Coloring oriented graphs embeddable on surfaces

Proposition d'un mémoire math-info couplé à un stage de recherche.

Mémoire math-info :
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Stage de recherche (Juin-Juillet 2020) :
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General context : The *chromatic number* $\chi(G)$ of a graph G is the minimum number of colors needed to color the vertices of G in such a way that two adjacent vertices receive distinct colors. It is certainly the most studied parameter in graph theory and many deep results and theories have been developped around this concept.

For a few years, more and more results are showing that the right concept to generalize chromatic number to directed graphs (a.k.a. digraphs) is the so-called notion of *dichromatic number*. The *dichromatic number* $\vec{\chi}(D)$ of a digraph D is the minimum number of colors needed to color the vertices of D in such a way that no directed cycle is monochromatic. In other words, it is the minimum number of acyclic subdigraphs needed to partition $V(D)$. Clearly, if G is the underlying graph of D (uv is an arc in G if (u, v) is an arc in D or (v, u) is an arc in D) then $\vec{\chi}(D) \leq \chi(G)$, and there is equality if D is a symmetric digraph (if (u, v) is an arc then (v, u) is an arc).

Nowadays, more and more efforts are made to extend coloring results on (undirected) graphs to digraphs through this notion. The goal of this internship is to participate to this effort and to concentrate on coloring oriented graphs (digraphs in which there are not opposite arcs, i.e. if (u, v) is an arc then (v, u) is not an arc) which are embeddable on a surface.

A graph is *embeddable* on a surface Σ if its vertices can be mapped onto distinct points of Σ and its edges onto simple curves of Σ joining the points onto which its endvertices are mapped, so that two edge curves do not intersect except in their common extremity.

Déroulement des stages :

The maximum dichromatic number of a digraph embeddable on a given surface Σ is equal to the maximum chromatic number of an (undirected) graphs embeddable in Σ, and thus is well-known. However, one expects that when considering only oriented graphs then the maximum is lower than this bound. As an example, the maximum dichromatic number of a planar digraph is 4, but we know that the maximum dichromatic number of a planar oriented graph is at most 3 and Neumann-Lara conjecture that it is at most 2.

During the mémoire, the student will have to understand a few classical theorems on the chromatic number on graphs on surfaces [4, 3, 8, 2, 5] and a few on the dichromatic number [1, 6, 7]. The aim of the stage will be to get upper bounds on the maximum dichromatic number of an oriented graph embeddable on a given surface Σ, which are lower than the maximum dichromatic number of a digraph embeddable on Σ.

Références

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