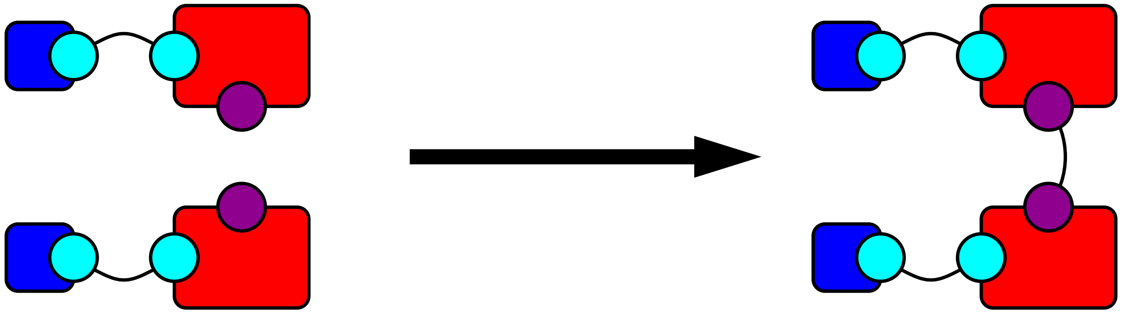
A gap between two worlds

Two levels of description:

1. Databases of proteins interactions in natural language
   + documented and detailed description
   + transparent description
   – cannot be interpreted
2. ODE-based models
   + can be integrated
   – opaque modelling process, models can hardly be modified
   – there are also some scalability issues.

Rule-based approach

We use site graph rewrite systems



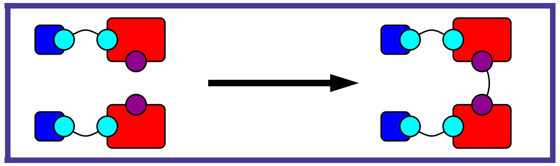
1. The description level matches with both
   - the observation level
   - and the intervention level

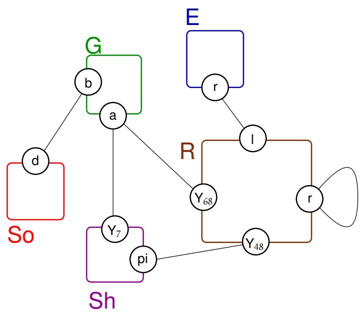
   of the biologist.

   We can tune the model easily.
2. Model description is very compact.

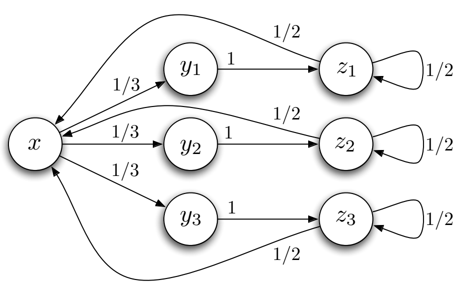
Semantics

Several semantics (qualititative and/or quantitative) can be defined.





Interaction map



CTMC

$$
\begin{cases}
\frac{dx_1}{dt} = -k_1 \cdot x_1 \cdot x_2 + k_{-1} \cdot x_3 \\
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\vdots \\
\frac{dx_n}{dt} = -k_1 \cdot x_1 \cdot c_2 + k_{-1} \cdot x_3
\end{cases}
$$

ODEs

Semantics

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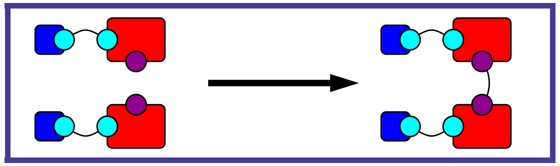
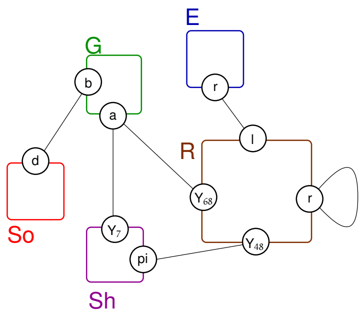
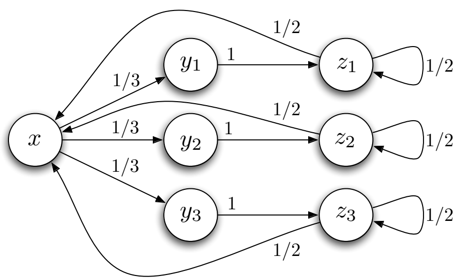




Interaction map



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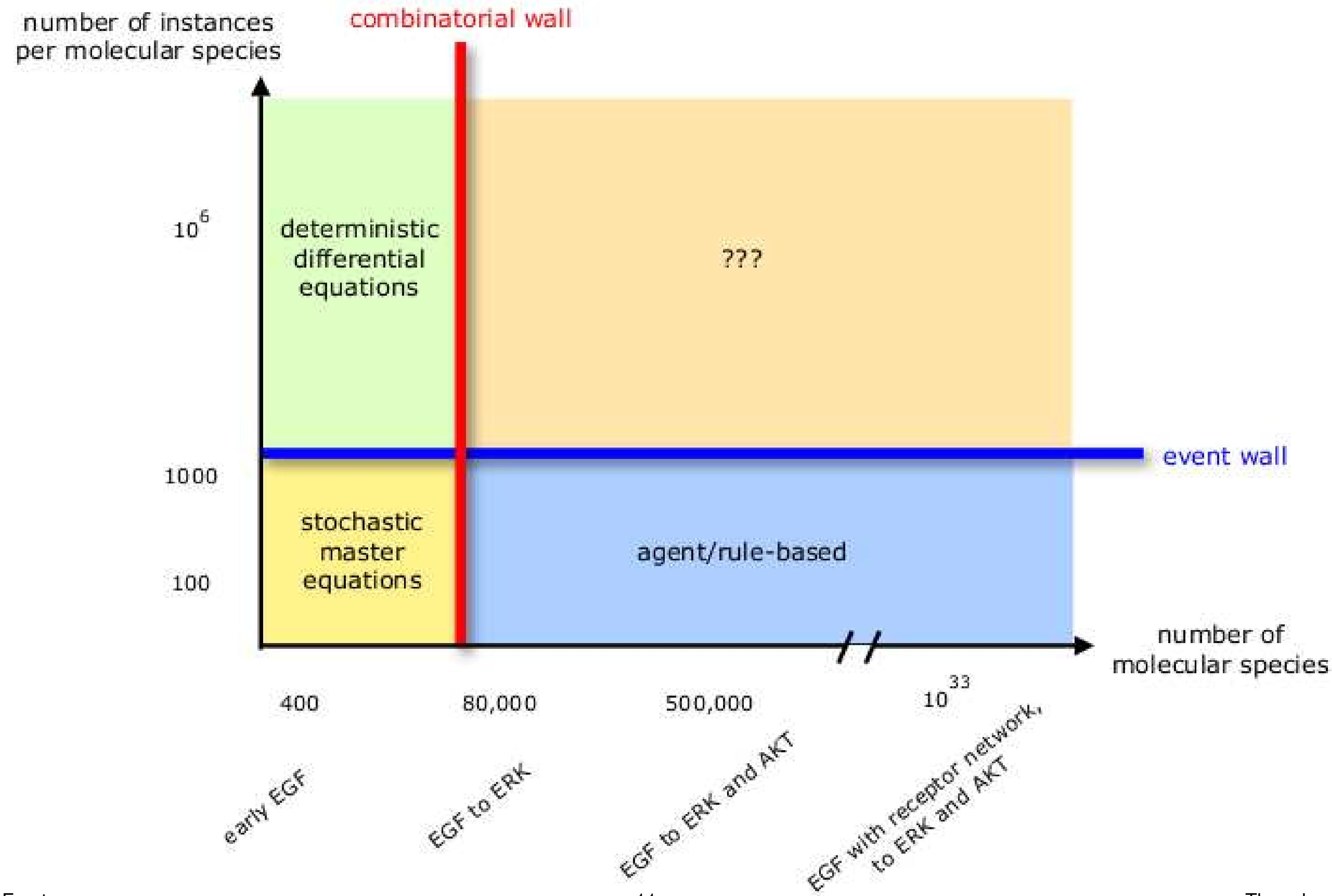
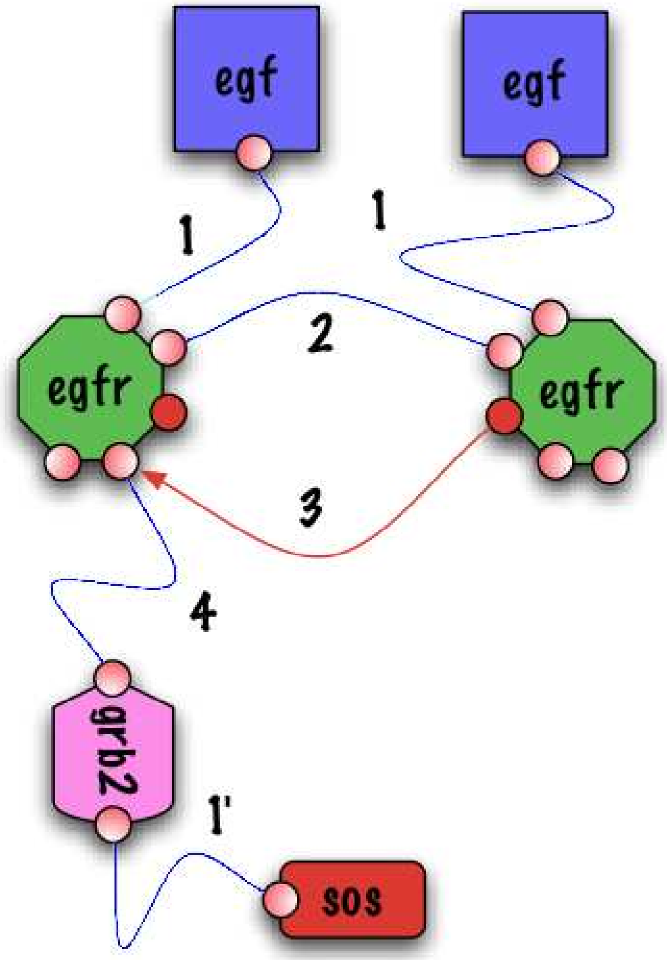
\end{cases}$$

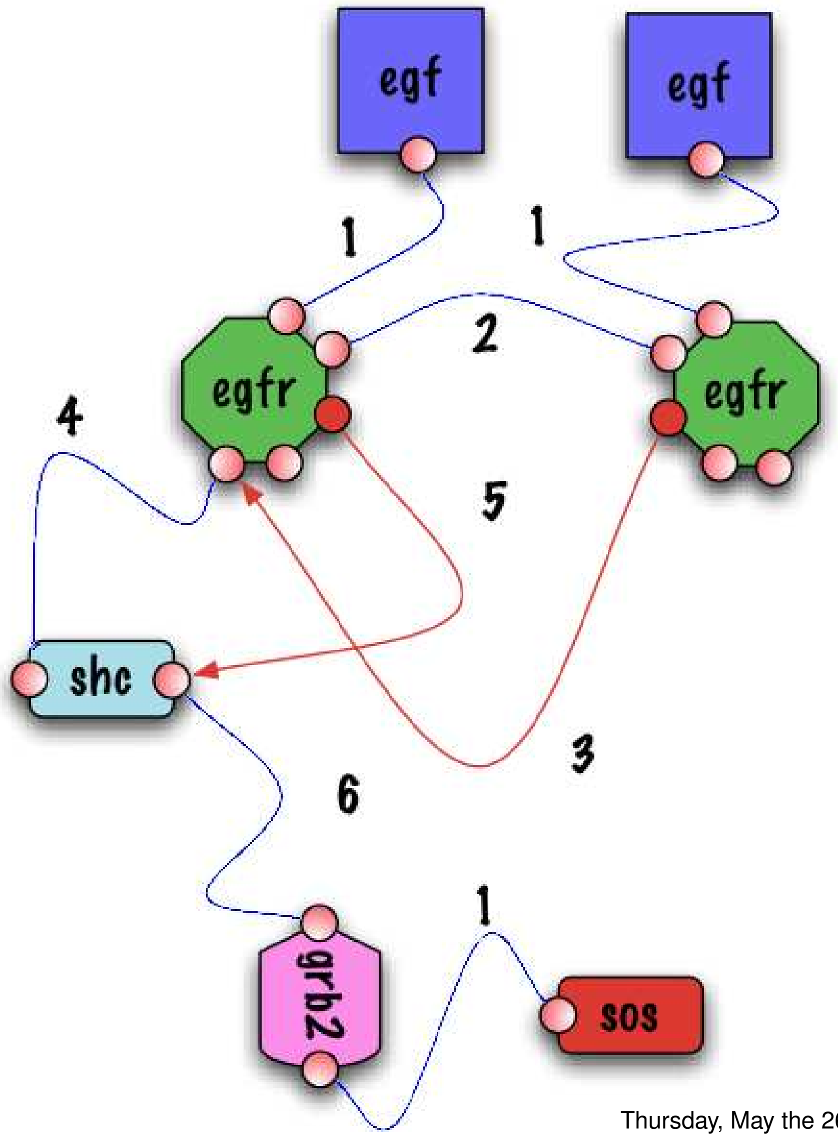
ODEs

Complexity walls



A breach in the wall(s) ?





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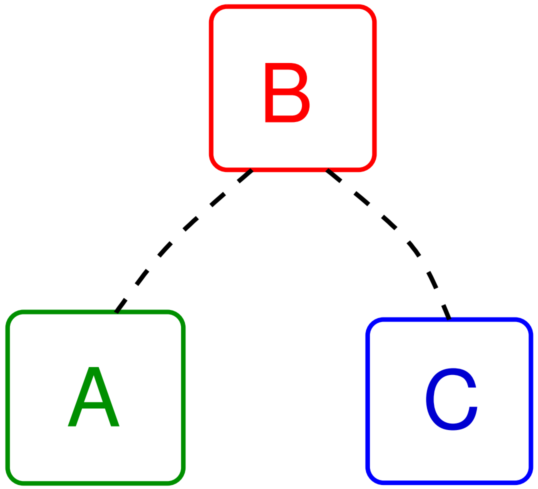
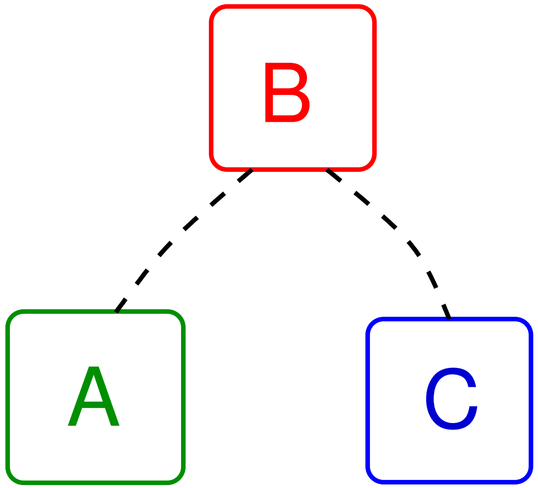
A simple adapter

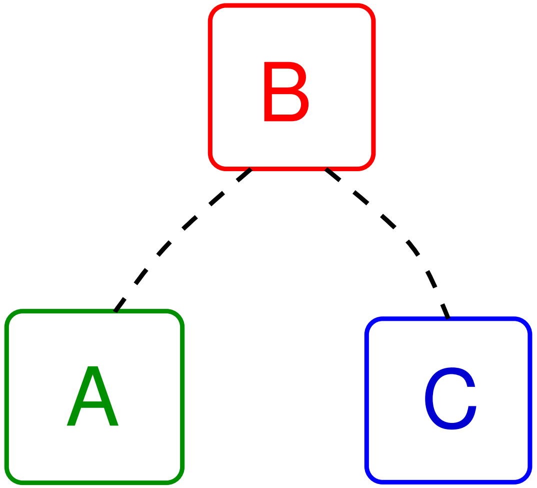


A simple adapter



$$A\,,\ \emptyset B\emptyset \longleftrightarrow AB\emptyset \qquad k^{AB},k_d^{AB}$$
$$A\,,\ \emptyset BC \longleftrightarrow ABC \qquad k^{AB},k_d^{AB}$$
$$\emptyset B\emptyset\,,\ C \longleftrightarrow \emptyset BC \qquad k^{BC},k_d^{BC}$$
$$AB\emptyset\,,\ C \longleftrightarrow ABC \qquad k^{BC},k_d^{BC}$$

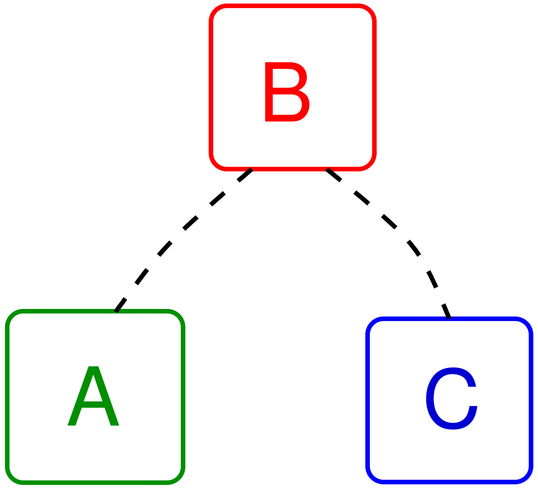
A simple adapter



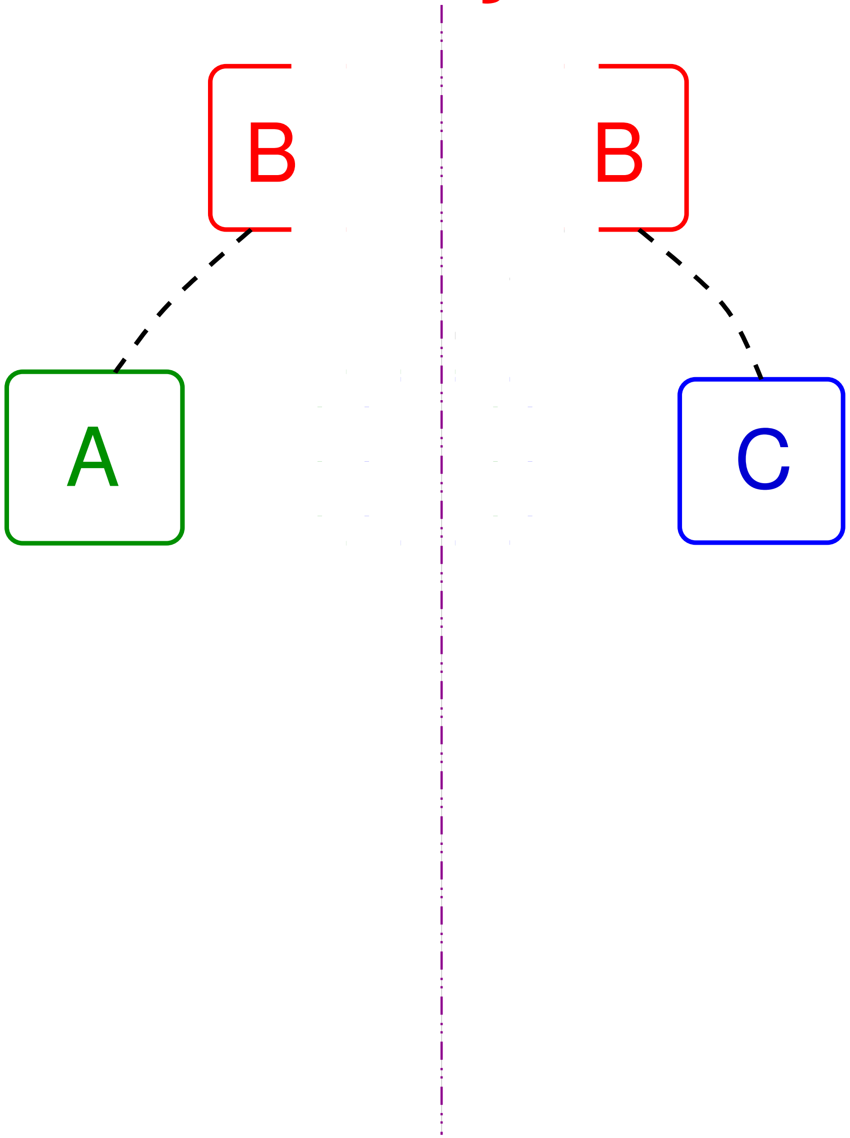
$$A \,,\, \emptyset B \emptyset \longleftrightarrow AB\emptyset \qquad k^{AB}, k_d^{AB}$$
$$A \,,\, \emptyset BC \longleftrightarrow ABC \qquad k^{AB}, k_d^{AB}$$
$$\emptyset B \emptyset \,,\, C \longleftrightarrow \emptyset BC \qquad k^{BC}, k_d^{BC}$$
$$AB\emptyset \,,\, C \longleftrightarrow ABC \qquad k^{BC}, k_d^{BC}$$

$$\begin{cases}
\frac{d[A]}{dt} = k_d^{AB}\cdot([AB\emptyset]+[ABC]) - [A]\cdot k^{AB}\cdot([\emptyset B\emptyset]+[\emptyset BC]) \\
\frac{d[C]}{dt} = k_d^{BC}\cdot([\emptyset BC]+[ABC]) - [C]\cdot k^{BC}\cdot([\emptyset B\emptyset]+[AB\emptyset]) \\
\frac{d[\emptyset B\emptyset]}{dt} = k_d^{AB}\cdot[AB\emptyset] + k_d^{BC}\cdot[\emptyset BC] - [\emptyset B\emptyset]\cdot([A]\cdot k^{AB} + [C]\cdot k^{BC}) \\
\frac{d[AB\emptyset]}{dt} = [A]\cdot k^{AB}\cdot[\emptyset B\emptyset] + k_d^{BC}\cdot[ABC] - [AB\emptyset]\cdot(k_d^{AB} + [C]\cdot k^{BC}) \\
\frac{d[\emptyset BC]}{dt} = k_d^{AB}\cdot[ABC] + [C]\cdot k^{BC}\cdot[\emptyset B\emptyset] - [\emptyset BC]\cdot(k_d^{BC} + [A]\cdot k^{AB}) \\
\frac{d[ABC]}{dt} = [A]\cdot k^{AB}\cdot[\emptyset BC] + [C]\cdot k^{BC}\cdot[AB\emptyset] - [ABC]\cdot(k_d^{AB} + k_d^{BC})
\end{cases}$$

