Forced structure in digraphs with large dichromatic number

Proposition d'un mémoire math-info couplé à un stage de recherche.

Mémoire math-info:
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Stage de recherche (Juin-Juillet 2020):
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General context: The *chromatic number* $\chi(G)$ of a graph G is the minimum number of colors needed to color the vertices of G in such a way that two adjacent vertices receive distinct colors. It is certainly the most studied parameter in graph theory and many deep results and theories have been developped around this concept.

In the last decade, many works have been devoted to generalize the chromatic number to directed graphs (a.k.a. digraphs) with the notion of *dichromatic number*. The *dichromatic number* $\vec{\chi}(D)$ of a digraph D is the minimum number of colors needed to color the vertices of D in such a way that no directed cycle is monochromatic. In other words, it is the minimum number of acyclic subdigraphs needed to partition $V(D)$. Clearly, if G is the underlying graph of D (uv is an arc in G if (u, v) is an arc in D or (v, u) is an arc in D) then $\vec{\chi}(D) \leq \chi(G)$, and there is equality if D is a symmetric digraph (if (u, v) is an arc then (v, u) is an arc).

The goal of the internship consists in answering the following question: "what can we say about a digraph with large dichromatic?". And more precisely "which structure can be contained in any digraph of large dichromatic number?" (or conversely which structure has to be forbidden to guarantee a bounded chromatic number?). Similar questions received a considerable attention for undirected graphs.

In graph theory, several possible notions of containment have been introduced. A digraph G *contains* a digraph H, if G contains H as a subdigraph, as an induced subdigraph, as a topological minor, as a butterfly minor, as a strong minor etc etc. During this internship, we propose to focus on the case where G contains a subdivision of D.

Goals:

A *subdivision* of a digraph H is a digraph obtained from H by replacing each edge by a directed path (possibly of length 1). We say that G *contains H as a subdivision* (also called a *topological minor*), if G contains a subdivision of H as a subdigraph. In other words, a subdivision of H can be obtained from G by deleting edges and deleting vertices.

The *symmetric complete graph* $\overleftrightarrow{K}_k$ is the digraph obtained from K_k by replacing each edge by a digon. In [1], it is proved that for every digraph D, if $\vec{\chi}(D) \geq 4^{k^2}$, then D must contain $\overleftrightarrow{K}_k$ (and thus any digraph on at most k vertices) as a subdivision. The intership can go in two directions:

- Obtaining a polynomial upper bound (or at least a better upper bound) for $\overleftrightarrow{K}_k$.
- Obtaining upper bounds for particular directed graphs.

During the first part of the internship, the student will read the existing literature related to that topic, e.g. [1, 2, 3, 4]. During the second part, the student will tackle questions related it.

The student should have a particular taste for combinatorics and structural graph theory. It is also possible to work on algorithmic questions related to that topic if the student is interested.

References

[1] P. Aboulker, N. Cohen, W. Lochet, F. Havet, P. Mourra, S. Thomassé Subdivisions in digraphs of large out-degree or large dichromatic number. Electronic journal of Combinatorics, Vol. 6, 3, 2019

[2] M. Axenovich, A. Girão, R. Snyder, L. Weber Strong complete minors in digraphs Submitted, 2020.

[3] L. Gishboliner, R. Steiner, T. Szabó. Dichromatic number and forced subdivisions Submitted, 2020.

[4] T. Mészáros, R. Steiner Complete minors in digraphs with given dichromatic number Submitted, 2021.