

A CONSTRUCTIVE CHARACTERIZATION OF THE LATTICES OF ALL RETRACTIONS, PRECLOSURE, QUASI-CLOSURE AND CLOSURE OPERATORS ON A COMPLETE LATTICE

BY

PATRICK COUSOT AND RADHIA COUSOT *

Université de Metz
Faculté des Sciences
Ile du Sauley
5700 Metz — França

1. Introduction

We give a constructive characterization of the complete lattices of all retractions, preclosure, quasi-closure and closure operators on a complete lattice. Our general approach is the following: in order to study the structure of the set $\Gamma \subseteq (L \rightarrow L)$ of operators ρ on a complete lattice L satisfying a given axiom A , we show that ρ has property A if and only if it is a fixed point of some monotone operator F on the complete lattice $(L \rightarrow L)$ proving that Γ is the set of fixed points of F . Then using Cousot & Cousot's constructive version of Tarski's lattice theoretical fixed point theorem, we constructively characterize the infimum, supremum, union and intersection of the complete lattice Γ which are defined by means of limits of stationary transfinite iteration sequences for F . Variants of this argument are used when F is a closure operator (in which case the constructive version of Tarski's theorem amounts to Ward's theorem) or when the operators with property A are the postfixes of F or the common fixed points of two functionals. The reasoning is repeated when Γ is characterized by means of more than one axiom.

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2. Preliminaries

1.1. Let $L(\sqsubseteq, \perp, \top, \sqcup, \sqcap)$ be a non-empty *complete lattice* with *partial ordering* \sqsubseteq , *least upper bound* \sqcup , *greatest lower bound* \sqcap . The *infimum* \perp of L is $\sqcap L$, the *supremum* \top of L is $\sqcup L$. (Birkhoff's standard reference book [2] provides the necessary background material).

1.2. Let $\theta \in (L \rightarrow M)$ be a total function from the complete lattice $L(\sqsubseteq, \perp, \top, \sqcup, \sqcap)$ to the complete lattice $M(\sqsubseteq', \perp', \top', \sqcup', \sqcap')$. θ is a *join-morphism* when $(\forall x, y \in L, \theta(x \sqcup y) = \theta(x) \sqcup' \theta(y))$. θ is a *complete join-morphism* when $(\forall S \subseteq L, \theta(\sqcup S) = \sqcup' \theta(S))$. The dual notions are the ones of *meet-morphism* and *complete meet-morphism*.

1.3. Using Church[3]'s lambda notation (so that $f \in (L \rightarrow M)$ is $\lambda x.f(x)$) let us recall that the set $(L \rightarrow L)$ of all operators on the complete lattice $L(\sqsubseteq, \perp, \top, \sqcup, \sqcap)$ is a complete lattice $(\sqsubseteq', \perp', \top', \sqcup', \sqcap')$ where $((f \sqsubseteq' g) \Leftrightarrow (\forall x \in L, f(x) \sqsubseteq g(x)))$, $\perp' = \lambda x. \perp$, $\top' = \lambda x. \top$, $\sqcup' = \lambda S. (\lambda x. \sqcup \{f(x) : f \in S\})$, $\sqcap' = \lambda S. (\lambda x. \sqcap \{f(x) : f \in S\})$. In the following we will omit the primes so that the distinction between $\sqsubseteq, \perp, \top, \sqcup, \sqcap$ and $\sqsubseteq', \perp', \top', \sqcup', \sqcap'$ will be contextual.

1.4. A *retraction* ρ on L is an operator on L (i. e. $\rho \in (L \rightarrow L)$) which is *monotone* (i. e. $\forall x, y \in L, (x \sqsubseteq y) \Rightarrow (\rho(x) \sqsubseteq \rho(y))$) and *idempotent* (i. e. $\rho = \rho \circ \rho$ that is $\forall x \in L, \rho(x) = \rho(\rho(x))$).

1.5. An *upper preclosure operator* $\bar{\rho}$ on L is monotone, idempotent and satisfies the *upper connectivity axiom* $(\forall x \in L, \bar{\rho}(x \sqcup \bar{\rho}(x)) = \bar{\rho}(x))$. The dual notion is the one of *lower preclosure operator* (Ladegaillerie, [6, Def. 1]).