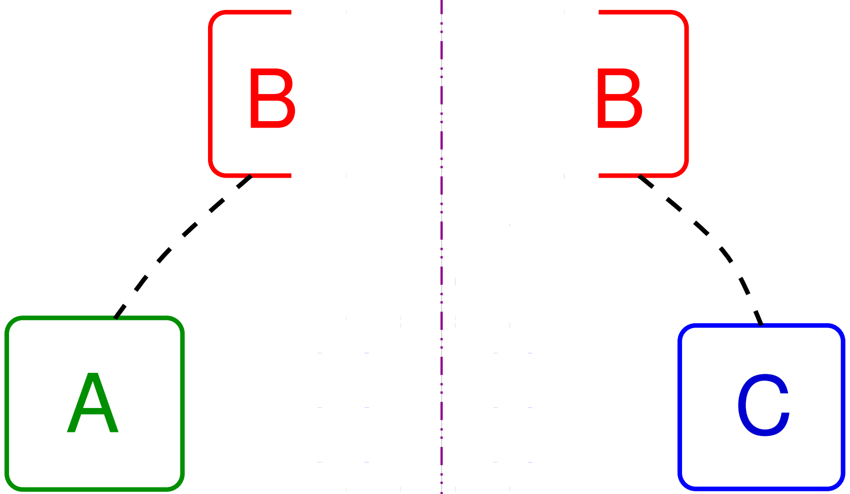
Two subsystems



Two subsystems



Two subsystems



$$[AB?] \stackrel{\Delta}{=} [AB\emptyset] + [ABC]$$

$$[\emptyset B?] \stackrel{\Delta}{=} [\emptyset B\emptyset] + [\emptyset BC]$$

$$\begin{cases} \frac{d[A]}{dt} = k_d^{AB} \cdot [AB?] - [A] \cdot k^{AB} \cdot [\emptyset B?] \\ \frac{d[AB?]}{dt} = [A] \cdot k^{AB} \cdot [\emptyset B?] - k_d^{AB} \cdot [AB?] \\ \frac{d[\emptyset B?]}{dt} = k_d^{AB} \cdot [AB?] - [A] \cdot k^{AB} \cdot [\emptyset B?] \end{cases}$$

$$[?BC] \stackrel{\Delta}{=} [\emptyset BC] + [ABC]$$

$$[?B\emptyset] \stackrel{\Delta}{=} [\emptyset B\emptyset] + [AB\emptyset]$$

$$\begin{cases} \frac{d[C]}{dt} = k_d^{BC} \cdot [?BC] - [C] \cdot k^{BC} \cdot [?B\emptyset] \\ \frac{d[?BC]}{dt} = [C] \cdot k^{BC} \cdot [?B\emptyset] - k_d^{BC} \cdot [?BC] \\ \frac{d[?B\emptyset]}{dt} = k_d^{BC} \cdot [?BC] - [C] \cdot k^{BC} \cdot [?B\emptyset] \end{cases}$$

Dependence index

We introduce:

$$[?B?] \stackrel{\Delta}{=} [?B\emptyset] + [?BC].$$

The binding with A and with C would be independent if, and only if:

$$\frac{[ABC]}{[?BC]} = \frac{[AB?]}{[?B?]}.$$

Thus we define the dependence index as follows:

$$X \stackrel{\Delta}{=} [ABC]\cdot[?B?] - [AB?]\cdot[?BC].$$

We have (after a short computation):

$$\frac{dX}{dt} = -X\cdot \left([A]\cdot k^{AB} + k_d^{AB} + [C]\cdot k^{BC} + k_d^{BC}\right)$$

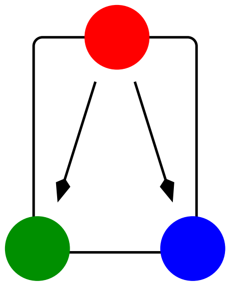
So the property:

$$[ABC] = \frac{[AB?]\cdot[?BC]}{[?B?]}$$

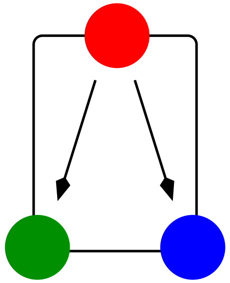
is an invariant (i.e. if it holds at time t, it holds at any time $t' \geq t$).

Overview

A system with a switch

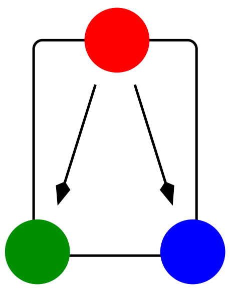


A system with a switch



$$\begin{array}{lcll}
(u,u,u) & \longrightarrow & (u,p,u) & k^c \\
(u,p,u) & \longrightarrow & (p,p,u) & k^l \\
(u,p,p) & \longrightarrow & (p,p,p) & k^l \\
(u,p,u) & \longrightarrow & (u,p,p) & k^r \\
(p,p,u) & \longrightarrow & (p,p,p) & k^r
\end{array}$$

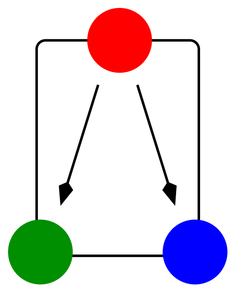
A system with a switch



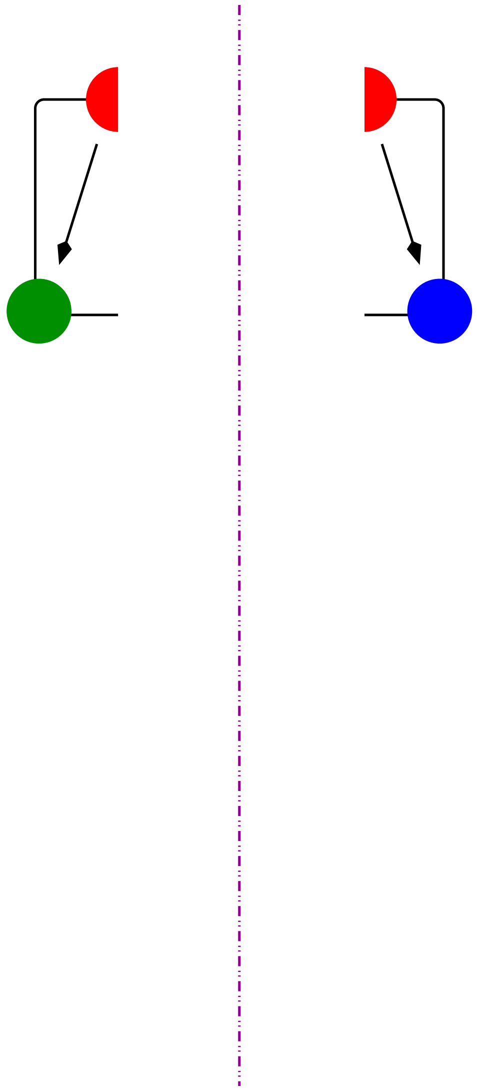
$$
\begin{array}{lllr}
(u,u,u) & \longrightarrow & (u,p,u) & k^c \\
(u,p,u) & \longrightarrow & (p,p,u) & k^l \\
(u,p,p) & \longrightarrow & (p,p,p) & k^l \\
(u,p,u) & \longrightarrow & (u,p,p) & k^r \\
(p,p,u) & \longrightarrow & (p,p,p) & k^r
\end{array}
$$

$$
\left\{
\begin{array}{l}
\frac{d[(u,u,u)]}{dt} = -k^c \cdot [(u,u,u)] \\
\frac{d[(u,p,u)]}{dt} = -k^l \cdot [(u,p,u)] + k^c \cdot [(u,u,u)] - k^r \cdot [(u,p,u)] \\
\frac{d[(u,p,p)]}{dt} = -k^l \cdot [(u,p,p)] + k^r \cdot [(u,p,u)] \\
\frac{d[(p,p,u)]}{dt} = k^l \cdot [(u,p,u)] - k^r \cdot [(p,p,u)] \\
\frac{d[(p,p,p)]}{dt} = k^l \cdot [(u,p,p)] + k^r \cdot [(p,p,u)]
\end{array}
\right.
$$

Two subsystems



Two subsystems



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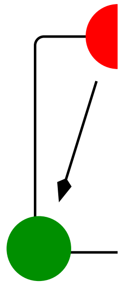
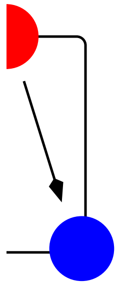




$$[(u,p,?)] \stackrel{\Delta}{=} [(u,p,u)] + [(u,p,p)]$$
$$[(p,p,?)] \stackrel{\Delta}{=} [(p,p,u)] + [(p,p,p)]$$

$$\begin{cases} \frac{d[(u,u,u)]}{dt} = -k^c \cdot [(u,u,u)] \\ \frac{d[(u,p,?)]}{dt} = -k^l \cdot [(u,p,?)] + k^c \cdot [(u,u,u)] \\ \frac{d[(p,p,?)]}{dt} = k^l \cdot [(u,p,?)] \end{cases}$$

$$[(?,p,u)] \stackrel{\Delta}{=} [(u,p,u)] + [(p,p,u)]$$
$$[(?,p,p)] \stackrel{\Delta}{=} [(u,p,p)] + [(p,p,p)]$$

$$\begin{cases} \frac{d[(u,u,u)]}{dt} = -k^c \cdot [(u,u,u)] \\ \frac{d[(?,p,u)]}{dt} = -k^r \cdot [(?,p,u)] + k^c \cdot [(u,u,u)] \\ \frac{d[(?,p,p)]}{dt} = k^r \cdot [(?,p,u)] \end{cases}$$

Dependence index

We introduce:

$$[(?,\mathrm{p},?)] \stackrel{\Delta}{=} [(?,\mathrm{p},\mathrm{u})] + [(?,\mathrm{p},\mathrm{p})]$$

The states of left site and right site would be independent if, and only if:

$$\frac{[(\mathrm{p},\mathrm{p},\mathrm{p})]}{[(\mathrm{p},\mathrm{p},?)]} = \frac{[(?,\mathrm{p},\mathrm{p})]}{[(?,\mathrm{p},?)]}.$$

Thus we define the dependence index as follows:

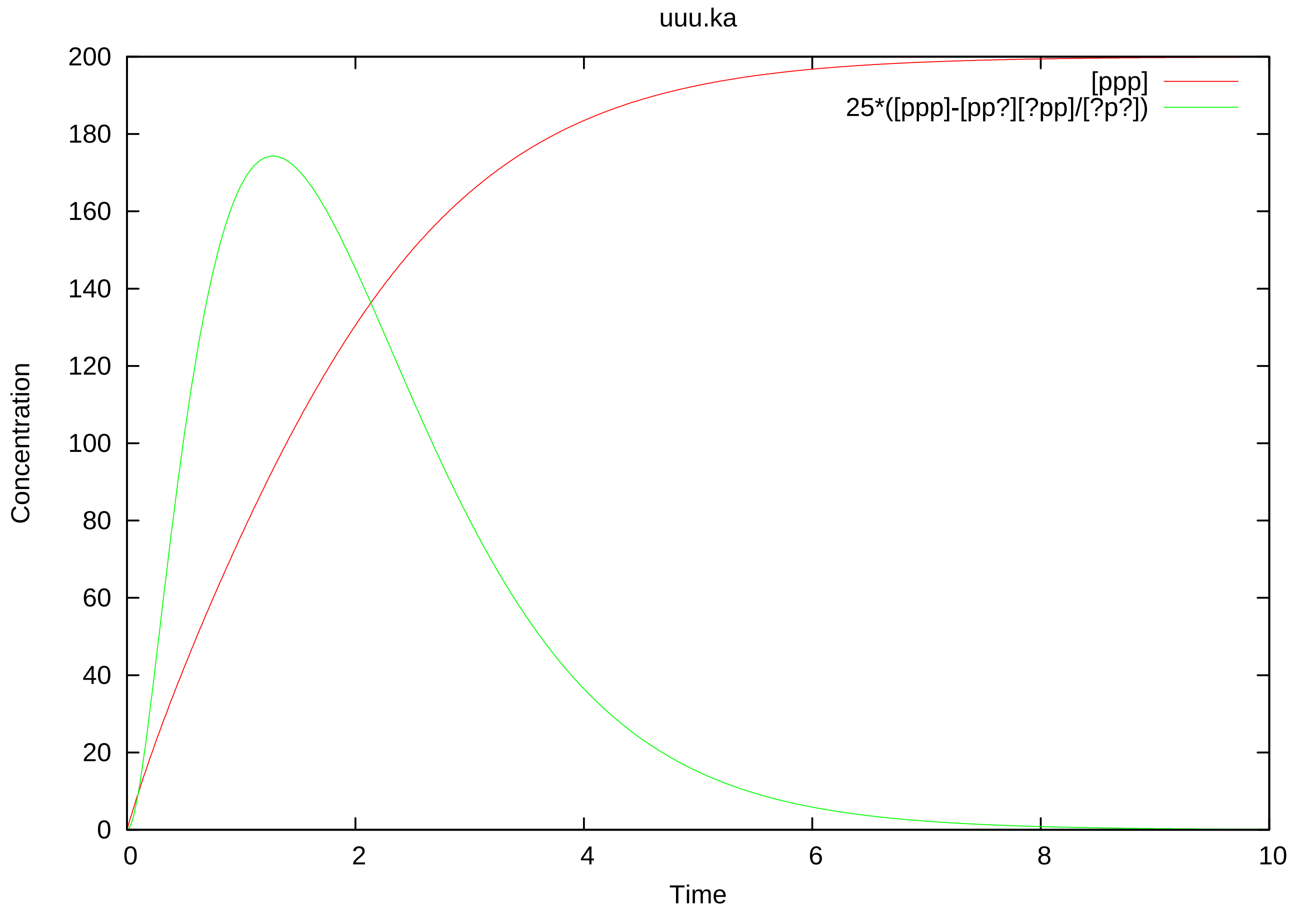
$$X \stackrel{\Delta}{=} [(\mathrm{p},\mathrm{p},\mathrm{p})]\cdot[(?,\mathrm{p},?)] - [(?,\mathrm{p},\mathrm{p})]\cdot[(\mathrm{p},\mathrm{p},?)].$$

We have (after a short computation):

$$\frac{dX}{dt} = -X \cdot \left(k^l + k^r\right) + k^c\cdot[(\mathrm{p},\mathrm{p},\mathrm{p})]\cdot[(\mathrm{u},\mathrm{u},\mathrm{u})].$$

As a consequence, the property $X = 0$ is not an invariant.
We can split the system into two subsystems,
but we cannot recombine both subsystems without errors.

Erroneous recombination



Concentrations evolution with respect to time ($[(u,u,u)](0) = 200$).

$[(p,p,p)]$ and $25 \cdot \left([(p,p,p)] - \frac{[(p,p,?)]\cdot[(?,p,p)]}{[(?,p,?)]}\right)$

Conclusion

- Independence:

  + the transformation is invertible:

    we can recover the concentration of any species;

  – it is a strong property

    which is hard to prove,

    which is hardly ever satisfied.

- Self-consistency:

  – some information is abstracted away

    we cannot recover the concentration of any species;

  + it is a weak property

    which is easy to ensure,

    which is easy to propagate;

  + it captures the essence of the kinetics of systems.

We are going to track the correlations that are read by the system.

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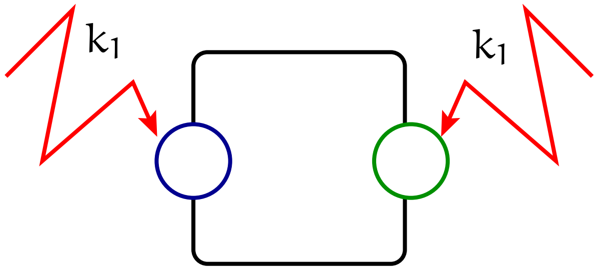
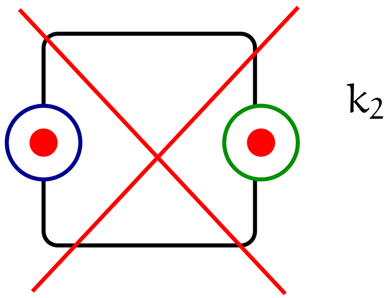
A model with symmetries



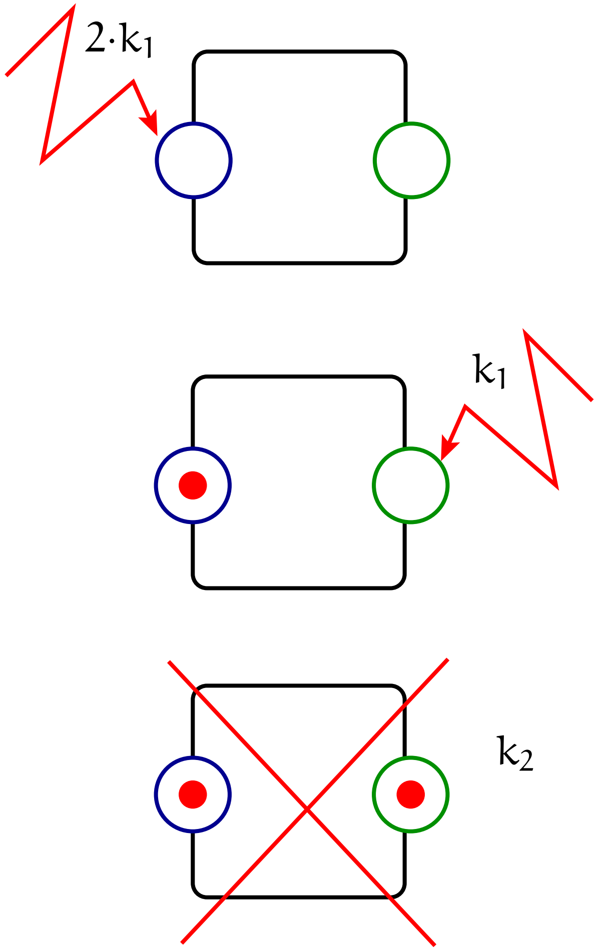
$$P \longrightarrow {}^{\star}P \quad k_1 \qquad\qquad P^{\star} \longrightarrow {}^{\star}P^{\star} \quad k_1$$

$$P \longrightarrow P^{\star} \quad k_1 \qquad\qquad {}^{\star}P \longrightarrow {}^{\star}P^{\star} \quad k_1$$



$${}^{\star}P^{\star} \longrightarrow \emptyset \quad k_2$$

Reduced model



$$P \longrightarrow {}^{\star}P \qquad 2\cdot k_1$$

$$^{\star}P \longrightarrow {}^{\star}P^{\star} \qquad k_1$$

$$^{\star}P^{\star} \longrightarrow \emptyset \qquad k_2$$

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Continuous differential semantics

Let $\mathcal{V}$, be a finite set of variables;
and $\mathbb{F}$, be a $\mathcal{C}^\infty$ mapping from $\mathcal{V} \rightarrow \mathbb{R}^+$ into $\mathcal{V} \rightarrow \mathbb{R}$,
as for instance,

- $\mathcal{V} \stackrel{\Delta}{=} \{[(u,u,u)], [(u,p,u)], [(p,p,u)], [(u,p,p)], [(p,p,p)]\}$,
- $\mathbb{F}(\rho) \stackrel{\Delta}{=} \begin{cases} [(u,u,u)] \mapsto -k^c \cdot \rho([(u,u,u)]) \\ [(u,p,u)] \mapsto -k^l \cdot \rho([(u,p,u)]) + k^c \cdot \rho([(u,u,u)]) - k^r \cdot \rho([(u,p,u)]) \\ [(u,p,p)] \mapsto -k^l \cdot \rho([(u,p,p)]) + k^r \cdot \rho([(u,p,u)]) \\ [(p,p,u)] \mapsto k^l \cdot \rho([(u,p,u)]) - k^r \cdot \rho([(p,p,u)]) \\ [(p,p,p)] \mapsto k^l \cdot \rho([(u,p,p)]) + k^r \cdot \rho([(p,p,u)]). \end{cases}$

The continuous semantics maps each initial state $X_0 \in \mathcal{V} \rightarrow \mathbb{R}^+$ to the maximal solution $X_{X_0} \in [0, T^{max}_{X_0}[\rightarrow (\mathcal{V} \rightarrow \mathbb{R}^+)$ which satisfies:

$$X_{X_0}(T) = X_0 + \int_{t=0}^{T} \mathbb{F}(X_{X_0}(t)) \cdot dt.$$

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7. Conclusion

Abstraction

An abstraction $(\mathcal{V}^\sharp, \psi, \mathbb{F}^\sharp)$ is given by:

- $\mathcal{V}^\sharp$: a finite set of observables,
- ψ: a mapping from $\mathcal{V} \to \mathbb{R}$ into $\mathcal{V}^\sharp \to \mathbb{R}$,
- $\mathbb{F}^\sharp$: a $\mathcal{C}^\infty$ mapping from $\mathcal{V}^\sharp \to \mathbb{R}^+$ into $\mathcal{V}^\sharp \to \mathbb{R}$;

such that:

- ψ is linear with positive coefficients,
  and for any sequence $(x_n) \in (\mathcal{V} \to \mathbb{R}^+)^{\mathbb{N}}$ such that $(||x_n||)$ diverges towards $+\infty$, then $(||\psi(x_n)||^\sharp)$ diverges as well
  (for arbitrary norms $||\cdot||$ and $||\cdot||^\sharp$),
- $\mathbb{F}^\sharp$ is ψ-complete, i.e. the following diagram commutes:

$$\begin{array}{ccc}
(\mathcal{V} \to \mathbb{R}^+) & \xrightarrow{\mathbb{F}} & (\mathcal{V} \to \mathbb{R}) \\
\psi \Big\downarrow \ell^* & & \Big\downarrow \ell^* \psi \\
(\mathcal{V}^\sharp \to \mathbb{R}^+) & \xrightarrow{\mathbb{F}^\sharp} & (\mathcal{V}^\sharp \to \mathbb{R})
\end{array}$$

i.e. $\psi \circ \mathbb{F} = \mathbb{F}^\sharp \circ \psi$.

Abstraction example

- $\mathcal{V} \stackrel{\Delta}{=} \{[(u,u,u)], [(u,p,u)], [(p,p,u)], [(u,p,p)], [(p,p,p)]\}$
- $\mathbb{F}(\rho) \stackrel{\Delta}{=} \begin{cases} [(u,u,u)] \mapsto -k^c \cdot \rho([(u,u,u)]) \\ [(u,p,u)] \mapsto -k^l \cdot \rho([(u,p,u)]) + k^c \cdot \rho([(u,u,u)]) - k^r \cdot \rho([(u,p,u)]) \\ [(u,p,p)] \mapsto -k^l \cdot \rho([(u,p,p)]) + k^r \cdot \rho([(u,p,u)]) \\ \ldots \end{cases}$
- $\mathcal{V}^\sharp \stackrel{\Delta}{=} \{[(u,u,u)], [(?,p,u)], [(?,p,p)], [(u,p,?)], [(p,p,?)]\}$
- $\psi(\rho) \stackrel{\Delta}{=} \begin{cases} [(u,u,u)] \mapsto \rho([(u,u,u)]) \\ [(?,p,u)] \mapsto \rho([(u,p,u)]) + \rho([(p,p,u)]) \\ [(?,p,p)] \mapsto \rho([(u,p,p)]) + \rho([(p,p,p)]) \\ \ldots \end{cases}$
- $\mathbb{F}^\sharp(\rho^\sharp) \stackrel{\Delta}{=} \begin{cases} [(u,u,u)] \mapsto -k^c \cdot \rho^\sharp([(u,u,u)]) \\ [(?,p,u)] \mapsto -k^r \cdot \rho^\sharp([(?,p,u)]) + k^c \cdot \rho^\sharp([(u,u,u)]) \\ [(?,p,p)] \mapsto k^r \cdot \rho^\sharp([(?,p,u)]) \\ \ldots \end{cases}$

(Completeness can be checked analytically.)

