# Hyperuniform systems via a stable matching between Poisson and lattice points

sujet de memoire L3 Math-Info

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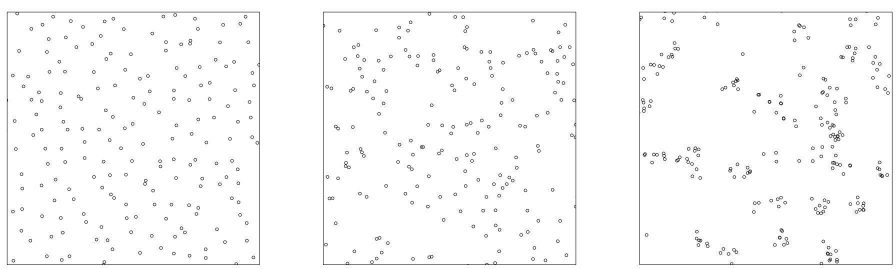
Roughly speaking, a set of particles (say in the $d$-dimensional Euclidean space) is spatially uniform if one does not observe clusters of points. Lattice points form in this regard form a more uniform pattern than independent points uniformly sampled (say in a large, bounded window), the latter being a prototype of the homogeneous Poisson point process. This simple intuition can be formalized in various ways giving rise to somewhat different mathematical notions of uniformity or clustering comparison; cf e.g. [1].



more uniform pattern    independent points    more clustering points

Figure 1: Clustering comparison. Figures borrowed from [1].

One of the most general definitions of uniformity uses the *variance of the number of points observed in the increasing window* to distinguish the following three cases:

- *Uniform patterns*, when the variance grows as the volume of the window. This is the case for Poisson point process (independent points).

- *Hyperuniform patterns*, when the variance grows slower than the window volume. This is the case for various perturbed lattice (crystal) models; cf [6]

- *Hyperfluctuating patterns*, when the variance of the number of points grows faster than the window volume. These are point patterns exhibiting some long range correlations, as e.g. the one formed by the crossings of independent, random (Poisson) lines.

Recently, an interesting construction of some hyperuniform patterns was proposed in [5], by matching Poisson to lattice points in the stable sense (of Gale and Shapley [2], see also [3]) meaning that matched points prefer to be close to each other.

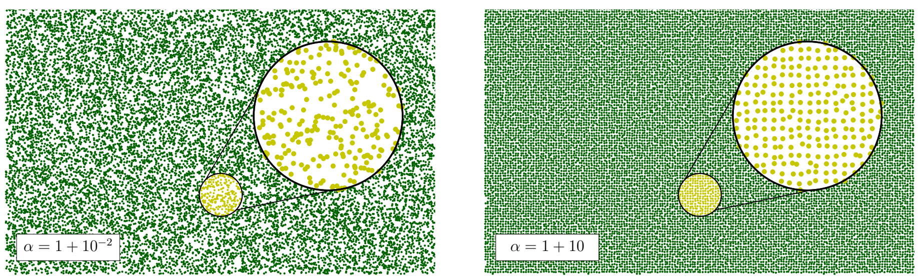



Figure 2: Stable matching of Poisson process of intensity $\alpha > 1$ to a unit square lattice. The matched Poisson points (a subset of all Poisson points) are those laying close to the lattice points in the stable sense. They form a hyperuniform pattern. Figures borrowed from [5]

The goal of the proposed subject is to get acquainted with the notions of hyperuniformity and hyperfluctuations, and the stable matching model leading to the hyperuniformity. Some mathematical results and/or statistical analysis of the simulated patterns (possibly using [4]) can be looked at more in details.

## References

[1] B. Błaszczyszyn and D. Yogeshwaran. Clustering comparison of point processes with applications to random geometric models. In V. Schmidt,

editor, *Stochastic Geometry, Spatial Statistics and Random Fields: Models and Algorithms*, volume 2120 of *Lecture Notes in Mathematics*, chapter 2, pages 31–71. Springer, 2014.

[2] David Gale and Lloyd S Shapley. College admissions and the stability of marriage. *The American Mathematical Monthly*, 69(1):9–15, 1962.

[3] Alexander E Holroyd, Robin Pemantle, Yuval Peres, Oded Schramm, et al. Poisson matching. *Annales de l'Institut Henri Poincaré, Probabilités et Statistiques*, 45(1):266–287, 2009.

[4] M.A. Klatt. Matchingpp — an R package for the stable matching of point patterns. GitHub repository, 2018.

[5] Michael Andreas Klatt, Günter Last, and D Yogeshwaran. Hyperuniform and rigid stable matchings. *arXiv preprint arXiv:1810.00265*, 2018.

[6] Salvatore Torquato. Hyperuniform states of matter. *Physics Reports*, 2018. arXiv:1801.06924.