1.6. A *quasi-closure operator* ρ on L is monotone, *comparing* (i. e. $\forall x \in L$, either $x \sqsubseteq \rho(x)$ or $\rho(x) \sqsubseteq x$) and satisfies the *connectivity axiom* $(\forall x \in L, \rho(x \sqcap \rho(x)) = \rho(x \sqcup \rho(x)))$, Bernard[1, p. 6]). Notice that monotony and connectivity axiom imply $(\forall x \in L, \rho(x) = \rho(x \sqcap \rho(x)) = \rho(x \sqcup \rho(x)))$ and using the comparing hypothesis this in turn implies idempotence.

1.7. An *upper closure operator* $\bar{\rho}$ on L is monotone, idempotent and *extensive* (i. e. $\lambda x. x \sqsubseteq \bar{\rho}$), (Moore[9]). Dually a *lower closure operator* $\underline{\rho} \in (L \rightarrow L)$ is monotone, idempotent and *reductive* (i. e. $\underline{\rho} \sqsubseteq \lambda x. x$).

1.8. Let $\bar{\rho}$ be an upper closure operator on $L(\sqsubseteq, \perp, \top, \sqcup, \sqcap)$. For all $x \in L$, the set $\{y \in \bar{\rho}(L) : x \sqsubseteq y\}$ is not empty and $\bar{\rho}(x)$ is its least element (Monteiro & Ribeiro[8, Th. 5.2]).

1.9. Let $R \subseteq L$ and $\bar{\rho} \in (L \rightarrow R)$ be such that for any $x \in L$, $\bar{\rho}(x)$ is the least element of the set $\{y \in R : x \sqsubseteq y\}$ then $\bar{\rho}$ is an upper closure operator and $R = \bar{\rho}(L)$, (Monteiro & Ribeiro[8, Th. 5.3]).

1.10. Let $\bar{\rho}$ be an upper closure operator on L , then the image $\bar{\rho}(L)$ of L by $\bar{\rho}$ is a complete lattice $\bar{\rho}(L)(\sqsubseteq, \bar{\rho}(\perp), \top, \lambda S. \bar{\rho}(\sqcup S), \sqcap)$ which is a complete sublattice of L if and only if $\bar{\rho}$ is a complete join-morphism, (Ward[14, Th. 4.1]).

1.11. Applying the duality principle it follows that if $R \subseteq L$ and $\bar{\rho}, \varrho \in (L \rightarrow L)$ are such that for any $x \in L$, $\bar{\rho}(x)$ is the least element of the set $\{y \in R : x \sqsubseteq y\}$ and $\varrho(x)$ is the greatest element of the set $\{y \in R : y \sqsubseteq x\}$ then $R = \bar{\rho}(L) = \varrho(L)$ is a complete sublattice $R(\sqsubseteq, \perp, \top, \sqcup, \sqcap)$ of L whereas $\bar{\rho}$ is a complete join-morphism and ϱ is a complete meet-morphism.

1.12. Let $\bar{\rho}_1$ and $\bar{\rho}_2$ be upper closure operators on L . Then according to Ore[12, p. 525], $\bar{\rho}_1 \circ \bar{\rho}_2$ and $\bar{\rho}_2 \circ \bar{\rho}_1$ are upper closure operators on L if and only if $\bar{\rho}_1$ and $\bar{\rho}_2$ are *commuting* (i. e. $\bar{\rho}_1 \circ \bar{\rho}_2 = \bar{\rho}_2 \circ \bar{\rho}_1$) in which case $\bar{\rho}_1 \circ \bar{\rho}_2(L) = \bar{\rho}_2 \circ \bar{\rho}_1(L) = \bar{\rho}_1(L) \cap \bar{\rho}_2(L)$.

1.13. Let $L(\sqsubseteq, \perp, \top, \sqcup, \sqcap)$ be a complete lattice, μ the smallest ordinal such that the class $\{\delta : \delta \in \mu\}$ has a cardinality greater than the cardinality of L and $f \in (L \rightarrow L)$. The *upper iteration sequence* for f starting with $d \in L$ is the μ -termed sequence $\langle X^\delta, \delta \in \mu \rangle$ of elements of L defined by transfinite recursion in the following way:

$$- X^0 = d$$

$$- X^\delta = f(X^{\delta-1}) \text{ for every successor ordinal } \delta \in \mu$$

$$- X^\delta = \sqcup_{\alpha < \delta} X^\alpha \text{ for every limit ordinal } \delta \in \mu$$