Abstract continuous trajectories

Let $(\mathcal{V}, \mathbb{F})$ be a concrete system;
Let $(\mathcal{V}^\sharp, \psi, \mathbb{F}^\sharp)$ be an abstraction of the concrete system $(\mathcal{V}, \mathbb{F})$;
Let $X_0 \in \mathcal{V} \to \mathbb{R}^+$ be an initial (concrete) state.
We know that the following system:

$$Y_{\psi(X_0)}(T) = \psi(X_0) + \int_{t=0}^{T} \mathbb{F}^\sharp \left(Y_{\psi(X_0)}(t)\right) \cdot dt$$

has a unique maximal solution $Y_{\psi(X_0)}$ such that $Y_{\psi(X_0)} = \psi(X_0)$.

Theorem 1 Moreover, this solution is the projection of the maximal solution X_{X_0} of the system

$$X_{X_0}(T) = X_0 + \int_{t=0}^{T} \mathbb{F} \left(X_{X_0}(t)\right) \cdot dt,$$

which satisfies $X_{X_0}(0) = X_0$.
(ie $Y_{\psi(X_0)} = \psi(X_{X_0})$)

Abstract continuous trajectories
Proof sketch

Given an abstraction $(\mathcal{V}^\sharp, \psi, \mathbb{F}^\sharp)$, we have:

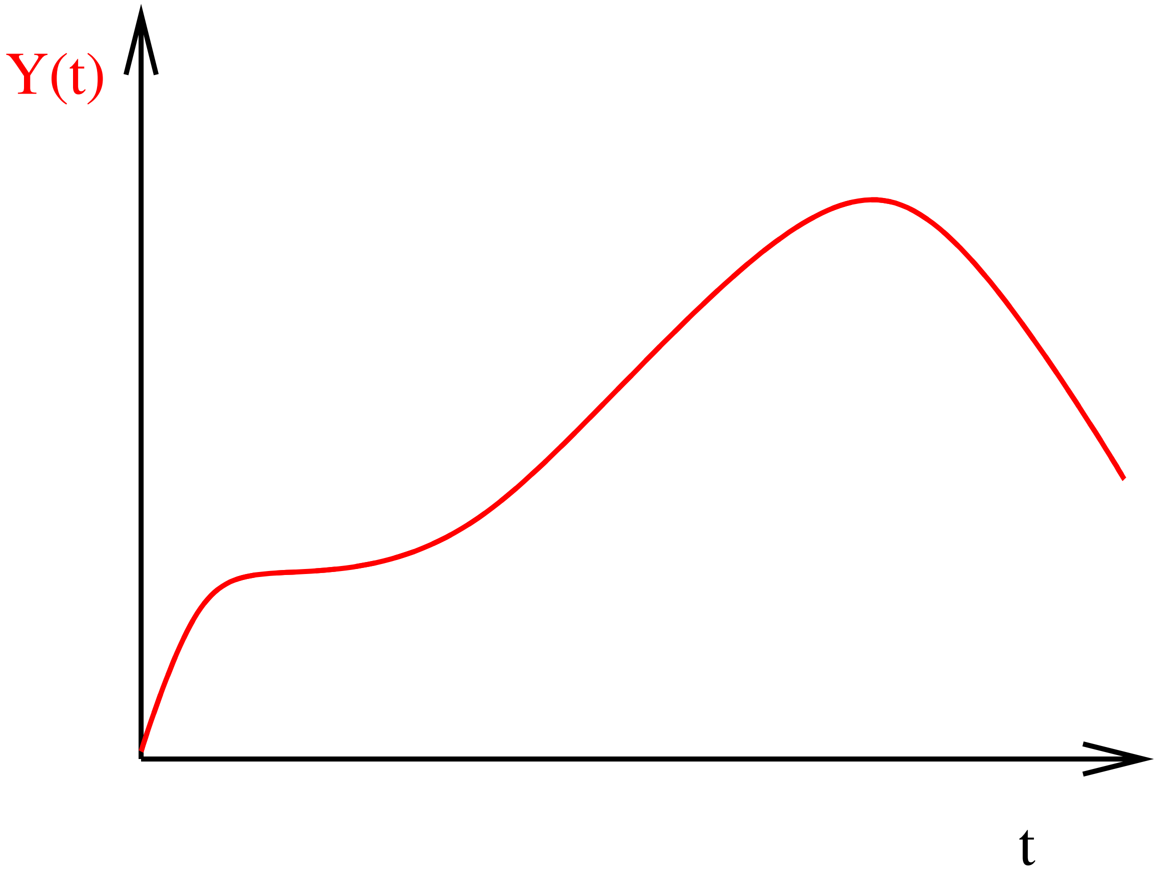
$$\begin{array}{rcl}
X_{X_0}(T) & = & X_0 + \int_{t=0}^{T} \mathbb{F}\left(X_{X_0}(t)\right) \cdot dt \\
\psi\left(X_{X_0}(T)\right) & = & \psi\left(X_0 + \int_{t=0}^{T} \mathbb{F}\left(X_{X_0}(t)\right) \cdot dt\right) \\
\psi\left(X_{X_0}(T)\right) & = & \psi(X_0) + \int_{t=0}^{T} [\psi \circ \mathbb{F}]\left(X_{X_0}(t)\right) \cdot dt \ \ (\psi \text{ is linear}) \\
\psi\left(X_{X_0}(T)\right) & = & \psi(X_0) + \int_{t=0}^{T} \mathbb{F}^\sharp\left(\psi\left(X_{X_0}(t)\right)\right) \cdot dt \ \ (\mathbb{F}^\sharp \text{ is } \psi\text{-complete})
\end{array}$$

We set $Y_0 \stackrel{\Delta}{=} \psi(X_0)$ and $Y_{Y_0} \stackrel{\Delta}{=} \psi \circ X_{X_0}$.
Then we have:

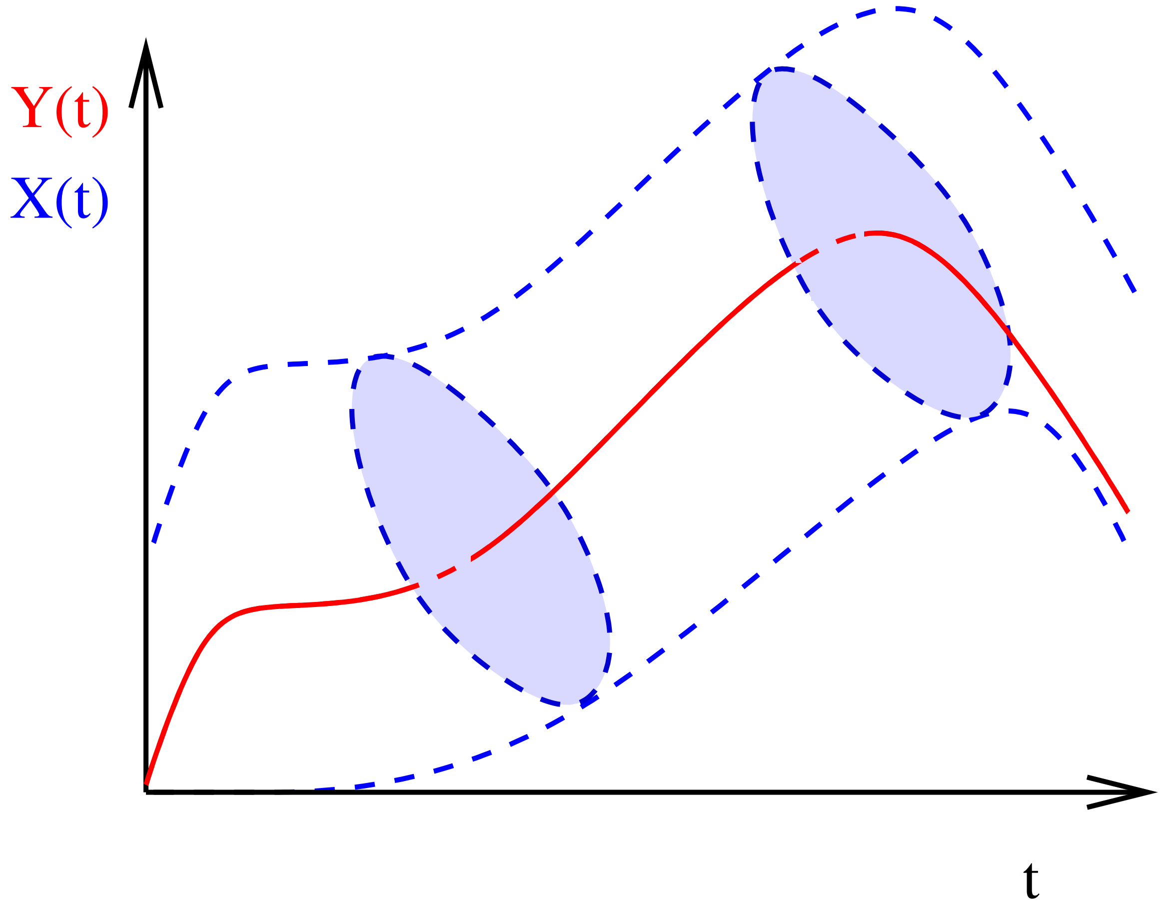
$$\boxed{Y_{Y_0}(T) = Y_0 + \int_{t=0}^{T} \mathbb{F}^\sharp\left(Y_{Y_0}(t)\right) \cdot dt}$$

The assumption about $\|\cdot\|$, $\|\cdot\|^\sharp$, and ψ ensures that $\psi \circ X_{X_0}$ is a maximal solution.

Fluid trajectories



Fluid trajectories



Overview

1. Context and motivations
2. Handmade ODEs
3. Abstract interpretation framework
   (a) Concrete semantics
   (b) Abstraction
   (c) Bisimulation
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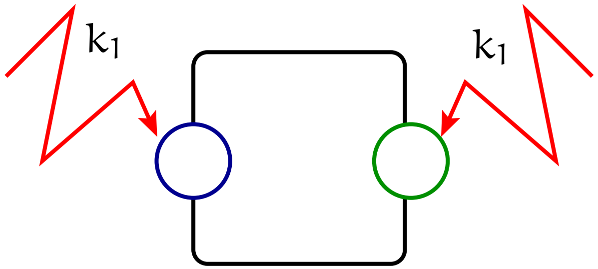
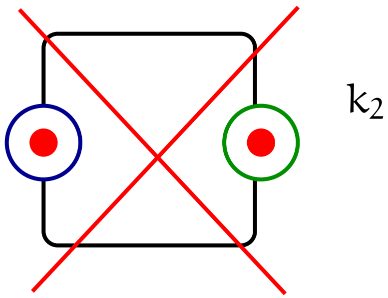
7. Conclusion

A model with symmetries



$$\begin{array}{llll} P & \longrightarrow & {}^{\star}P & k_1 \\ P & \longrightarrow & P^{\star} & k_1 \end{array} \qquad \begin{array}{llll} P^{\star} & \longrightarrow & {}^{\star}P^{\star} & k_1 \\ {}^{\star}P & \longrightarrow & {}^{\star}P^{\star} & k_1 \end{array}$$



$$ {}^{\star}P^{\star} \longrightarrow \emptyset \quad k_2 $$

Differential equations

- Initial system:

$$\frac{d}{dt}\begin{bmatrix} P \\ {}^{\star}P \\ P^{\star} \\ {}^{\star}P^{\star} \end{bmatrix} = \begin{bmatrix} -2\cdot k_1 & 0 & 0 & 0 \\ k_1 & -k_1 & 0 & 0 \\ k_1 & 0 & -k_1 & 0 \\ 0 & k_1 & k_1 & -k_2 \end{bmatrix} \cdot \begin{bmatrix} P \\ {}^{\star}P \\ P^{\star} \\ {}^{\star}P^{\star} \end{bmatrix}$$

- Reduced system:

$$\frac{d}{dt}\begin{bmatrix} P \\ {}^{\star}P + P^{\star} \\ 0 \\ {}^{\star}P^{\star} \end{bmatrix} = \begin{bmatrix} -2\cdot k_1 & 0 & 0 & 0 \\ 2\cdot k_1 & -k_1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & k_1 & 0 & -k_2 \end{bmatrix} \cdot \begin{bmatrix} P \\ {}^{\star}P + P^{\star} \\ 0 \\ {}^{\star}P^{\star} \end{bmatrix}$$

Differential equations

- Initial system:

$$\frac{d}{dt}\begin{bmatrix} P \\ {}^{\star}P \\ P^{\star} \\ {}^{\star}P^{\star} \end{bmatrix} = \begin{bmatrix} -2\cdot k_1 & 0 & 0 & 0 \\ k_1 & -k_1 & 0 & 0 \\ k_1 & 0 & -k_1 & 0 \\ 0 & k_1 & k_1 & -k_2 \end{bmatrix} \cdot \begin{bmatrix} P \\ {}^{\star}P \\ P^{\star} \\ {}^{\star}P^{\star} \end{bmatrix}$$

- Reduced system:

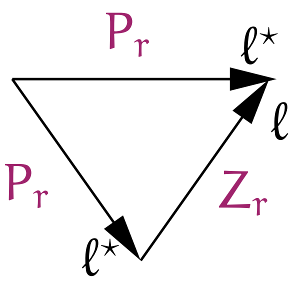
$$\frac{d}{dt}\begin{bmatrix} P \\ {}^{\star}P + P^{\star} \\ 0 \\ {}^{\star}P^{\star} \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{P} \cdot \begin{bmatrix} -2\cdot k_1 & 0 & 0 & 0 \\ k_1 & -k_1 & 0 & 0 \\ k_1 & 0 & -k_1 & 0 \\ 0 & k_1 & k_1 & -k_2 \end{bmatrix} \cdot \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{Z} \cdot \begin{bmatrix} P \\ {}^{\star}P + P^{\star} \\ 0 \\ {}^{\star}P^{\star} \end{bmatrix}$$

Pair of projections induced by an equivalence relation among variables

Let r be an idempotent mapping from $\mathcal{V}$ to $\mathcal{V}$.
We define two linear projections $P_r, Z_r \in (\mathcal{V} \to \mathbb{R}^+) \to (\mathcal{V} \to \mathbb{R}^+)$ by:

- $P_r(\rho)(V) = \begin{cases} \sum\{\rho(V') \mid r(V') = r(V)\} & \text{when } V = r(V) \\ 0 & \text{when } V \neq r(V); \end{cases}$
- $Z_r(\rho) = \begin{cases} V \mapsto \rho(V) & \text{when } V = r(V) \\ V \mapsto 0 & \text{when } V \neq r(V). \end{cases}$

We notice that the following diagram commutes:



Induced bisimulation

The mapping r induces a bisimulation,

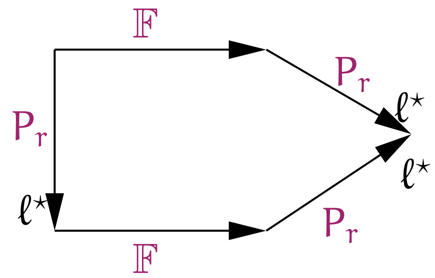
$\stackrel{\Delta}{\Longleftrightarrow}$

for any $\sigma, \sigma' \in \mathcal{V} \rightarrow \mathbb{R}^{+}$, $P_r(\sigma) = P_r(\sigma') \implies P_r(\mathbb{F}(\sigma)) = P_r(\mathbb{F}(\sigma'))$.

Indeed the mapping r induces a bisimulation,

$\Longleftrightarrow$

for any $\sigma \in \mathcal{V} \rightarrow \mathbb{R}^{+}$, $P_r(\mathbb{F}(\sigma)) = P_r(\mathbb{F}(P_r(\sigma)))$.



Induced abstraction

Under these assumptions $(r(\mathcal{V}), P_r, P_r \circ \mathbb{F} \circ Z_r)$ is an abstraction of $(\mathcal{V}, \mathbb{F})$:

As proved in the following commutative diagram:

$$\begin{tikzcd}
\cdot \arrow[rrr, "\mathbb{F}"] \arrow[d, "P_r"'] \arrow[dr, "P_r"] & & & \cdot \arrow[d, "P_r"] \\
\ell^\star \arrow[r, "Z_r"'] & \cdot \arrow[r, "\mathbb{F}"'] & \cdot \arrow[r, "P_r"'] & \ell^\star
\end{tikzcd}$$

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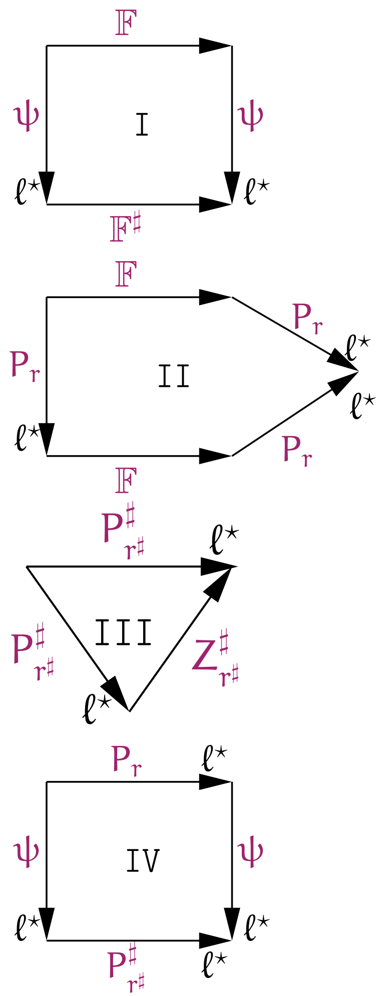
Abstract projection

We assume that we are given:

- a concrete system $(\mathcal{V}, \mathbb{F})$;
- an abstraction $(\mathcal{V}^\sharp, \psi, \mathbb{F}^\sharp)$ of $(\mathcal{V}, \mathbb{F})$ $(\mathtt{I})$;
- a mapping r over $\mathcal{V}$ which induces a bisimulation $(\mathtt{II})$;
- (P_r, Z_r), the pair of projections induced by r;
- a mapping $r^\sharp$ over $\mathcal{V}^\sharp$;
- $(P^\sharp_{r^\sharp}, Z^\sharp_{r^\sharp})$, the pair of projections over $\mathcal{V}^\sharp$ induced by $r^\sharp$ $(\mathtt{III})$;

such that: $\psi \circ P_r = P^\sharp_{r^\sharp} \circ \psi$ $(\mathtt{IV})$.



Combination of abstractions

Under these assumptions, $(r^\sharp(\mathcal{V}^\sharp), P^\sharp_{r^\sharp} \circ \psi, P^\sharp_{r^\sharp} \circ \mathbb{F}^\sharp \circ Z^\sharp_{r^\sharp})$ is an abstraction of $(\mathcal{V}, \mathbb{F})$,
as proved in the following commutative diagram:

$$
\begin{tikzcd}
{} \arrow[rrrr, "\mathbb{F}"] \arrow[dr, "P_r"] \arrow[dd, "\psi"'] \arrow[ddr, phantom, "\mathrm{IV}"] & & \arrow[d, phantom, "\mathrm{II}"] & & {} \arrow[dl, "P_r"'] \arrow[dd, "\psi"] \arrow[ddl, phantom, "\mathrm{IV}"] \\
& {} \arrow[r, "\mathbb{F}"] \arrow[dd, "\psi"] \arrow[ddr, phantom, "\mathrm{I}"] & {} \arrow[r, "P_r"] \arrow[dd, "\psi"] \arrow[ddr, phantom, "\mathrm{IV}"] & {} \arrow[ddr, "\psi"] & \\
\ell^\star \arrow[d, "P^\sharp_{r^\sharp}"'] \arrow[dr, "P^\sharp_{r^\sharp}"] \arrow[d, phantom, shift left=3ex, "\mathrm{III}"] & & & & \ell^\star \arrow[d, "P^\sharp_{r^\sharp}"] \\
\ell^\star \arrow[r, "Z^\sharp_{r^\sharp}"'] & {} \arrow[r, "\mathbb{F}^\sharp"'] & {} \arrow[rr, "P^\sharp_{r^\sharp}"'] & & \ell^\star
\end{tikzcd}
$$

Overview

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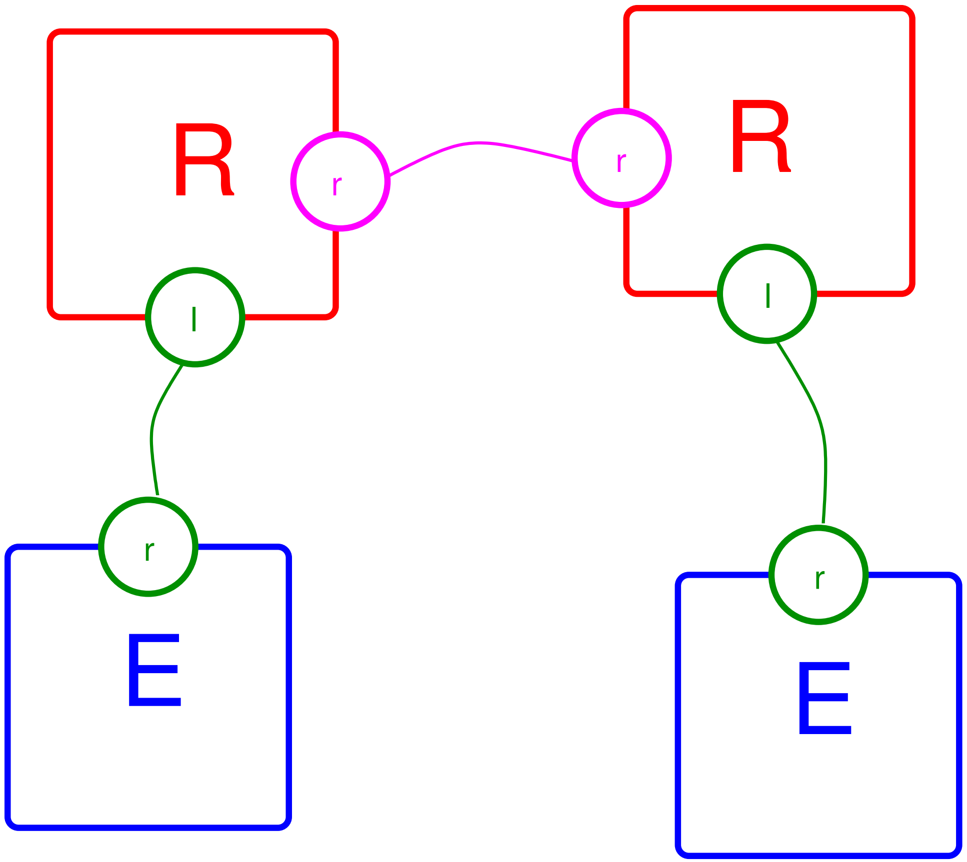
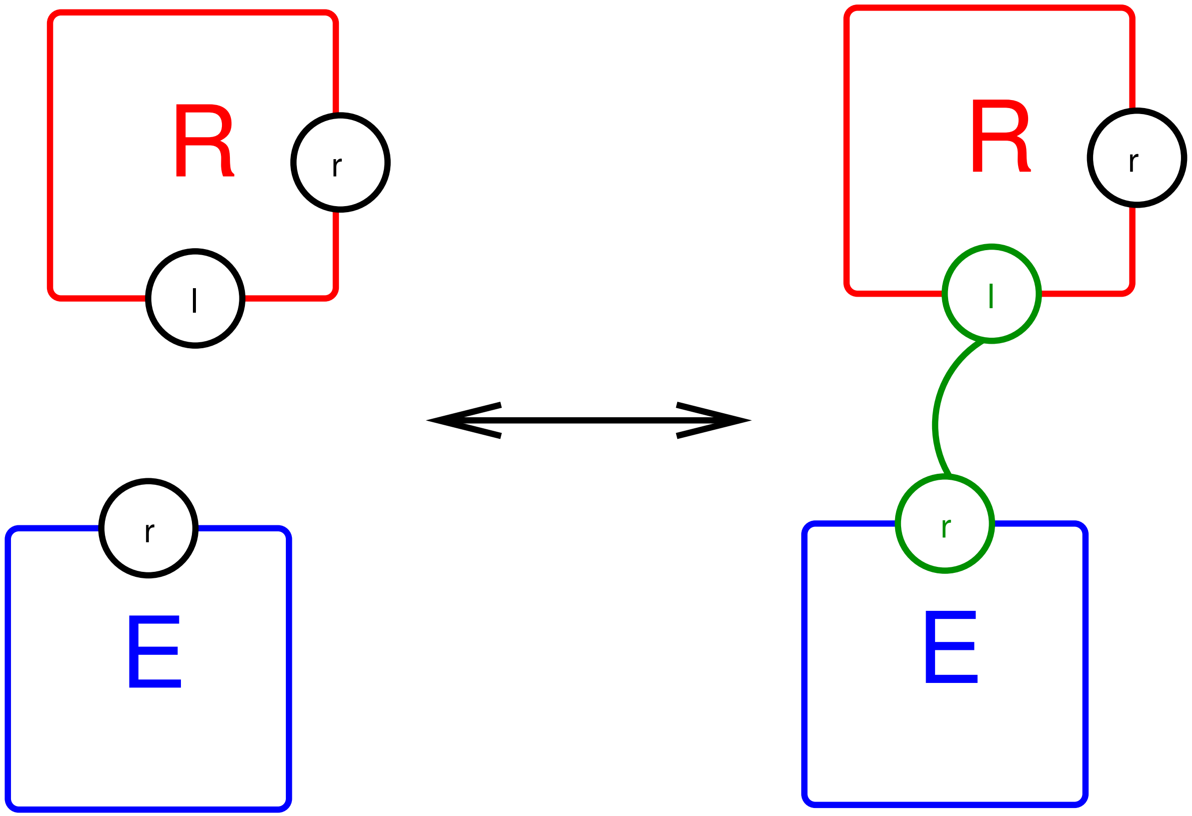
A species



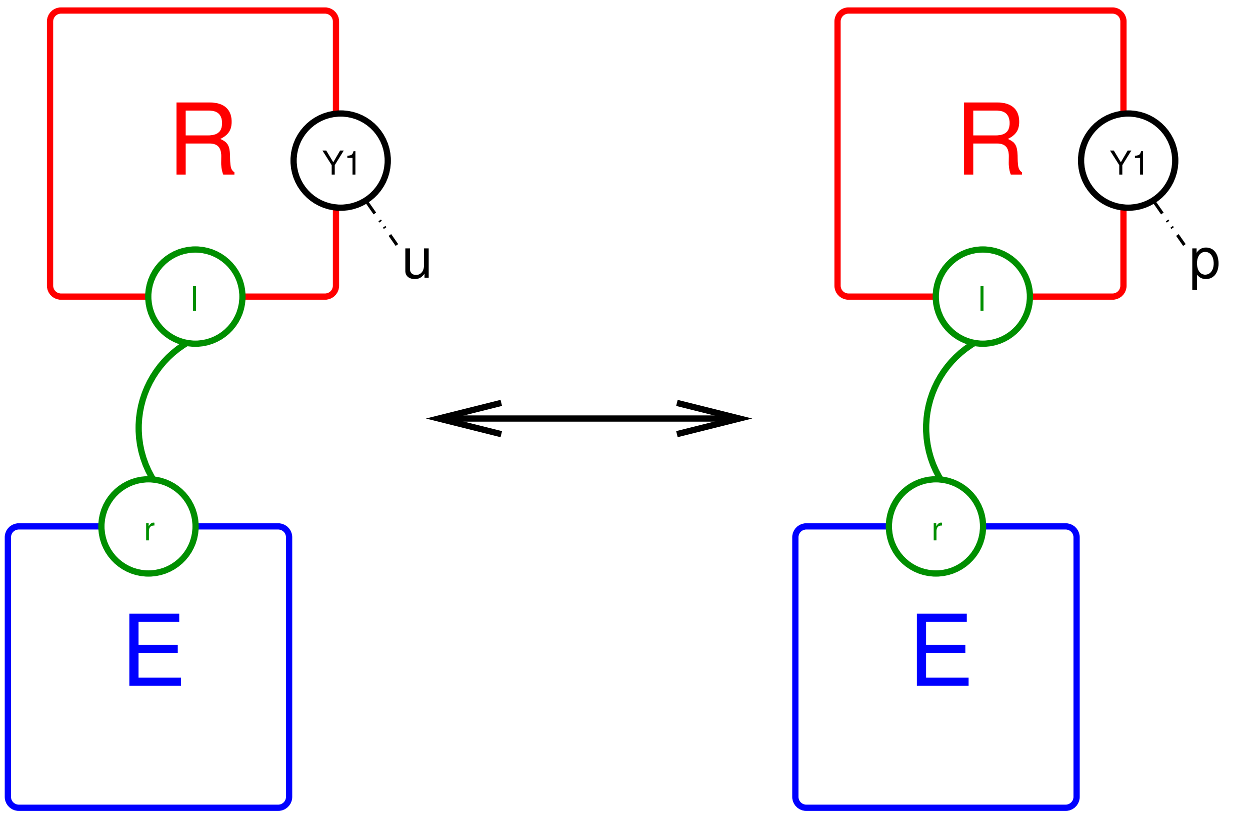
E(r!1), R(l!1,r!2), R(r!2,l!3), E(r!3)

A Unbinding/Binding Rule



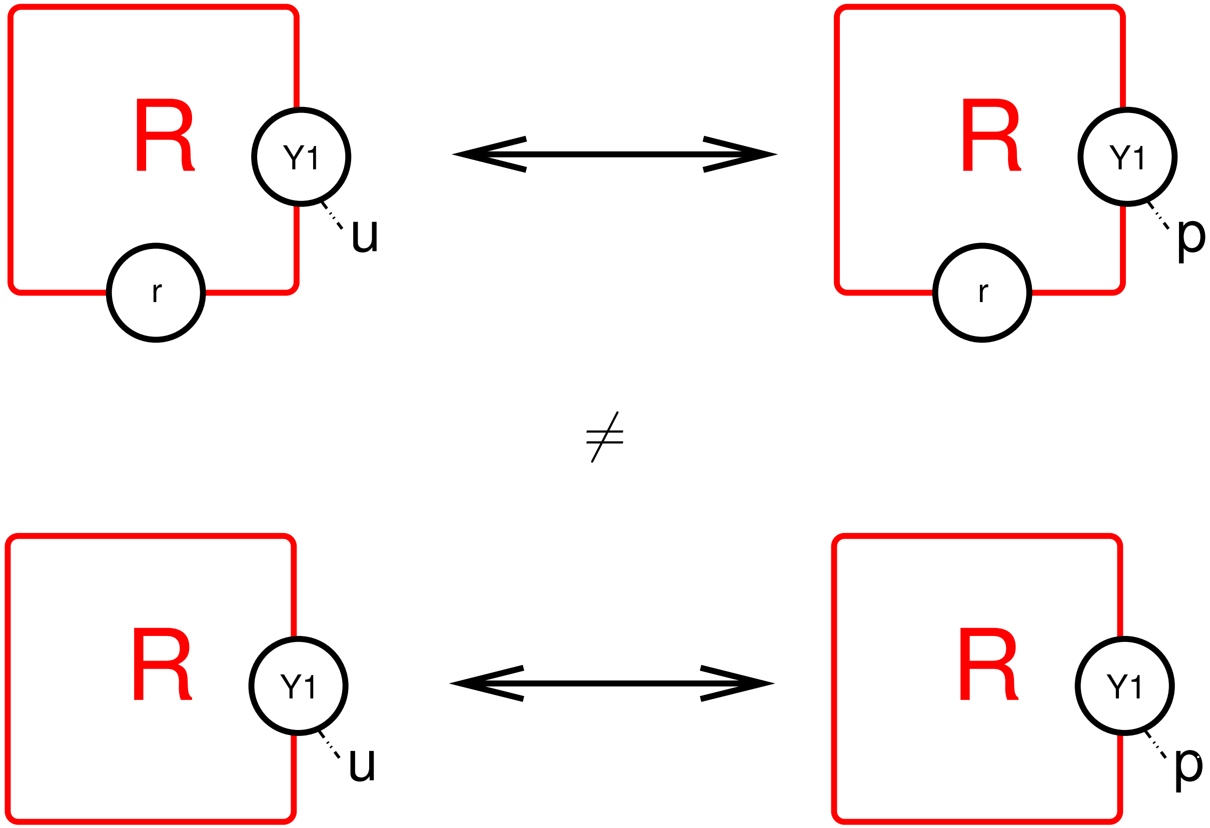
E(r), R(l,r) $\longleftrightarrow$ E(r!1), R(l!1,r)

Internal state



$$R(Y1\sim u, l!1), E(r!1) \longleftrightarrow R(Y1\sim p, l!1), E(r!1)$$

Don’t care, Don’t write



Overview

1. Context and motivations
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Requirements

1. Reachable species

   A set $\mathcal{R}$ of connected site-graphs such that:

   - $\mathcal{R}$ is finite;
   - $\mathcal{R}$ is closed with respect to rule application: i.e. applying a rule with a tuple of site-graphs in $\mathcal{R}$ gives a tuple of site-graphs in $\mathcal{R}$;

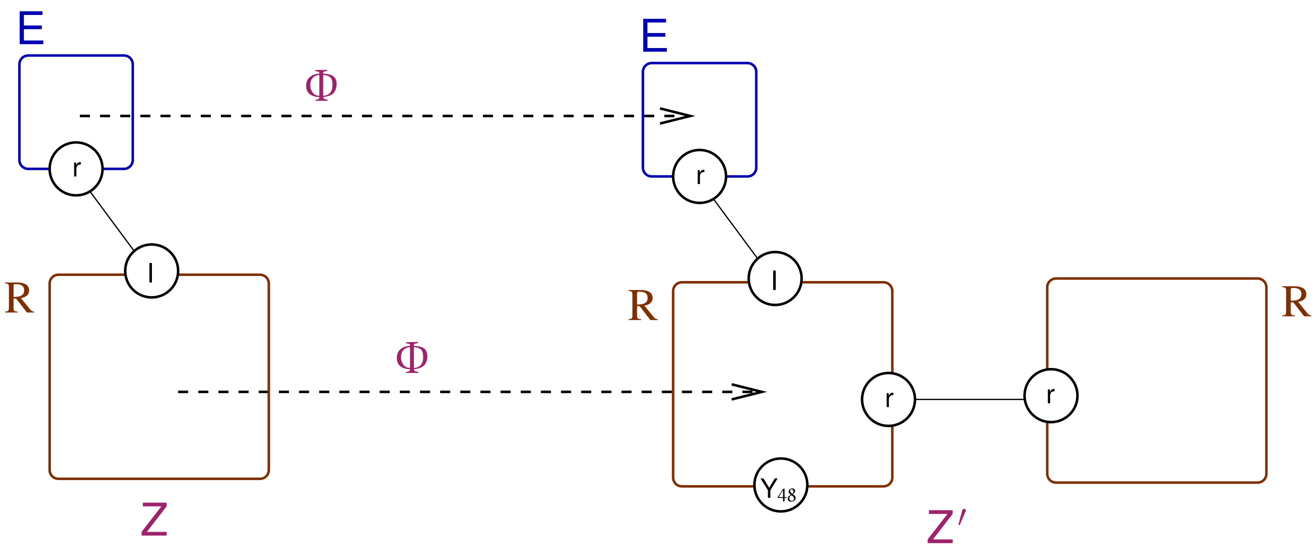
2. Rules are associated with kinetic factors

   - the unit depends on the arity of the rule as follows:

   $$\left(\frac{L}{mol}\right)^{arity-1} \cdot s^{-1}$$

   where *arity* is the number of connected components in the lhs.

Embedding



We write $Z \lhd_{\Phi} Z'$ iff:

- Φ is a site-graph morphism:
  - i is less specific than $\Phi(i)$,
  - if there is a link between (i, s) and (i', s'), then there is a link between $(\Phi(i), s)$ and $(\Phi(i'), s')$.
- Φ is an into map (injective):
  - $\Phi(i) = \Phi(i')$ implies that $i = i'$.

Differential system

Let us consider a rule *rule*:

$$\textit{lhs} \rightarrow \textit{rhs} \quad k.$$

1. We write *lhs* as a multi-set $\{C_i\}$ of non empty connected components.
2. A ground instantiation of the rule *rule* is defined by a tuple $(r_i, \Phi i)$ such that $\forall i, r_i \in \mathcal{R}$ and $C_i \lhd_{\Phi i} r_i$.
3. The ground instantiation can be written as follows:

$$r_1, \ldots, r_m \rightarrow p_1, \ldots, p_n \quad k.$$

4. The activity of a ground instantiation is defined as:

$$\textit{act}_{(r_i,\Phi i)} = \frac{k \cdot \prod [r_i]}{\sharp\{\Phi \mid \textit{lhs} \lhd_{\Phi} \textit{lhs}\}}.$$

5. Each ground instantiation induces the following contributions:

$$\frac{d[r_i]}{dt} \stackrel{+}{=} -\textit{act}_{(r_i,\Phi i)}, \qquad \frac{d[p_i]}{dt} \stackrel{+}{=} \textit{act}_{(r_i,\Phi i)}.$$

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    (a) Fragments
    (b) Soundness criteria
    (c) Abstract counterpart
    (d) Symmetries between sites
7. Conclusion

Abstract domain

We are looking for suitable pair $(\mathcal{V}^\sharp, \psi)$ (such that $\mathbb{F}^\sharp$ exists)

The set of linear variable replacements is too big to be explored.

We introduce a specific shape on $(\mathcal{V}^\sharp, \psi)$ so as:

- restrict the exploration;
- drive the intuition;
- having efficient way to find suitable abstractions $(\mathcal{V}^\sharp, \psi)$ and to compute $\mathbb{F}^\sharp$.

Our choice might be not optimal, but we can live with that.

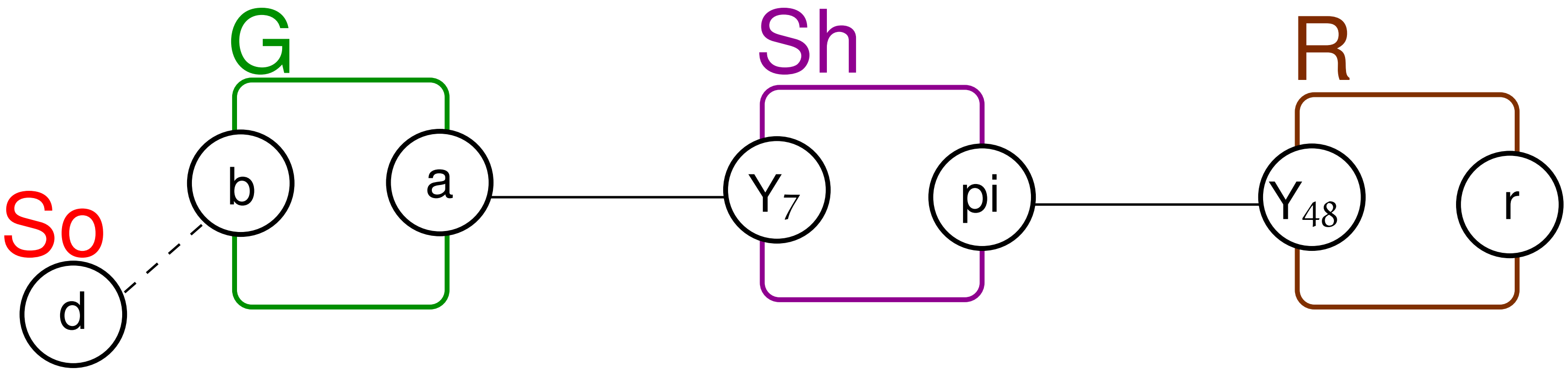
Partial species

Fragments are well-chosen partial species.

A partial species $X \in \mathcal{P}$ is a connected site-graph such that:

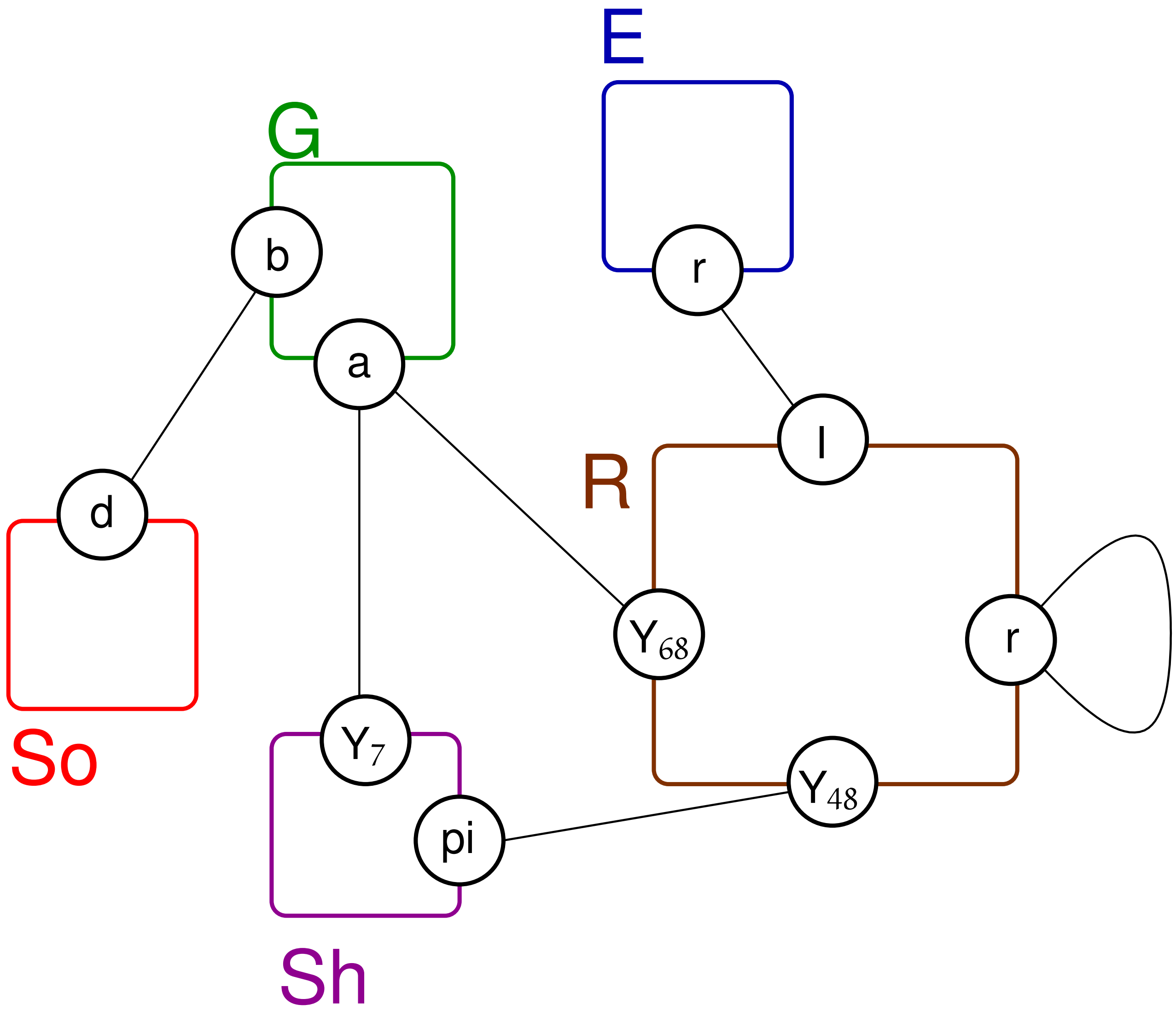
- the set of the sites of each node of type A is a subset of the set of the sites of A;
- sites are free, bound to an other site, or tagged with a binding type.