(The dual *lower iteration sequence* is defined by:

$$- X^\delta = \sqcap_{\alpha < \delta} X^\alpha \text{ for every limit ordinal } \delta \in \mu)$$

1.14. We say that the sequence $\langle X^\delta, \delta \in \mu \rangle$ is *stationary* if and only if $(\exists \varepsilon \in \mu : (\forall \beta \in \mu, (\varepsilon \leq \beta) \Rightarrow (X^\varepsilon = X^\beta)))$ in which case the

limit of the sequence is X^ε . We denote by $luis(f)(d)$ the limit of a stationary upper iteration sequence for f starting with d (dually $llis(f)(d)$).

1.15. Let f be a monotone operator on $L(\sqsubseteq, \perp, \top, \sqcup, \sqcap)$. The sets of *prefixed points*, *fixed points* and *postfixed points* of f are respectively $prefp(f) = \{x \in L : x \sqsubseteq f(x)\}$, $fp(f) = \{x \in L : x = f(x)\}$, $postfp(f) = \{x \in L : f(x) \sqsubseteq x\}$.

1.16. The set of postfixed points of a monotone operator f on a complete lattice $L(\sqsubseteq, \perp, \top, \sqcup, \sqcap)$ is a complete lattice $postfp(f)$ ($\sqsubseteq, lfp(f), \top, \lambda S.luis(\lambda z.z \sqcup f(z))(\sqcup S), \sqcap$) which is the image of L by the upper closure operator $\lambda d.luis(\lambda z.z \sqcup f(z))(d)$. The *least fixed point* of f is $lfp(f) = luis(f)(\perp)$, (Cousot & Cousot[4, Th. 4.2]).

1.17. Let f be a monotone operator on the complete lattice $L(\sqsubseteq, \perp, \top, \sqcup, \sqcap)$. Then $\lambda x.luis(\lambda y.y \sqcup f(y))(x) = \lambda x.luis(\lambda y.x \sqcup f(y))(x) = \lambda x.lfp(\lambda y.x \sqcup f(y))$ is an upper closure operator on L greater than or equal to f , (Cousot & Cousot[4, Th.4.1. & 4.2]).

1.18. Cousot & Cousot[4, Th.5.1]'s constructive version of Tarski[13, Th.1]'s lattice theoretical fixed point theorem states that the set $fp(f)$ of fixed points of the monotone operator f on $L(\sqsubseteq, \perp, \top, \sqcup, \sqcap)$ is a complete lattice $fp(f)$ ($\sqsubseteq, luis(f)(\perp), llis(f)(\top), \lambda S.luis(f)(\sqcup S), \lambda S.llis(f)(\sqcap S)$). $fp(f)$ is the image of L by the upper preclosure operator $llis(f) \circ luis(\lambda z.z \sqcup f(z))$ and the image of L by the lower preclosure operator $luis(f) \circ llis(\lambda z.z \sqcap f(z))$. Moreover $luis(f) \circ llis(\lambda z.z \sqcap f(z)) \sqsubseteq llis(f) \circ luis(\lambda z.z \sqcup f(z))$.

1.19. Let \underline{F} and \overline{F} be monotone operators on $L(\sqsubseteq, \perp, \top, \sqcup, \sqcap)$ such that $\underline{F} \sqsubseteq \overline{F}$ and $\overline{F} \circ \underline{F} \sqsubseteq \underline{F} \circ \overline{F}$. Then the set $fp(\underline{F}, \overline{F}) = \{x \in L : x = \underline{F}(x) = \overline{F}(x)\}$ of common fixed points of \overline{F} and \underline{F} is a non-empty complete lattice $fp(\underline{F}, \overline{F})$ ($\sqsubseteq, luis(\overline{F})(\perp), llis(\underline{F})(\top), \lambda S.luis(\overline{F})(\sqcup S), \lambda S.llis(\underline{F})(\sqcap S)$) which is the image of L by the lower preclosure operator $luis(\overline{F}) \circ llis(\lambda z.z \sqcap \underline{F}(z))$ and the image of L by the upper preclosure operator $llis(\underline{F}) \circ luis(\lambda z.z \sqcup \overline{F}(z))$. Moreover $luis(\overline{F}) \circ llis(\lambda z.z \sqcap \underline{F}(z)) \sqsubseteq llis(\underline{F}) \circ luis(\lambda z.z \sqcup \overline{F}(z))$, (Cousot & Cousot [4, Th. 6.1]).

2. The lattice of all monotone operators on a complete lattice

DEFINITION 2.1 $\overline{M}, \underline{M} \in ((L \rightarrow L) \rightarrow (L \rightarrow L))$

$$\overline{M} = \lambda f. (\lambda x. \sqcup \{f(y) : (y \in L) \text{ and } (y \sqsubseteq x)\})$$

$$\underline{M} = \lambda f. (\lambda x. \sqcap \{f(y) : (y \in L) \text{ and } (x \sqsubseteq y)\})$$

THEOREM 2.2. *If $f \in (L \rightarrow L)$ then $\overline{M}(f)$ is the least monotone operator on L greater than or equal to f and $\underline{M}(f)$ is the greatest monotone operator on L less than or equal to f .*

PROOF: Let $a, b \in L$ be such that $a \sqsubseteq b$. For all $y \in L$, $(y \sqsubseteq a)$ implies $(y \sqsubseteq b)$ so that $\sqcup \{f(y) : y \sqsubseteq a\} \sqsubseteq \sqcup \{f(y) : y \sqsubseteq b\}$ proving that $\overline{M}(f)(a) \sqsubseteq \overline{M}(f)(b)$. Since \sqsubseteq is reflexive, $\forall a \in L$ we have $f(a) \sqsubseteq \sqcup \{f(y) : y \sqsubseteq a\} = \overline{M}(f)(a)$ that is $f \sqsubseteq \overline{M}(f)$ proving that $\forall f \in (L \rightarrow L)$, $\overline{M}(f)$ is a monotone operator on L greater than or equal to f .

Let g be a monotone operator on L greater than or equal to f . For all $y \in L$, $f(y) \sqsubseteq g(y)$ so that $\forall a \in L$, $\overline{M}(f)(a) = \sqcup \{f(y) : y \sqsubseteq a\} \sqsubseteq \sqcup \{g(y) : y \sqsubseteq a\} \sqsubseteq \sqcup \{g(y) : g(y) \sqsubseteq g(a)\} = g(a)$ proving that $\overline{M}(f)$ is the least monotone operator on L greater than or equal to f .

By duality, $\underline{M}(f)$ is the greatest monotone operator on L less than or equal to f . *End of proof.*

From 2.2 and 1.11 we get:

COROLLARY 2.3 \overline{M} is an upper closure operator on $(L \rightarrow L)$ which is a complete join-morphism and \underline{M} is a lower closure operator on $(L \rightarrow L)$ which is a complete meet-morphism.

The set of all monotone operators on the complete lattice L is a complete sublattice $\overline{M}(L \rightarrow L) = \underline{M}(L \rightarrow L) (\sqsubseteq, \perp, \top, \sqcup, \sqcap)$ of $(L \rightarrow L) (\sqsubseteq, \perp, \top, \sqcup, \sqcap)$.

3. The lattice of all extensive operators on a complete lattice

DEFINITION 3.1 $\overline{E}, \underline{E} \in ((L \rightarrow L) \rightarrow (L \rightarrow L))$

$$\overline{E} = \lambda f. (\lambda x. (x \sqcup f(x)))$$

$$\underline{E} = \lambda f. (\lambda x. (x \sqcap f(x)))$$

LEMMA 3.2 *If $f \in (L \rightarrow L)$ then $\bar{E}(f)$ is the least extensive operator on L greater than or equal to f . \bar{E} is a complete join-morphism.*

COROLLARY 3.3 *\bar{E} is an upper closure operator on $(L \rightarrow L)$. The set of all extensive operators on the complete lattice L is a complete sublattice $\bar{E}(L \rightarrow L) (\sqsubseteq, \lambda x.x, \top, \sqcup, \sqcap)$, of $(L \rightarrow L)$.*

LEMMA 3.4

- $\bar{M} \circ \bar{E} = \bar{E} \circ \bar{M}$ is an upper closure operator on $(L \rightarrow L)$
- $\bar{M}(L \rightarrow L) \cap \bar{E}(L \rightarrow L) = \bar{M} \circ \bar{E}(L \rightarrow L) = \bar{E} \circ \bar{M}(L \rightarrow L)$ is a complete sublattice $(\sqsubseteq, \lambda x.x, \top, \sqcup, \sqcap)$ of $(L \rightarrow L)$.