For instance:

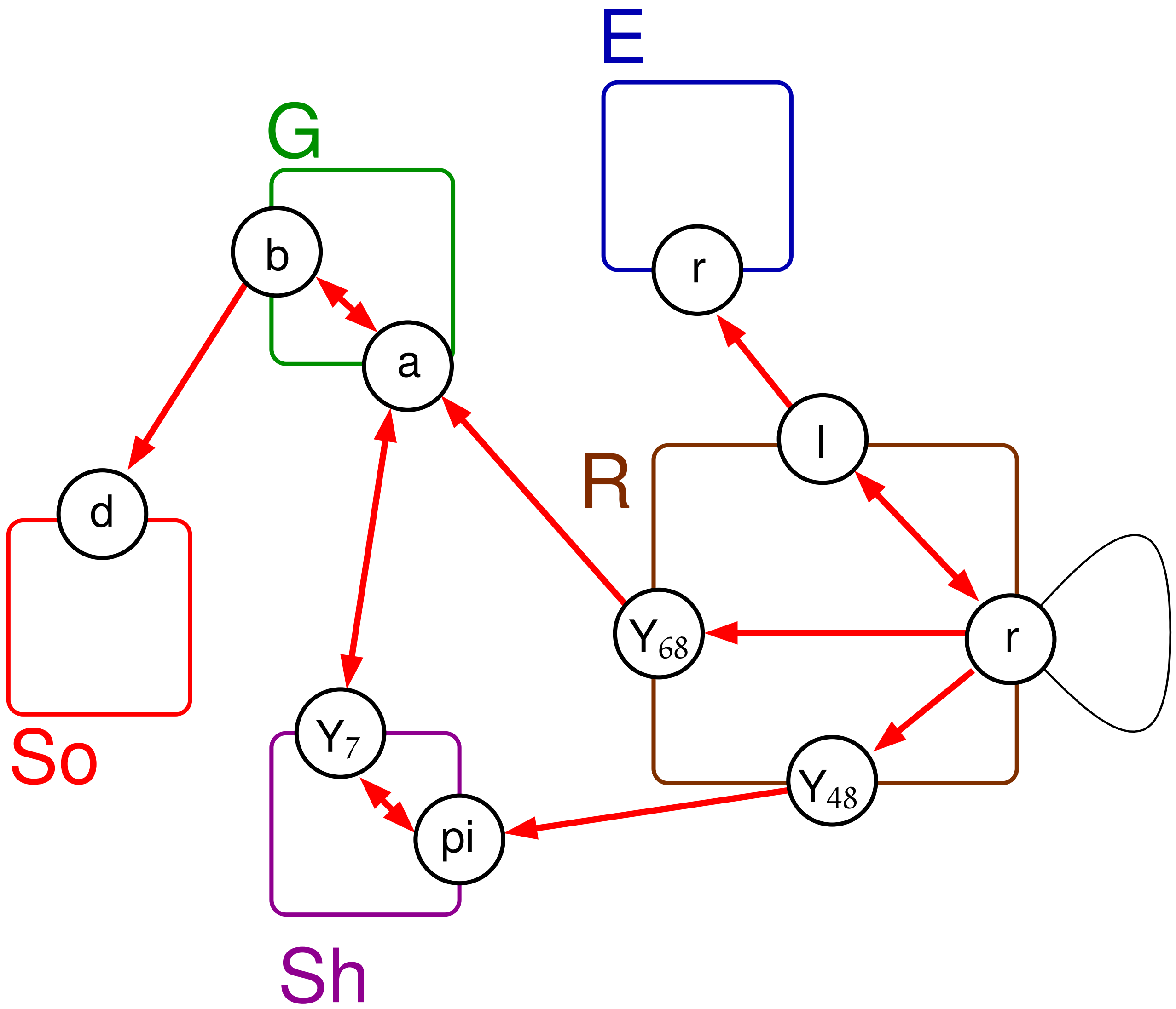


G(b!d.So,a!1),Sh(Y_7!1,pi!2),R(Y_{48}!2,r)

Contact map



Annotated contact map

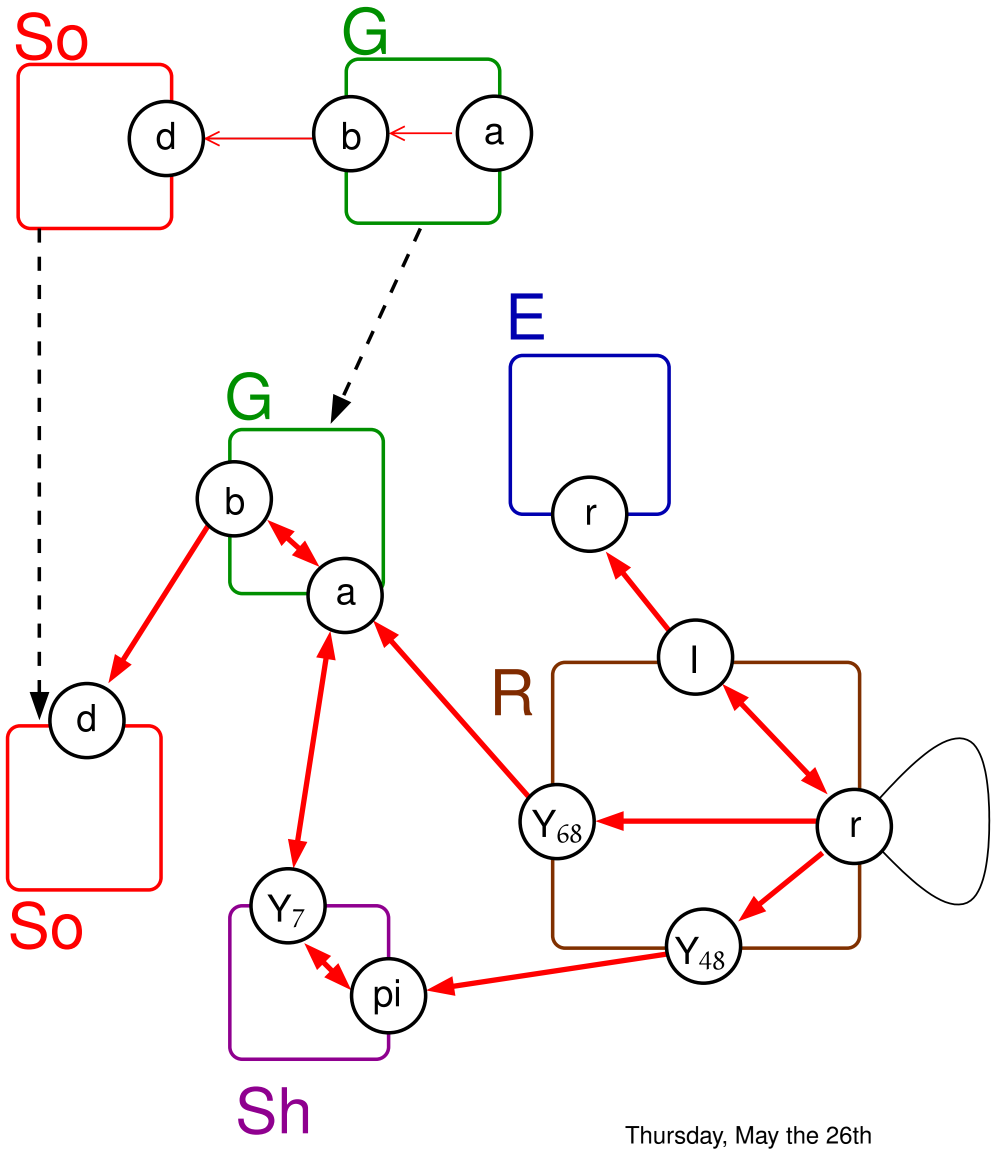


Fragments and prefragments

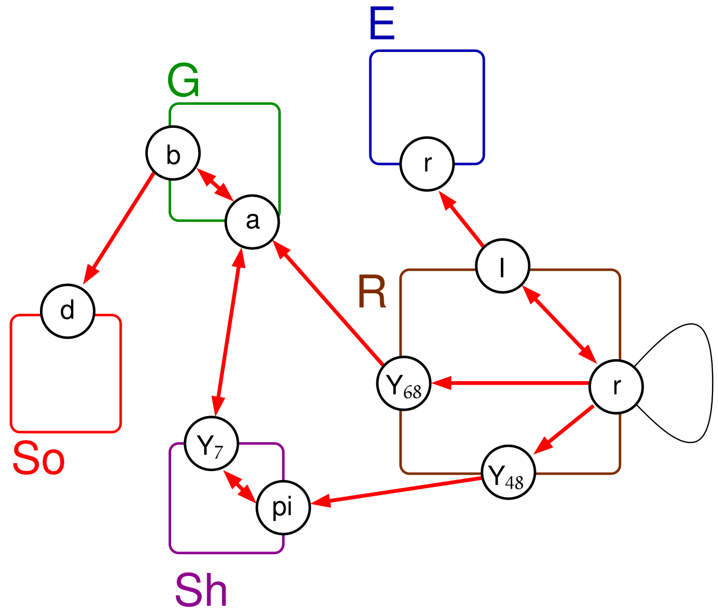
A prefragment is a connected site-graph which can be annotated with a binary relation $\rightarrow$ over the sites, such that:

1. There would be a site which is reachable from each other sites, via the reflexive and transitive closure of $\rightarrow$;
2. Any relation over sites can be projected over a relation on the annotated interaction map.

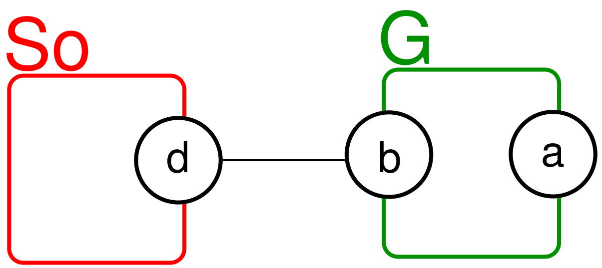
A fragment is a maximal prefragment (for the embedding order).

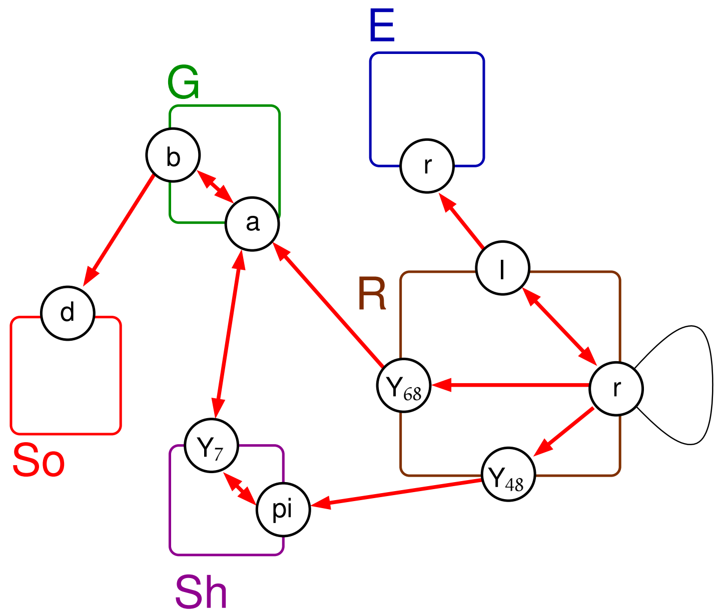


Are they fragments ?

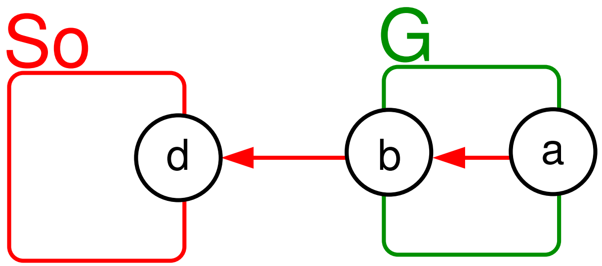


Are they fragments ?

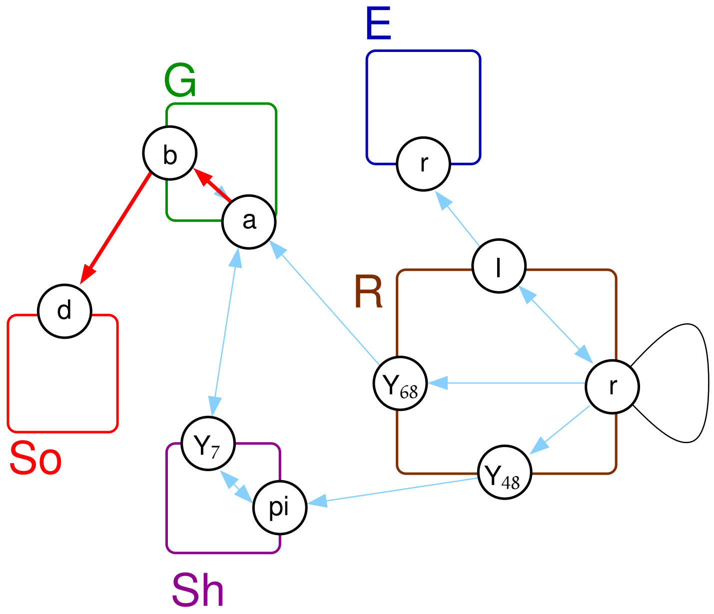




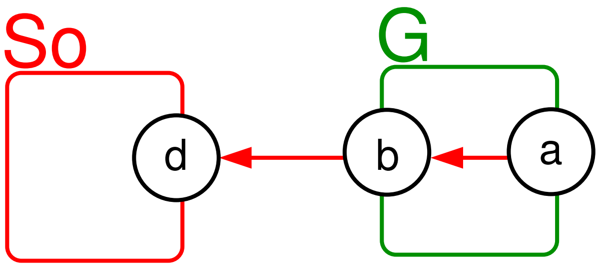
Are they fragments ?



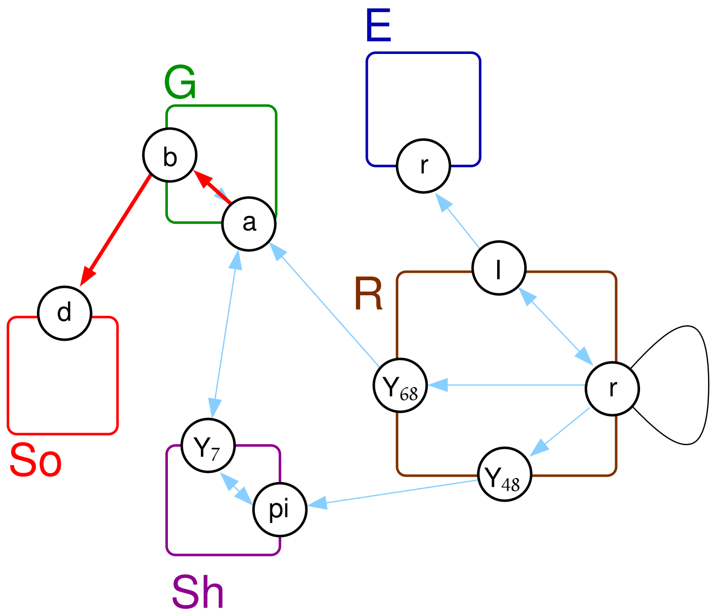
Thus, it is a prefragment.



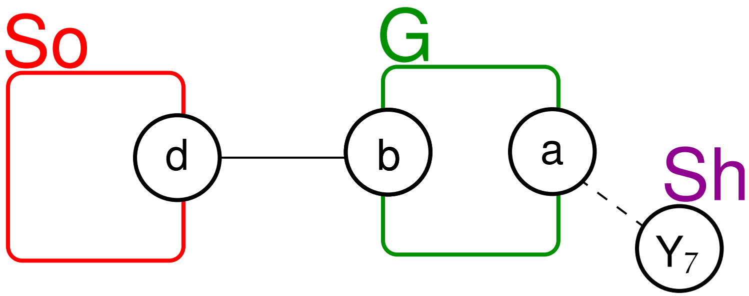
Are they fragments ?

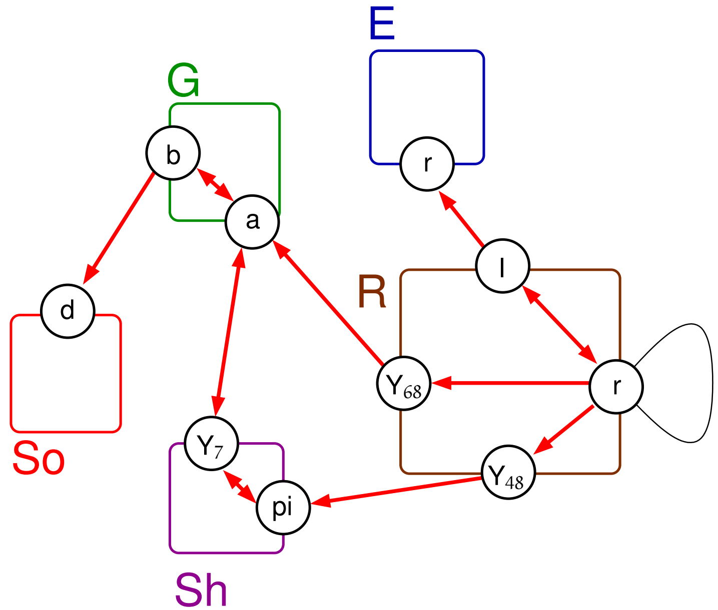


It is maximally specified.
Thus it is a fragment.

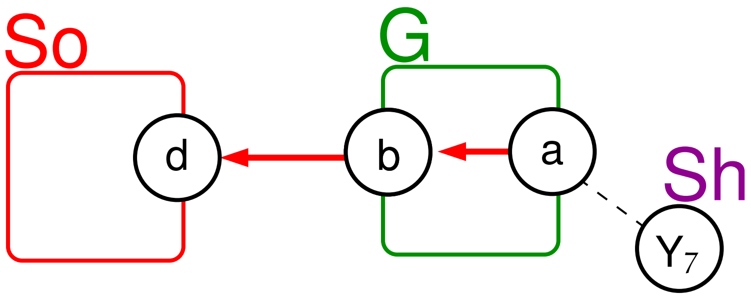


Are they fragments ?

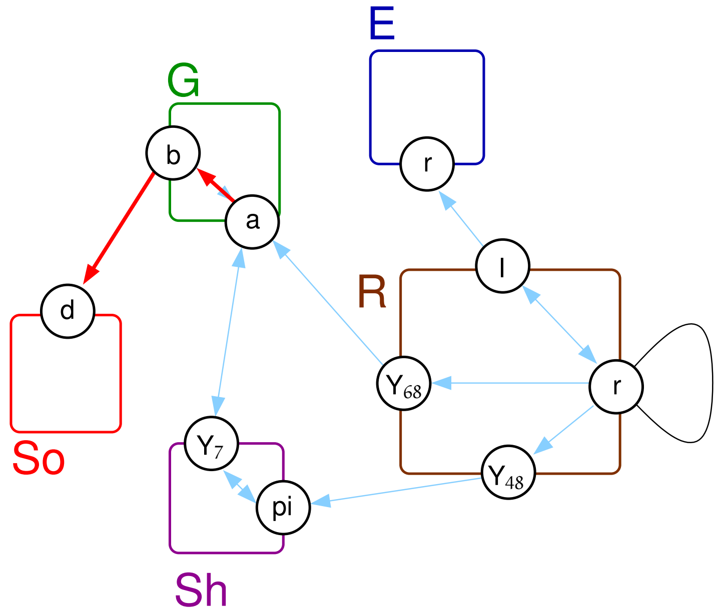




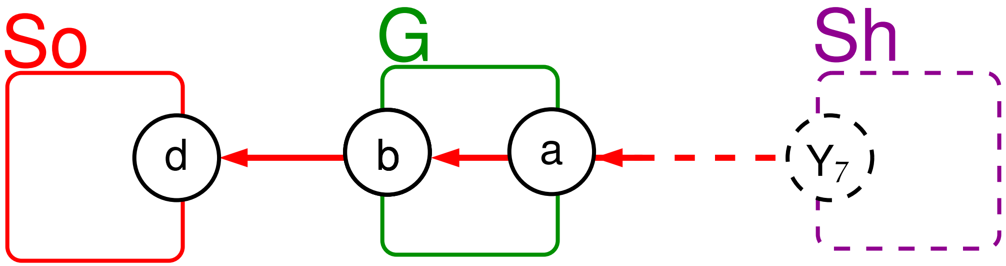
Are they fragments ?

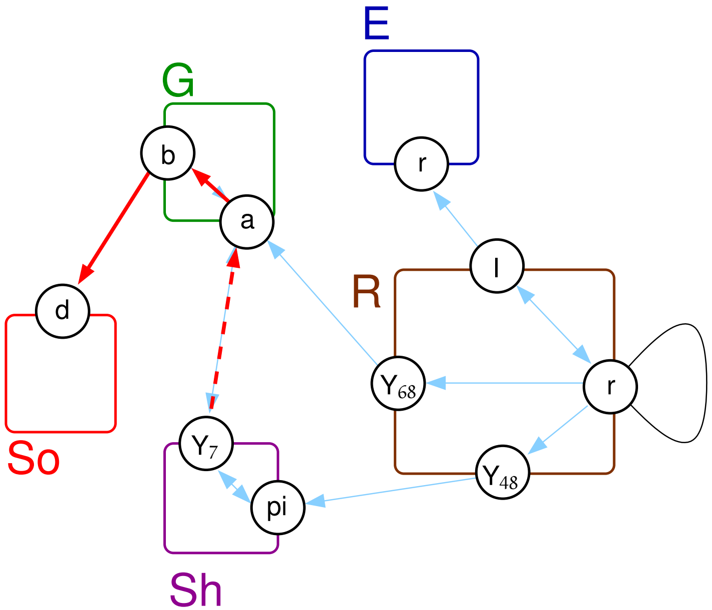


Thus, it is a prefragment.



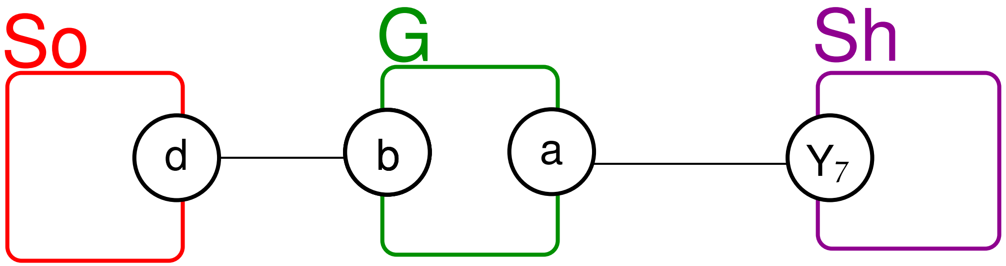
Are they fragments ?

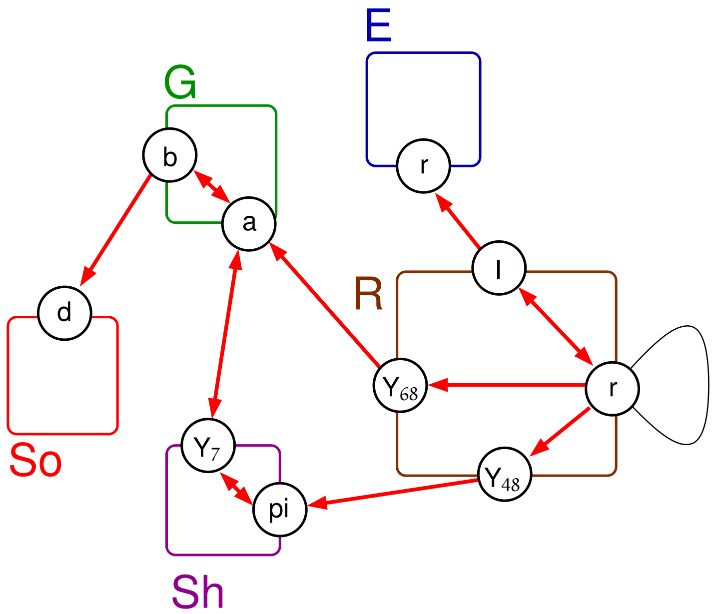




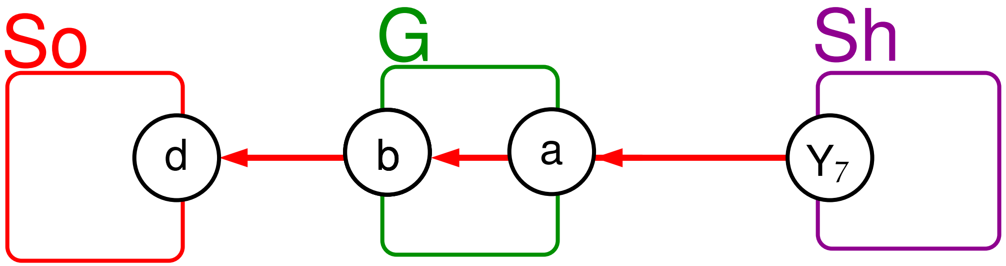
It can be refined into another prefragment.
Thus, it is not a fragment.