PROOF: Since \sqsubseteq is reflexive, for any $f \in (L \rightarrow L)$ and $x \in L$ we have $x \sqsubseteq \sqcup \{y \sqcup f(y) : y \sqsubseteq x\}$ so that $\bar{M}(\bar{E}(f))(x) = \sqcup \{y \sqcup f(y) : y \sqsubseteq x\} = x \sqcup (\sqcup \{y \sqcup f(y) : y \sqsubseteq x\}) = \sqcup \{x \sqcup y \sqcup f(y) : y \sqsubseteq x\} = \sqcup \{x \sqcup f(y) : y \sqsubseteq x\} = x \sqcup (\sqcup \{f(y) : y \sqsubseteq x\}) = \bar{E}(\bar{M}(f))(x)$. Therefore \bar{M} and \bar{E} are commuting and the lemma follows from 1.12 and 1.10. *End of proof.*

4. The lattice of all upper closure operators on a complete lattice

LEMMA 4.1 *Let F be the operator on $(\bar{M}(L \rightarrow L) \cap \bar{E}(L \rightarrow L))$ defined by $\lambda g.g \circ g$. Then*

- 1 — F is monotone
- 2 — $f p(F) = \text{post} f p(F)$

PROOF: 1. If $f \sqsubseteq g$ then $f \in \bar{M}(L \rightarrow L)$ implies $F(f) = f \circ f \sqsubseteq f \circ g$. Also $f \circ g \sqsubseteq g \circ g = F(g)$ proving that F is monotone.

2. Let f be a postfix point of F then $f \circ f \sqsubseteq f$. Since $f \in \bar{E}(L \rightarrow L)$ we have $\lambda x.x \sqsubseteq f$ so that $f \in \bar{M}(L \rightarrow L)$ implies $f \sqsubseteq f \circ f$. By antisymmetry $f \circ f = f$. *End of proof.*

DEFINITION 4.2

$$\bar{I} \in ((\bar{M}(L \rightarrow L) \cap \bar{E}(L \rightarrow L)) \rightarrow (\bar{M}(L \rightarrow L) \cap \bar{E}(L \rightarrow L)))$$

$$\bar{I} = \lambda f.luis(\lambda g.g \circ g)(f) = \lambda f.luis(\lambda g.g \sqcup g \circ g)(f)$$

$$\bar{C} \in ((L \rightarrow L) \rightarrow (L \rightarrow L))$$

$$\bar{C} = \bar{I} \circ \bar{M} \circ \bar{E} = \bar{I} \circ \bar{E} \circ \bar{M}$$

Ward[14,Th.4.2]'s theorem states that the set Γ of all upper closure operators on a complete lattice $L(\sqsubseteq, \perp, \top, \sqcup, \sqcap)$ is a complete lattice $\Gamma(\sqsubseteq, \lambda x.x, \top, \lambda S. \sqcap \{\eta \in \Gamma : (\forall \rho \in S, \rho \sqsubseteq \eta)\}, \sqcap)$. We give now a constructive version of this theorem:

THEOREM 4.3

- 1 — $\bar{C}(L \rightarrow L)(\sqsubseteq, \lambda x.x, \top, \lambda S.luis(\lambda g.g \circ g)(\sqcup S), \sqcap)$ is the complete lattice of all upper closure operators on the complete lattice L
- 2 — \bar{C} is an upper closure operator on $(L \rightarrow L)$
- 3 — $\forall f \in (L \rightarrow L), \bar{C}(f)$ is the least upper closure operator on L greater than or equal to f .

PROOF: According to definition 1.7 the set of all upper closure operators on the complete lattice $L(\sqsubseteq, \perp, \top, \sqcup, \sqcap)$ is the set of elements of $(\bar{M}(L \rightarrow L) \cap \bar{E}(L \rightarrow L))$ which are idempotent that is to say which are fixed points of $\lambda g.g \circ g$. In compliance with lemmas 4.1 and 3.4 $\lambda g.g \circ g$ is a monotone operator on the complete lattice $(\bar{M}(L \rightarrow L) \cap \bar{E}(L \rightarrow L))(\sqsubseteq, \lambda x.x, \perp, \sqcup, \sqcap)$ and $fp(\lambda g.g \circ g) = postfp(\lambda g.g \circ g)$. Therefore 1.16 implies that the set of all upper closure operators on L is the complete lattice $postfp(\lambda g.g \circ g)(\sqsubseteq, luis(\lambda g.g \circ g)(\lambda x.x) = \lambda x.x, \top, \lambda S.luis(\lambda g.g \circ g)(\sqcup S), \sqcap)$ which is the image of $\bar{M}(L \rightarrow L) \cap \bar{E}(L \rightarrow L)$ by the upper closure operator \bar{I} that is (lemma 3.4 and definition 4.2) $\bar{I}(\bar{M}(L \rightarrow L) \cap \bar{E}(L \rightarrow L)) = \bar{I} \circ \bar{M} \circ \bar{E}(L \rightarrow L) = \bar{C}(L \rightarrow L)$. \bar{C} is monotone and extensive since it is the composition of upper closure operators. $\forall f \in (L \rightarrow L), \bar{C}(f)$ is extensive, monotone and idempotent, hence $\bar{C}(f) = \bar{E} \circ \bar{C}(f) = \bar{M} \circ \bar{C}(f) = \bar{I} \circ \bar{C}(f) = \bar{I} \circ \bar{M} \circ \bar{E} \circ \bar{C}(f) = \bar{C} \circ \bar{C}(f)$ proving that \bar{C} is idempotent whence an upper closure operator. Then 4.3.3 follows from 1.8. *End of proof.*

COROLLARY 4.4

- 1 — $\bar{C} = \lambda f.(luis(\bar{E}(\bar{M}(f))))$
- 2 — $\bar{C} = \lambda f.(\lambda x.lfp(\lambda y.x \sqcup \bar{M}(f)(y)))$
- 3 — $\bar{C}(L \rightarrow L)(\sqsubseteq, \lambda x.x, \top, \lambda S.luis(\sqcup S), \sqcap)$
- 4 — $\bar{C}(L \rightarrow L)(\sqsubseteq, \lambda x.x, \top, \lambda S.(\lambda x.lfp(\lambda y.x \sqcup (\sqcup S)(y))), \sqcap)$

PROOF: It follows from 1.17 that $\forall f \in (L \rightarrow L), luis(\bar{E}(\bar{M}(f))) = \lambda x.lfp(\lambda y.x \sqcup \bar{M}(f)(y))$ is an upper closure operator on L greater

than or equal to $\overline{M}(f)$ whence according to theorem 2.2 greater than or equal to f . Let $\overline{\rho} \in \overline{C}(L \rightarrow L)$ be such that $f \sqsubseteq \overline{\rho}$. Then $\overline{E}(\overline{M}(f)) \sqsubseteq \overline{E}(\overline{M}(\overline{\rho})) = \overline{\rho}$ so that by monotony $luis(\overline{E}(\overline{M}(f))) \sqsubseteq Luis(\overline{\rho})$. Moreover $luis(\overline{\rho}) = \overline{\rho}$ since $\overline{\rho}$ is idempotent proving that $luis(\overline{E}(\overline{M}(f)))$ is the least upper closure operator on L greater than or equal to f and therefore $\overline{C} = \lambda f. (luis(\overline{E}(\overline{M}(f)))) = \lambda f. (\lambda x. lfp(\lambda y. x \sqcup \overline{M}(f)(y)))$, (1.17).

4.4.3 and 4.4.4 result from 1.10 remarking that whenever $S \subseteq \overline{C}(L \rightarrow L)$ then $\sqcup S$ is monotone and extensive (theorems 2.3 and 3.3) so that $luis(\overline{E}(\overline{M}(\sqcup S))) = Luis(\sqcup S)$ and $\lambda x. lfp(\lambda y. x \sqcup \overline{M}(\sqcup S)(y)) = \lambda x. lfp(\lambda y. x \sqcup (\sqcup S)(y))$. *End of proof.*

It should be noted that given a complete lattice $L(\sqsubseteq, \perp, \top, \sqcup, \sqcap)$ and $f \in (L \rightarrow L)$ there does not necessarily exist an upper closure operator on L less than or equal to f . A counter-example is given by $L = \{\perp, \top\}$ where $\perp \sqsubseteq \perp \sqsubseteq \top \sqsubseteq \top$ and f defined by $f(\perp) = \top$ and $f(\top) = \perp$. The upper closure operators on L are $\overline{\rho}_1$ (such that $\overline{\rho}_1(\perp) = \perp$ and $\overline{\rho}_1(\top) = \top$) and $\overline{\rho}_2$ (such that $\overline{\rho}_2(\perp) = \overline{\rho}_2(\top) = \top$) and neither $\overline{\rho}_1 \sqsubseteq f$ nor $\overline{\rho}_2 \sqsubseteq f$.