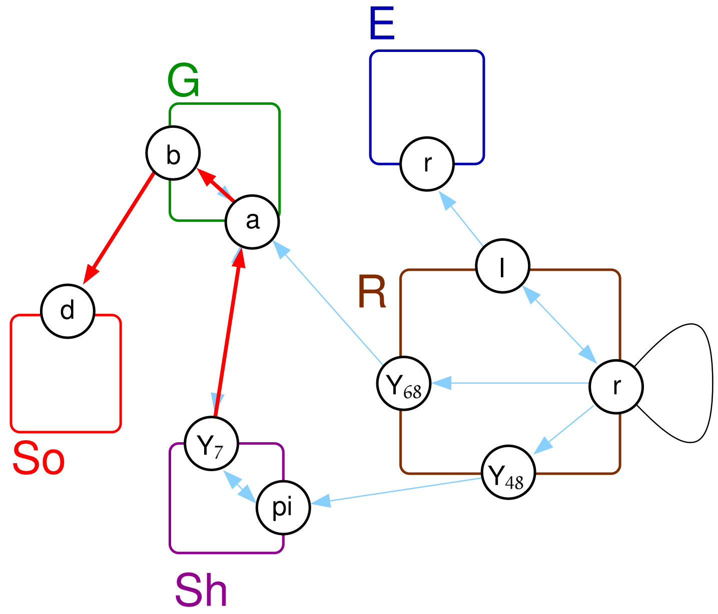
Are they fragments ?





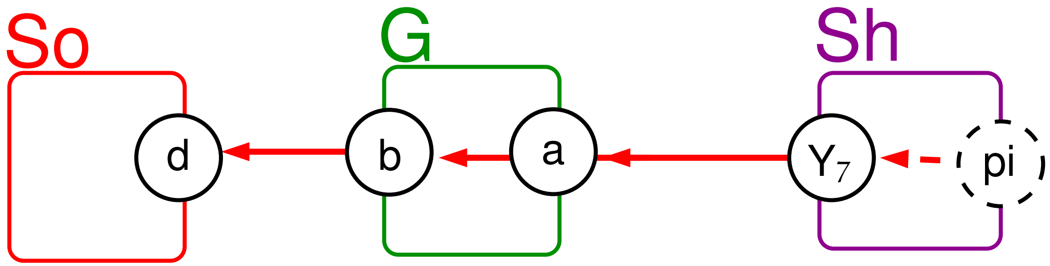
Are they fragments ?

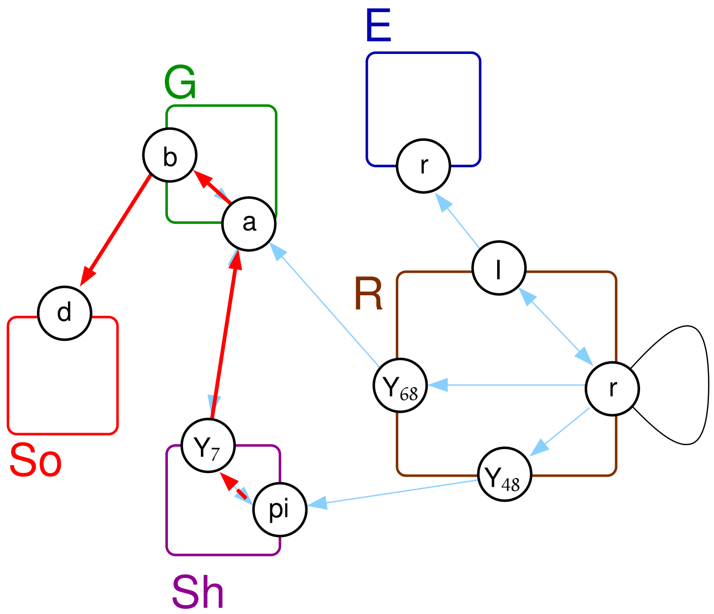




Thus, it is a prefragment.

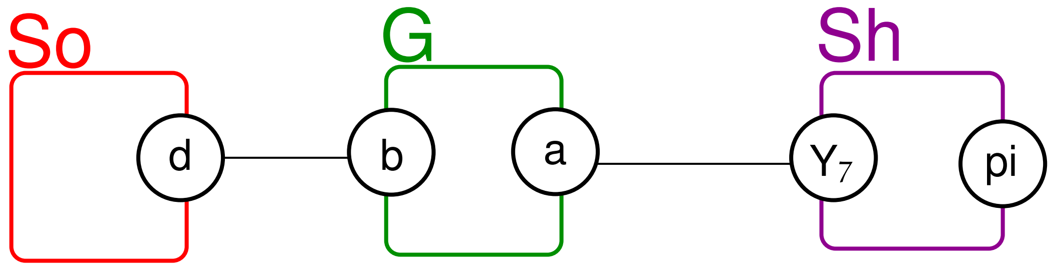
Are they fragments ?

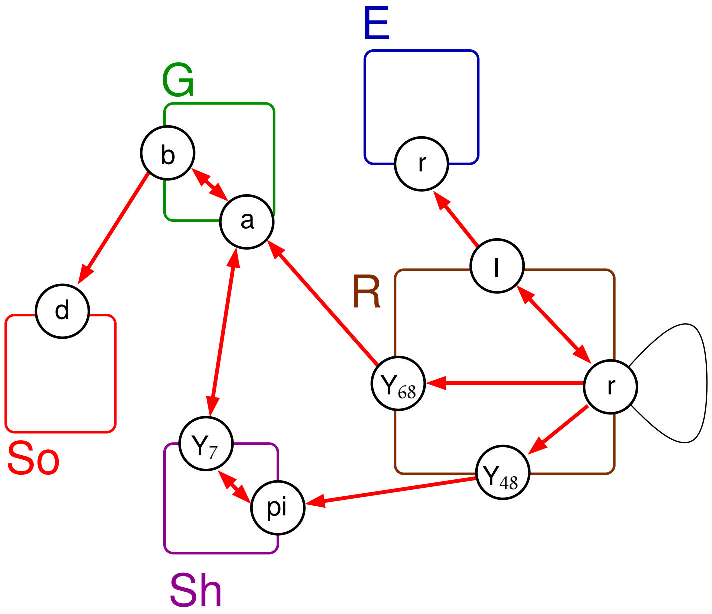




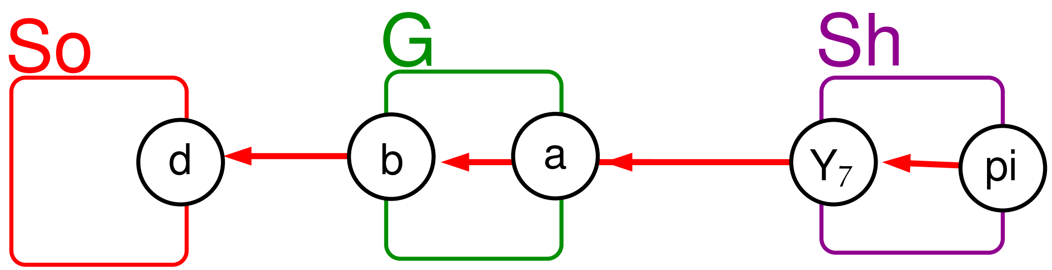
It can be refined into another prefragment.
Thus, it is not a fragment.

Are they fragments ?

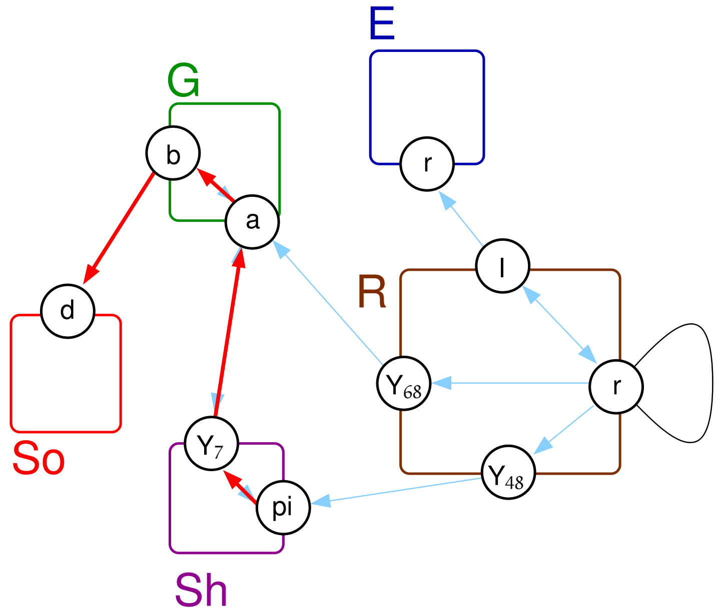




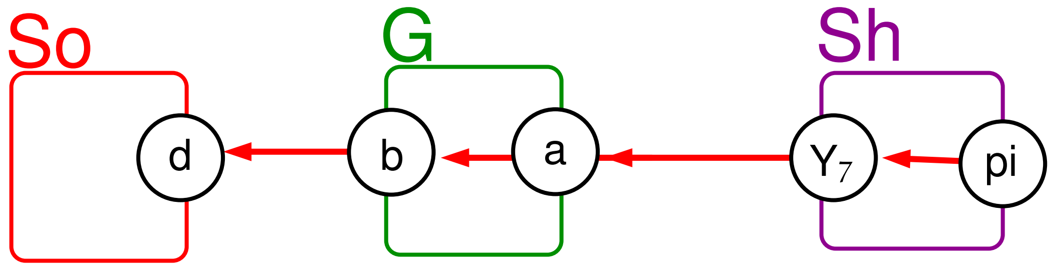
Are they fragments ?

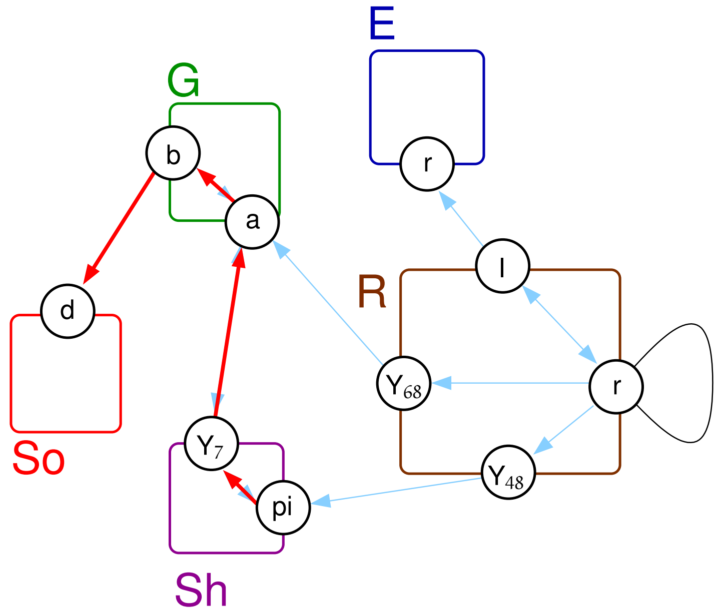


Thus, it is a prefragment.



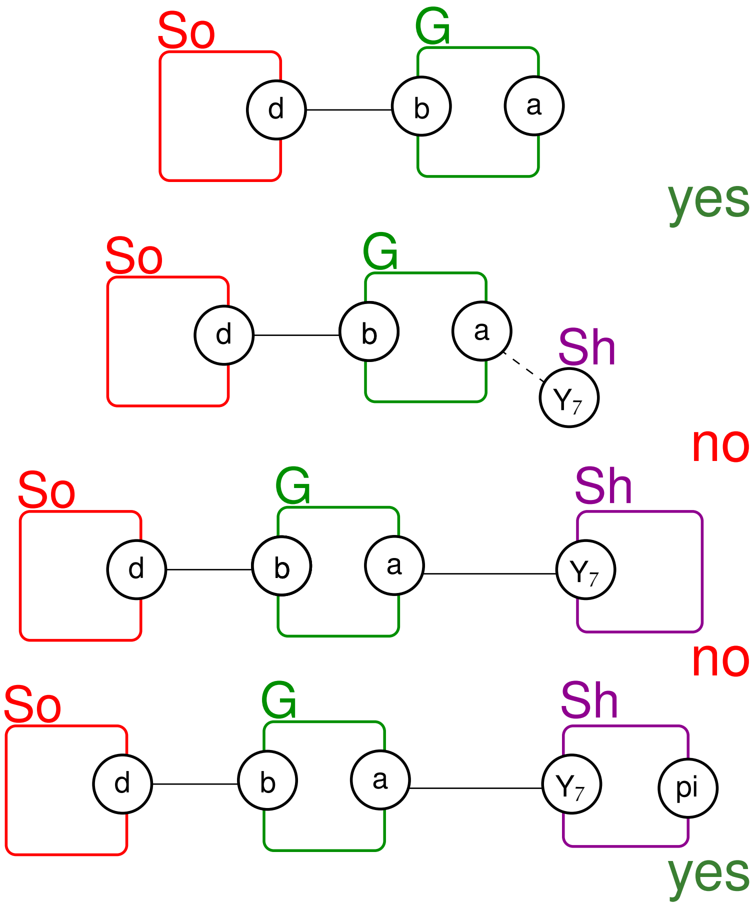
Are they fragments ?

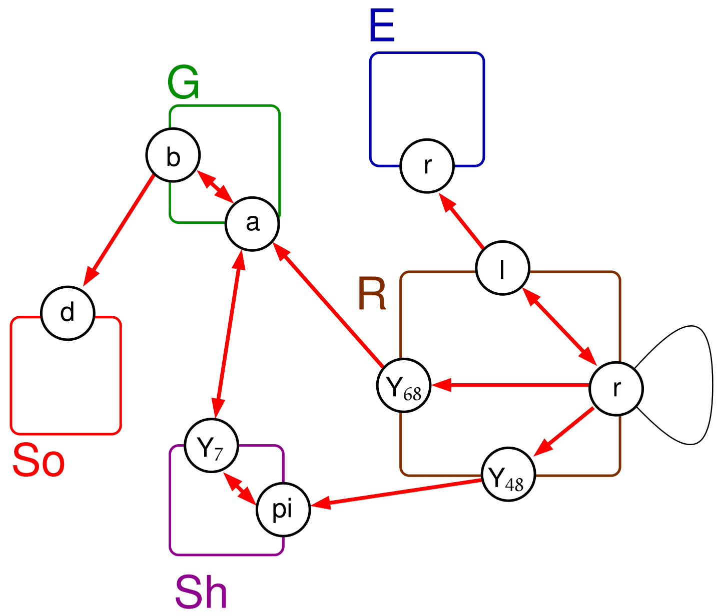




It is maximally specified.
Thus it is a fragment.

Are they fragments ?





Basic properties

1. We call a sub-fragment any partial species which can be embedded into a fragment.

   Property 1 (sub-fragment) The concentration of any sub-fragment can be expressed as a linear combination of the concentration of some fragments.

2. We consider two norms $||\cdot||$ on $\mathcal{V} \rightarrow \mathbb{R}^+$ and $||\cdot||^\sharp$ on $\mathcal{V}^\sharp \rightarrow \mathbb{R}^+$.

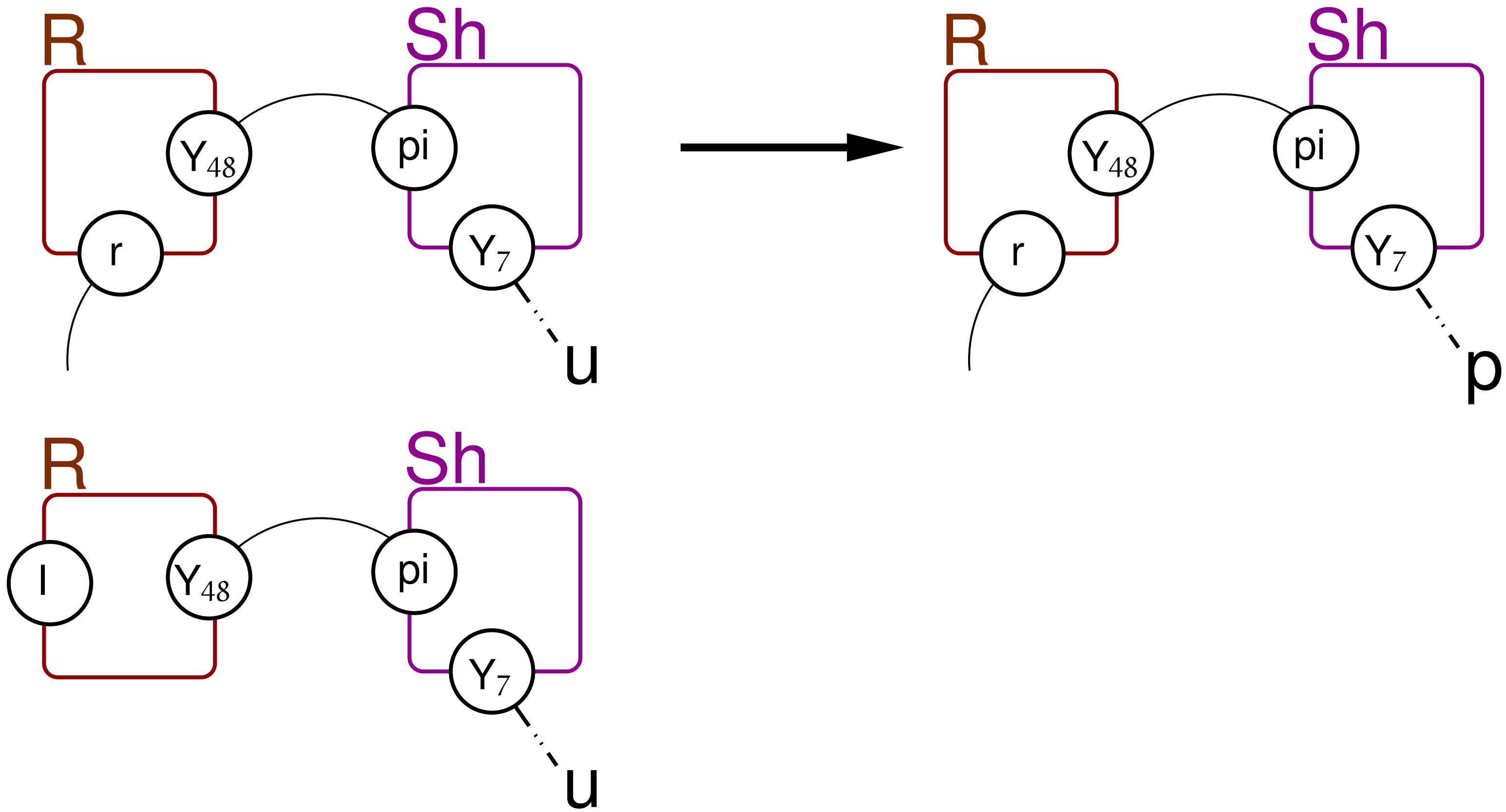
   Property 2 (non-degenerescence) Given a sequence of valuations $(x_n)_{n\in\mathbb{N}} \in (\mathcal{V} \rightarrow \mathbb{R}^+)^{\mathbb{N}}$ such that $||x_n||$ diverges toward $+\infty$, then $||\phi(x_n)||^\sharp$ diverges toward $+\infty$ as well.

Which other properties do we need so that the function $\mathbb{F}^\sharp$ can be defined ?

Overview

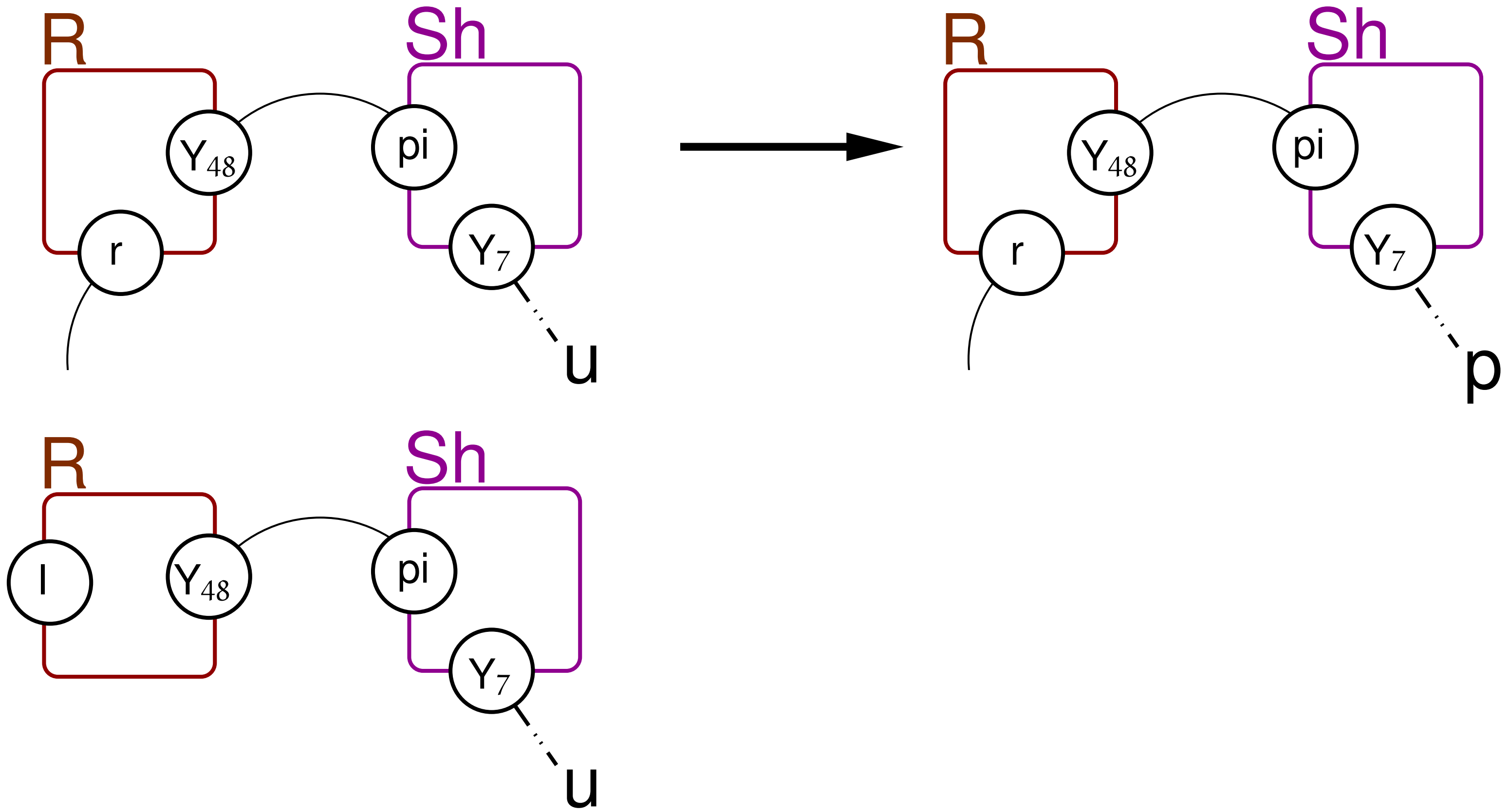
1. Context and motivations
2. Handmade ODEs
3. Abstract interpretation framework
4. Kappa
5. Concrete semantics
6. Abstract semantics
   (a) Fragments
   (b) Soundness criteria
   (c) Abstract counterpart
   (d) Symmetries between sites
7. Conclusion

Fragments consumption



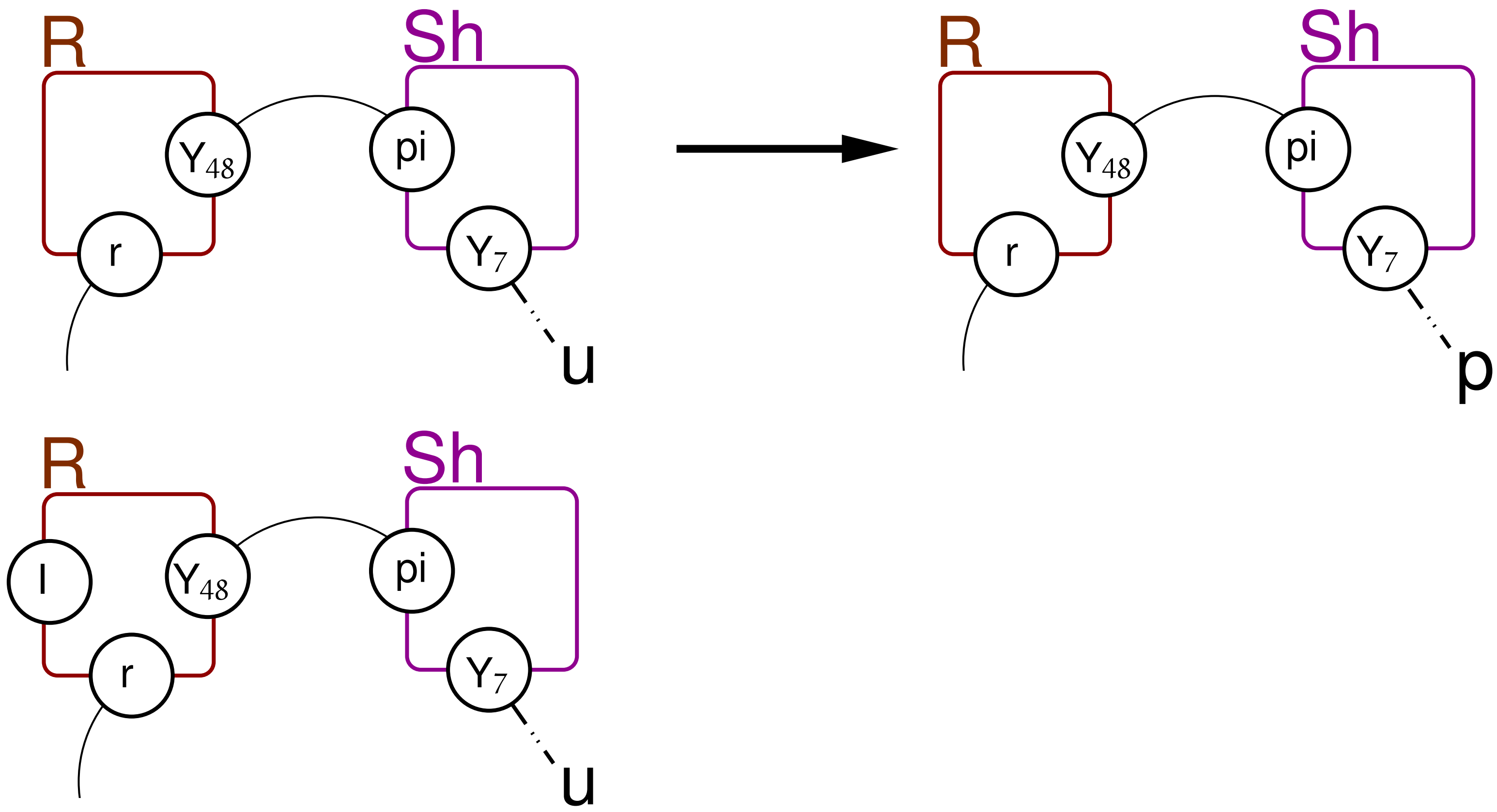
Can we express the amount (per time unit) of this fragment (bellow) concentration that is consumed by this rule (above)?

Fragments consumption



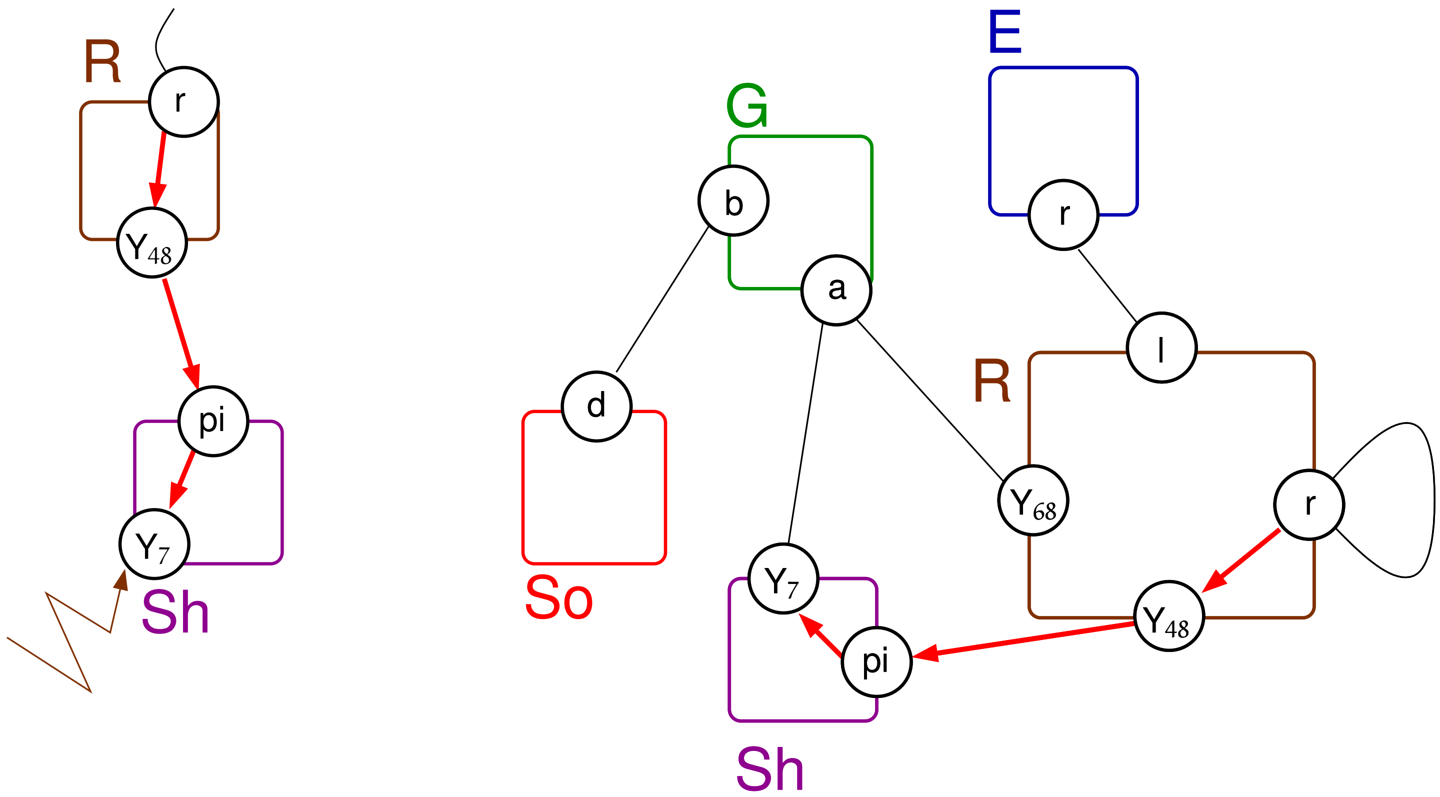
No, because we have abstracted away the correlation between the state of the site r and the state of the site l.

Fragments consumption
Proper intersection



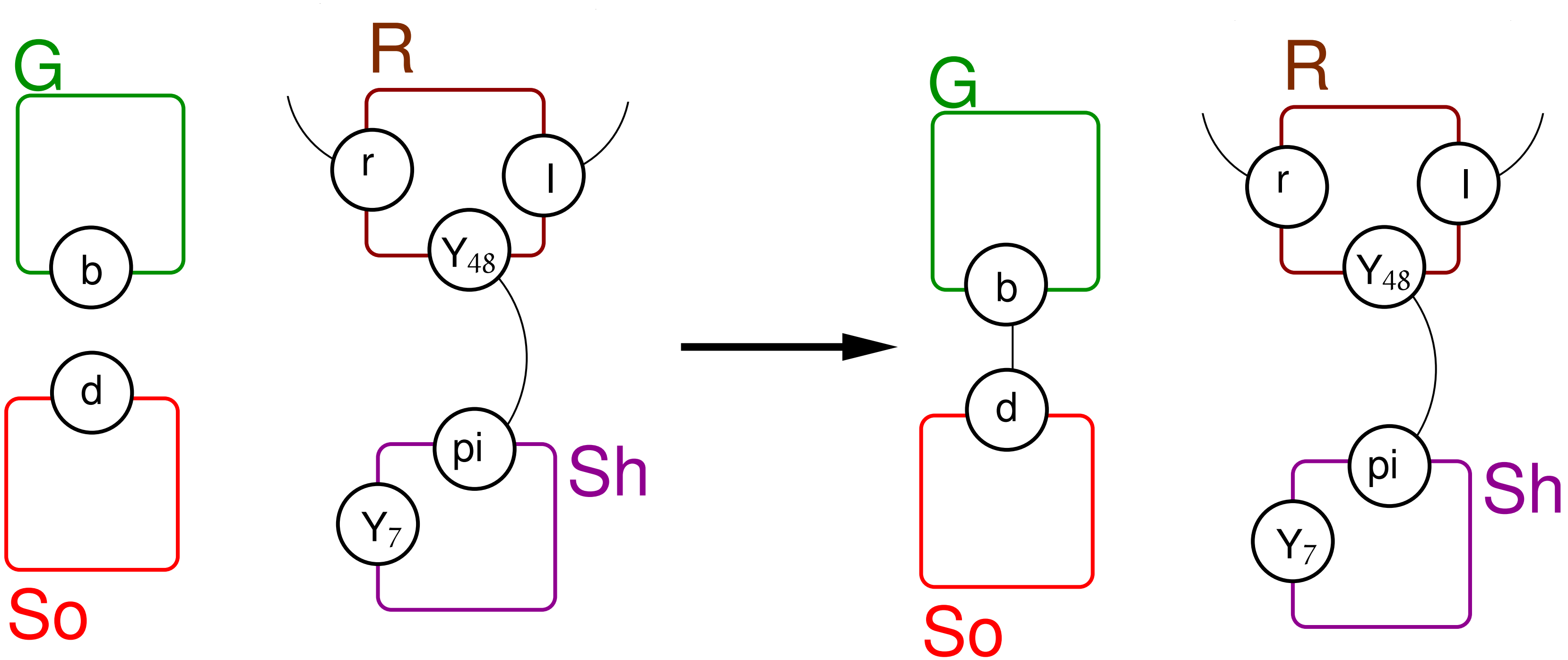
Whenever a fragment intersects a connected component of a lhs on a modified site, then the connected component must be embedded in the fragment!

Fragment consumption
Syntactic criteria



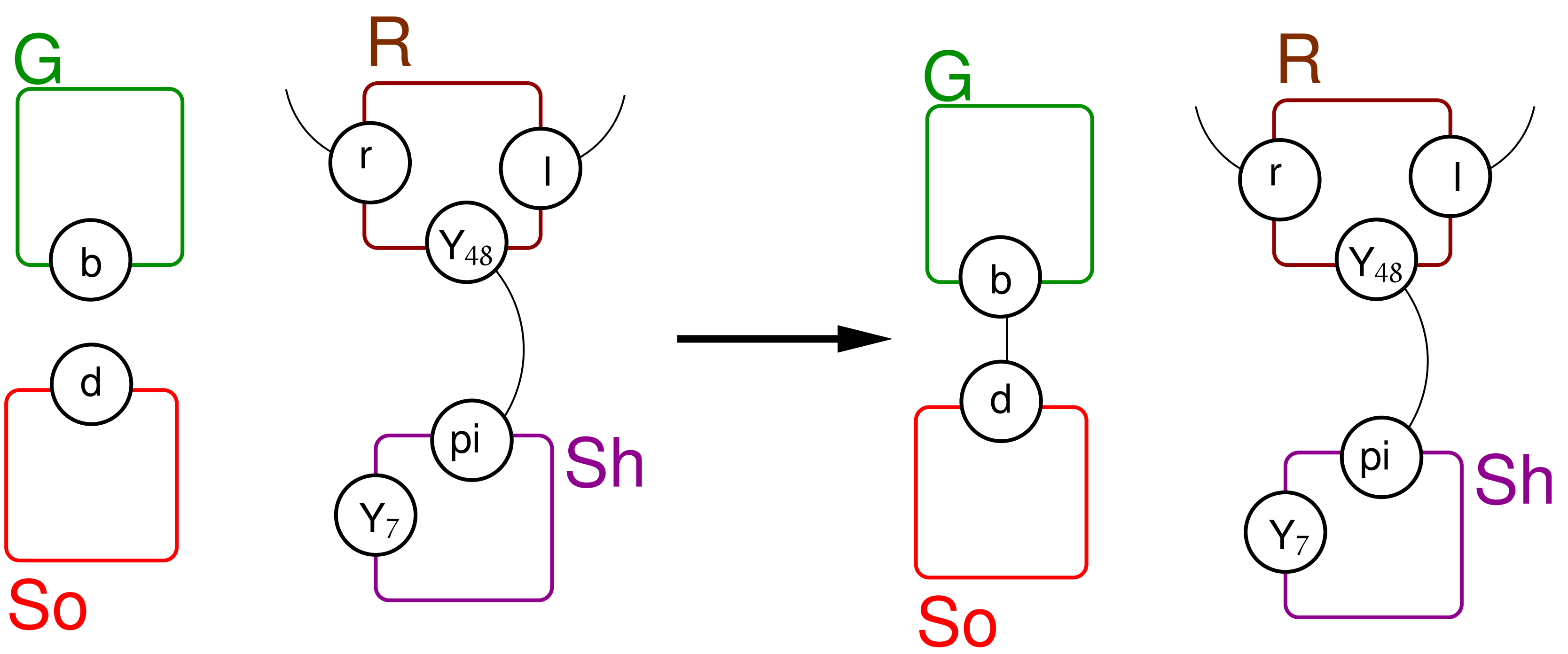
We reflect, in the annotated contact map, each path that stems from a tested site to a modified site (in the lhs of a rule).

Connected components



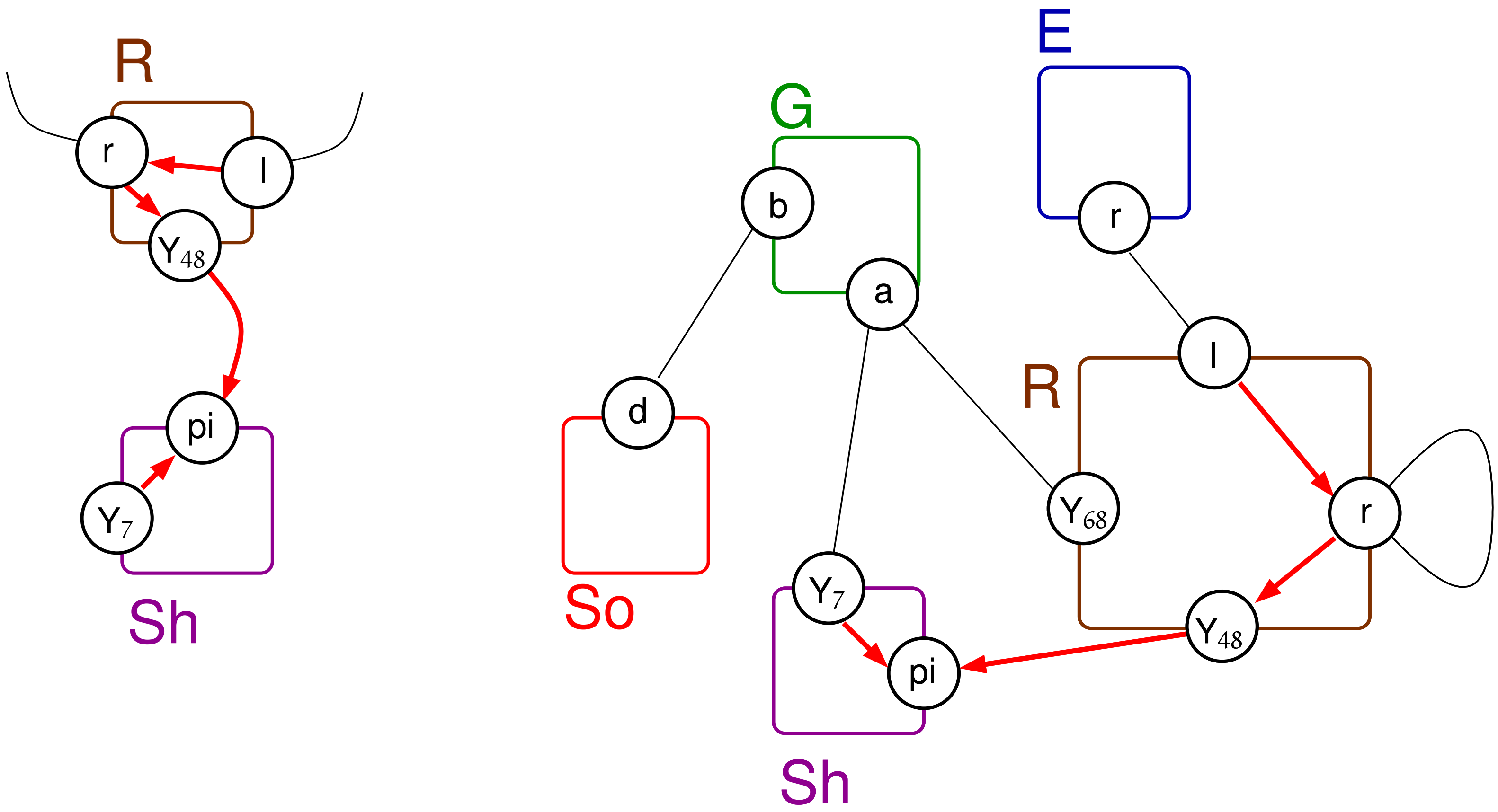
We need to express the “concentration” of any connected component of a lhs with respect to the “concentration” of fragments.

Connected components
Sub-fragment



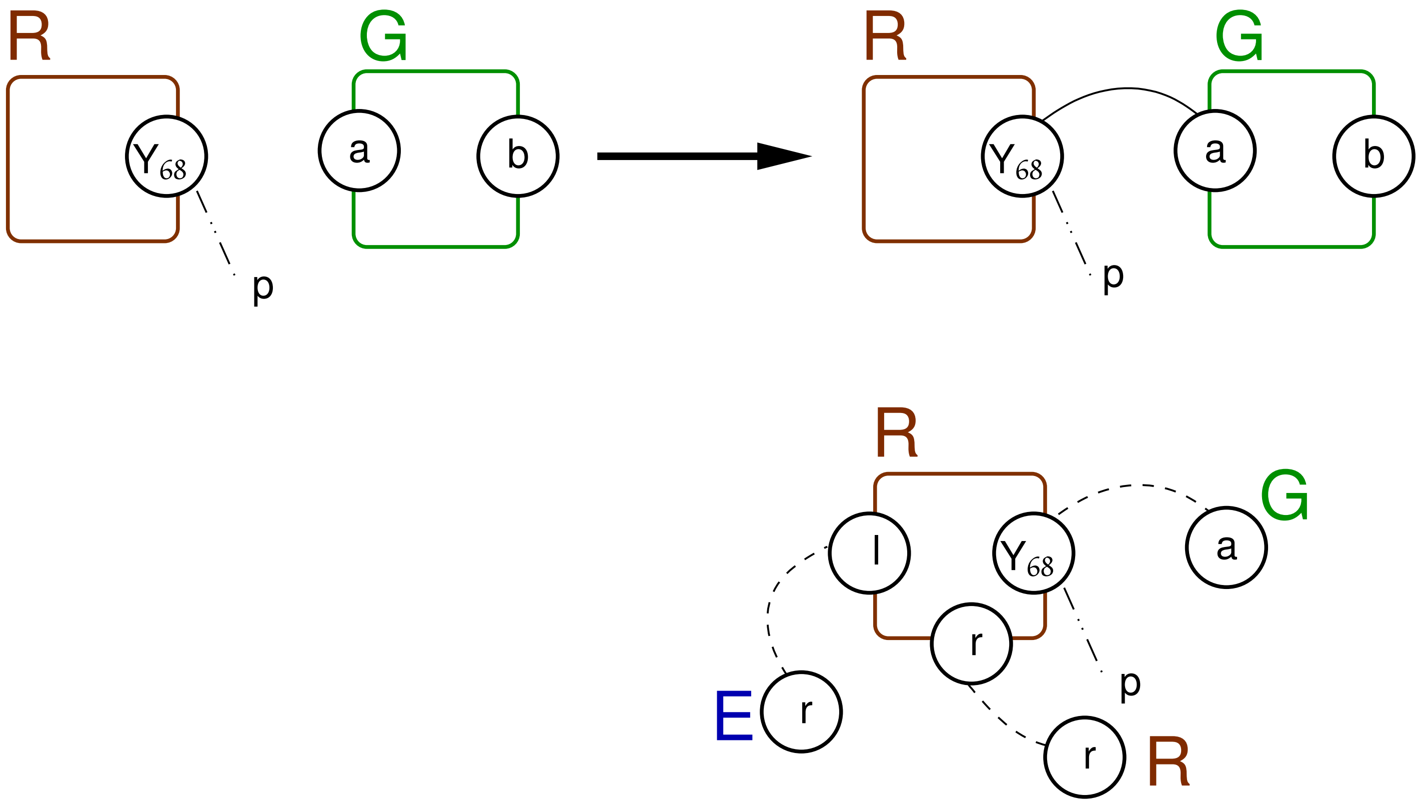
Each connected component of a lhs must be a sub-fragment.

Connected components
Syntactic criteria



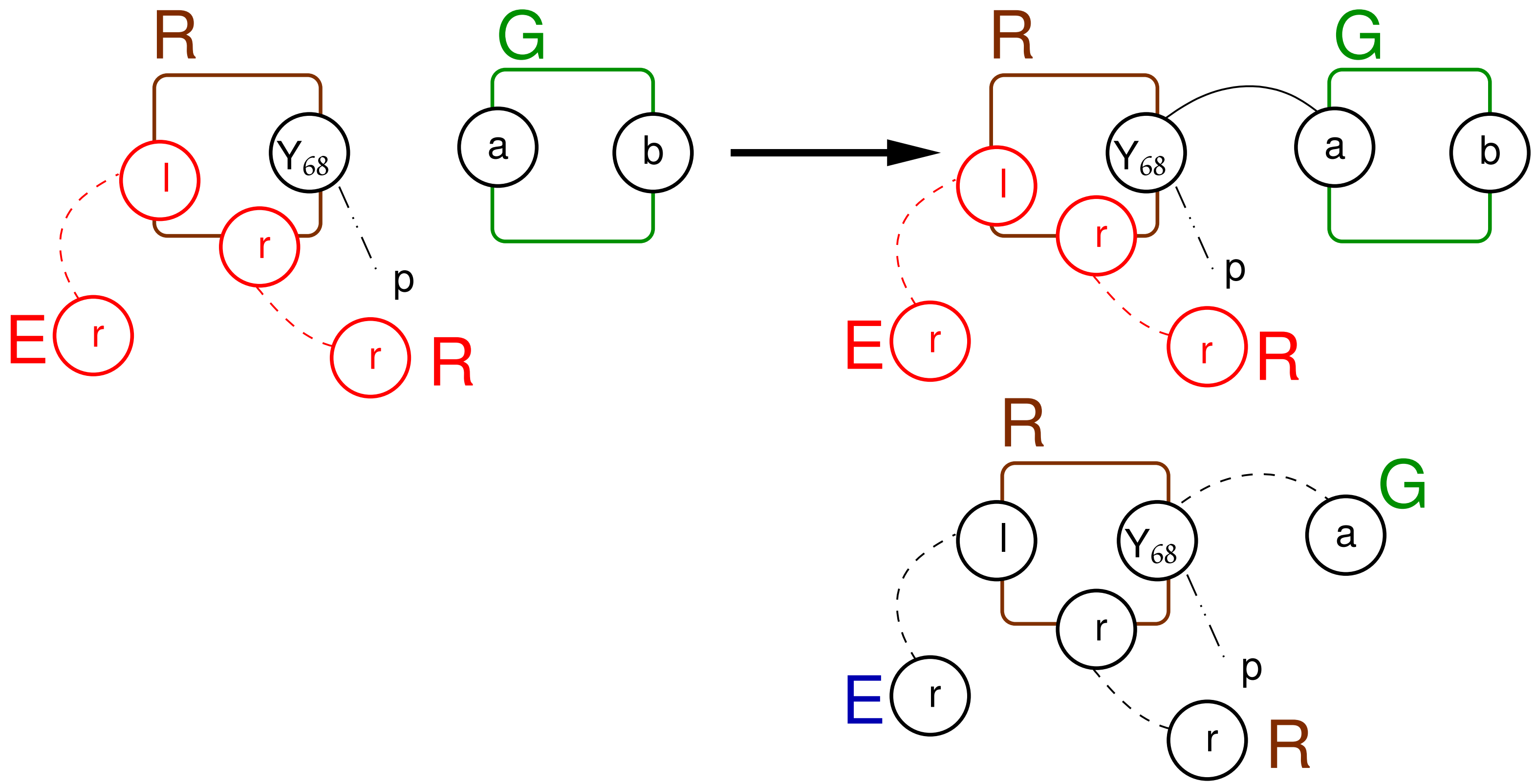
For each connected component of a lhs, there must exists a spanning tree, which reflects it-self in the annotated contact map.

Fragments production



Can we express the amount (per time unit) of this fragment (bellow) concentration that is produced by the rule (above)?

Fragments production
Proper intersection (bis)



Yes, if the connected components of the lhs of the refinement are sub-fragments.
This is already satisfied thans to the previous syntactic criteria.

Fragment properties

If:

- an annotated contact map satisfies the syntactic criteria,
- fragments are defined by this annotated contact map,
- we know the concentration of fragments;

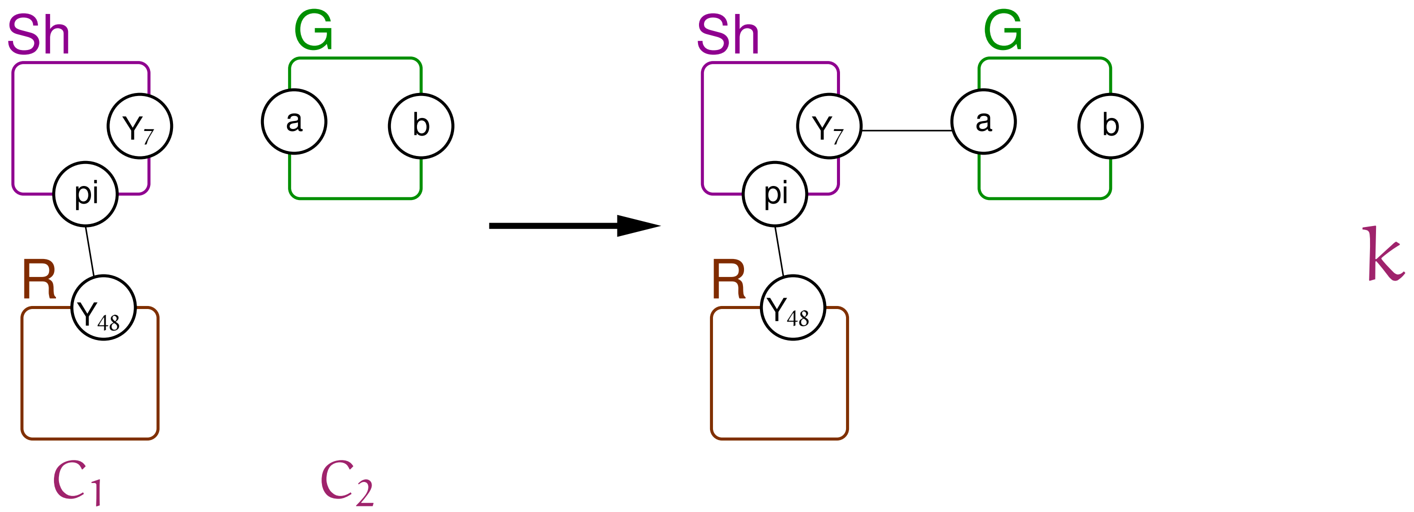
then:

- we can express the concentration of any connected component occuring in lhss,
- we can express fragment proper consumption,
- we can express fragment proper production (eg. see the LICS'2010 paper),
- WE HAVE A CONSTRUCTIVE DEFINITION FOR $\mathbb{F}^{\sharp}$.

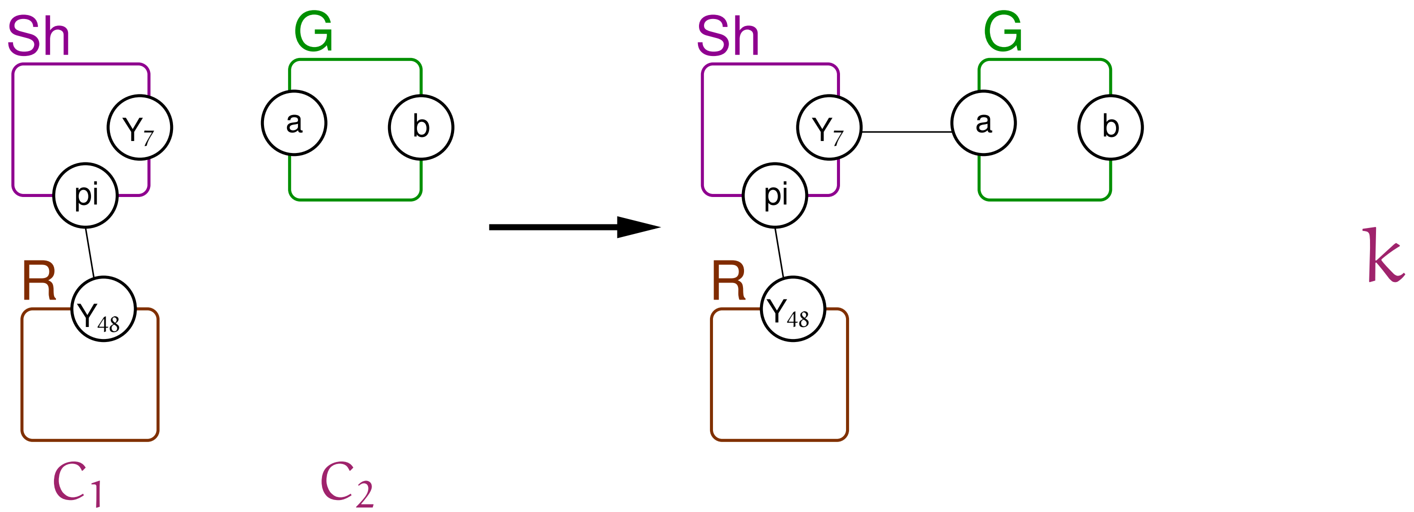
Overview

A binding rule

Let us abstract the contribution of a binding rule:



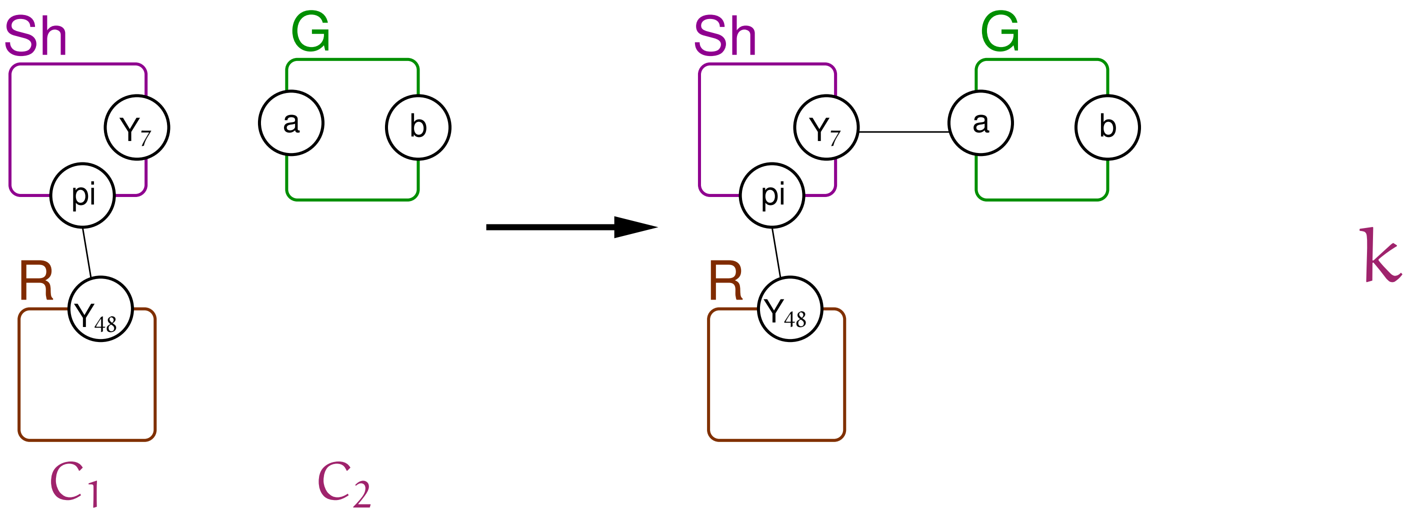
A binding rule: reactants



For any (F, Φ) such that $C_i \lhd_\Phi F$,

$$\frac{d[F]}{dt} \stackrel{+}{=} -\frac{k \cdot [F] \cdot [C_{3-i}]}{\sharp\{\Phi' \mid C_1, C_2 \lhd_{\Phi'} C_1, C_2\}}.$$

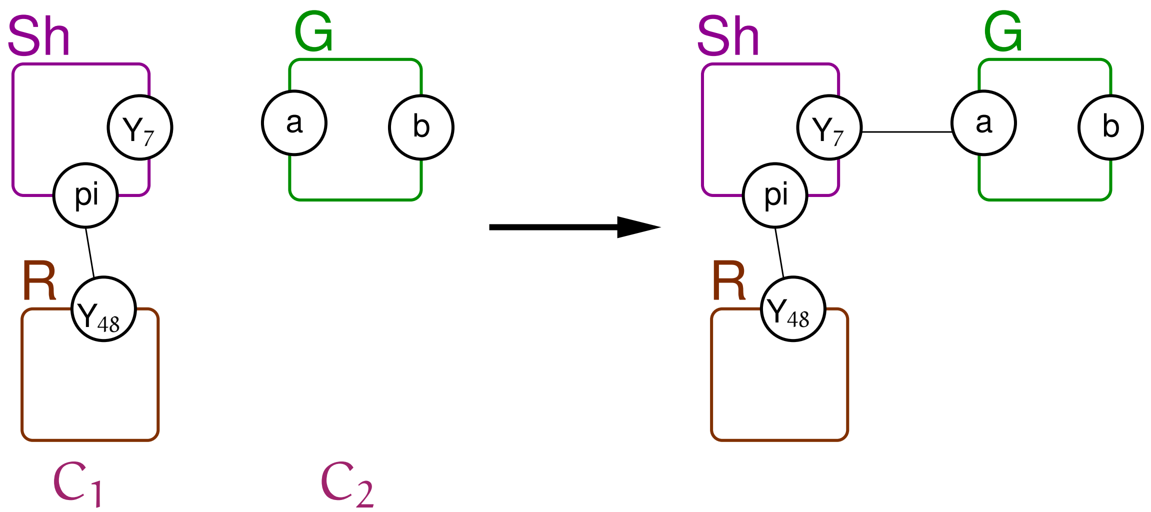
Binding rules: products



If the edge is solid, for any (F_1, Φ_1) and (F_2, Φ_2), such that $C_1 \lhd_{\Phi_1} F_1$ and $C_2 \lhd_{\Phi_2} F_2$,

$$\frac{d[F_1 - F_2]}{dt} = \frac{k \cdot [F_1] \cdot [F_2]}{\sharp\{\Phi' \mid C_1, C_2 \lhd_{\Phi'} C_1, C_2\}}$$

Binding rules: products



k

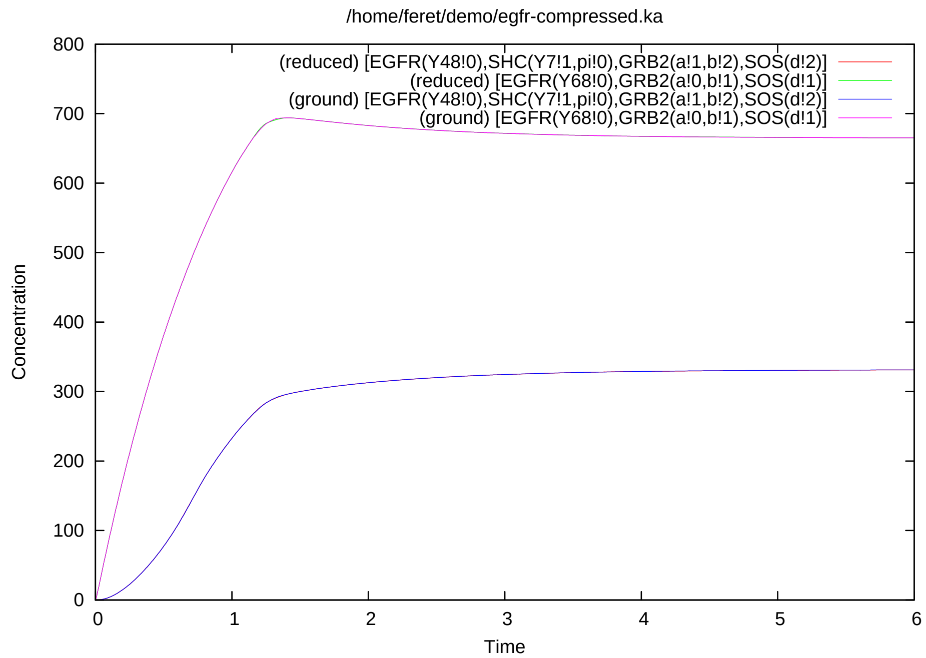
If the edge is dotted, for any (F, Φ) such that $C_i \lhd_\Phi F$,

$$\frac{d[F-]}{dt} \stackrel{+}{=} \frac{k \cdot [F] \cdot [C_{3-i}]}{\sharp\{\Phi' \mid C_1, C_2 \lhd_{\Phi'} C_1, C_2\}}.$$

Overview

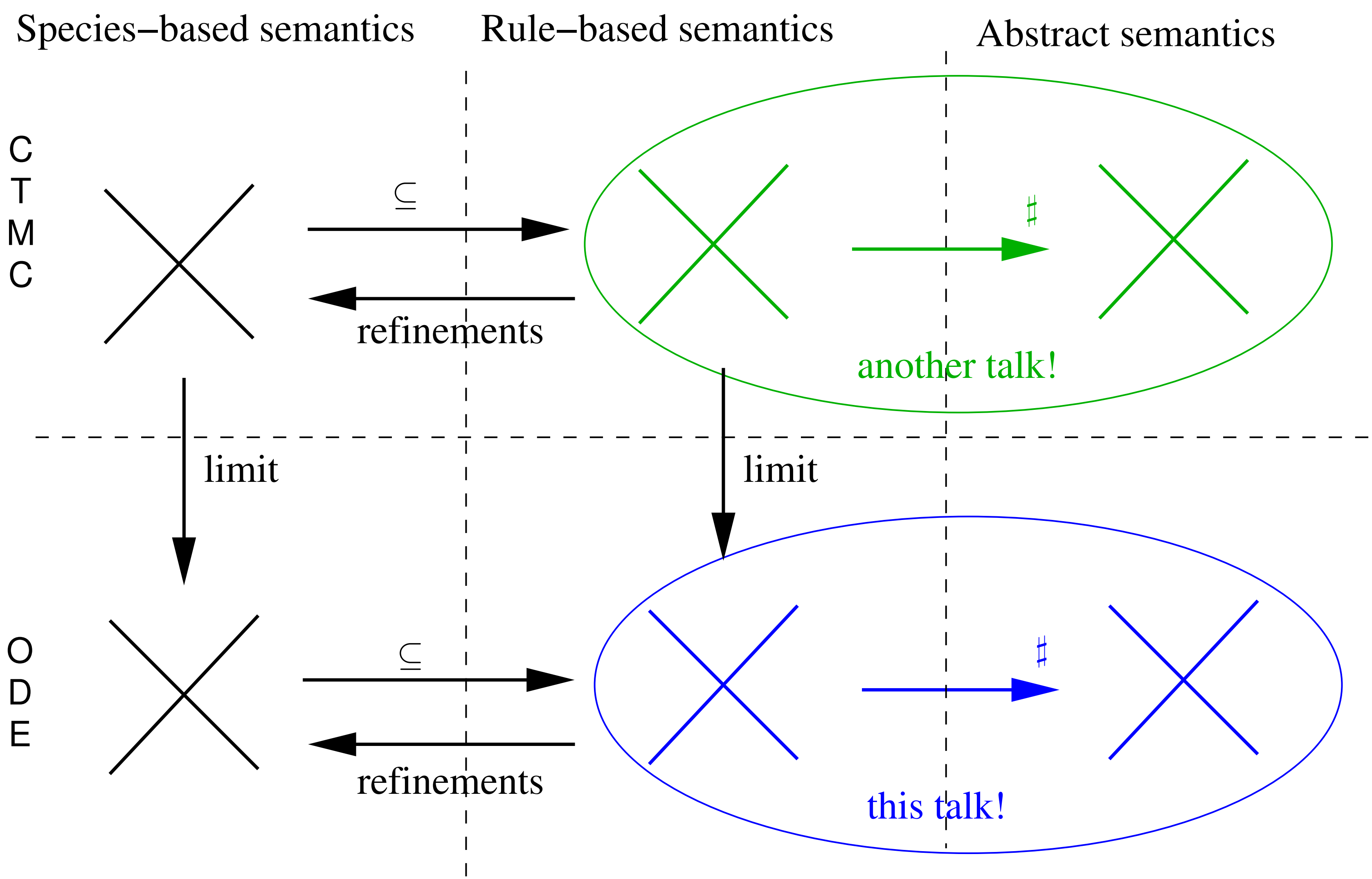
Experimental results

Model	early EGF	EGF/Insulin	SFB
#species	356	2899	$\sim 2.10^{19}$
#fragments (ODEs)	38	208	$\sim 2.10^{5}$
#fragments (CTMC)	356	618	$\sim 2.10^{19}$



Both differential semantics
(4 curves with match pairwise)

Related issues I: Semantics comparisons



Related issues II: Semantics approximations

1. ODE approximations:
   - Concrete definition of the control flow and hierarchy of abstractions.
     A notion of control flow which would be invariant by:
     - neutral rule refinement;
     - compilation of a Kappa system into a Kappa system with only one agent type.

   Joint work with Ferdinanda Camporesi (Bologna)

2. Stochastic semantics approximations:
   - Can we design abstraction ?
   - Find the adequate soundness criteria.

   Joint work with Thomas Henzinger (IST-Vienna), Heinz Koeppl (ETH-Zurich), Tatjana Petrov (EPFL)

Call for paper/participation

Second Workshop on Static Analysis and Systems Biology
(SASB 2011)
(co-chaired with Andre Levchenko)
13th Sept 2011, Venice
http://www.di.ens.fr/sasb2011

Invited speakers:

- Boris Kholodenko
- Edda Klipp
- Jean Krivine