Monteiro [7], Iseki [5] and Morgado ([10, 11]) have given several characterizations of the upper closure operators on a partially ordered set by means of a single axiom. We now formulate a characterization of the upper closure operators on a complete lattice using only one axiom:

COROLLARY 4.5 *If $L(\sqsubseteq, \perp, \top, \sqcup, \sqcap)$ is a complete lattice and $\overline{\rho}$ is an operator on L , then $\overline{\rho}$ is an upper closure operator on L if and only if:*

$$\overline{\rho} = \lambda x. \sqcup \{y \sqcup \overline{\rho}(y) : (y \in L) \text{ and } (y \sqsubseteq x)\}$$

PROOF: If $\overline{\rho}$ is a closure operator then it is monotone (so that $\overline{M}(\overline{\rho}) = \overline{\rho}$ (2.2)), extensive (so that $\overline{E}(\overline{\rho}) = \overline{\rho}$) and idempotent (so that $\overline{\rho} = \overline{\rho} \circ \overline{\rho}$) therefore $\overline{\rho} = \overline{M}(\overline{E}(\overline{\rho} \circ \overline{\rho})) = \lambda x. \sqcup \{y \sqcup \overline{\rho}(y) : (y \in L) \text{ and } (y \sqsubseteq x)\}$. Reciprocally, if $\overline{\rho}$ is equal to the monotone operator $\overline{M}(\overline{E}(\overline{\rho} \circ \overline{\rho}))$ then it is monotone. Also since \overline{M} and \overline{E} commute (3.4) $\overline{\rho}$ is equal to the extensive operator $\overline{E}(\overline{M}(\overline{\rho} \circ \overline{\rho}))$ whence it is extensive. Since $\overline{\rho} \circ \overline{\rho}$ is the composition of monotone and extensive operators it is monotone and extensive and therefore $\overline{\rho} = \overline{M}(\overline{E}(\overline{\rho} \circ \overline{\rho})) = \overline{\rho} \circ \overline{\rho}$ (3.2, 2.2) proving that $\overline{\rho}$ is idempotent whence an upper closure operator. *End of proof.*

5. The lattice of all retractions on a complete lattice

DEFINITION 5.1

$$\bar{J}, \underline{J} \in (\overline{M(L \rightarrow L)} \rightarrow \overline{M(L \rightarrow L)})$$

$$\bar{J} = \lambda f. (llis(\lambda g. g \circ g) \circ luis(\lambda g. g \sqcup g \circ g)(f))$$

$$\underline{J} = \lambda f. (luis(\lambda g. g \circ g) \circ llis(\lambda g. g \sqcap g \circ g)(f))$$

$$\bar{R}, \underline{R} \in ((L \rightarrow L) \rightarrow (L \rightarrow L))$$

$$\bar{R} = \bar{J} \circ \bar{M}$$

$$\underline{R} = \underline{J} \circ \underline{M}$$

THEOREM 5.2 *The set of all retractions on a complete lattice $L(\sqsubseteq, \perp, \top, \sqcup, \sqcap)$ is a complete lattice $\bar{R}(L \rightarrow L)(\sqsubseteq, \perp, \top, \lambda S. luis(\lambda g. g \circ g)(\sqcup S), \lambda S. llis(\lambda g. g \circ g)(\sqcap S))$ which is the image of $(L \rightarrow L)$ by the retractions \bar{R} and \underline{R} such that $\underline{R} \sqsubseteq \bar{R}$.*

$\forall f \in (L \rightarrow L)$, if ρ is a retraction on L greater than or equal to (less than or equal to) f , then $\bar{R}(f)$ ($\underline{R}(f)$) is less than or equal to (greater than or equal to) ρ .

PROOF: The set of all retractions on L is the set of monotone operators on L (i. e. elements of $\overline{M(L \rightarrow L)} = \overline{M(L \rightarrow L)}$) which are idempotent that is to say fixed points of $\lambda g. g \circ g$. Let $f, h \in \overline{M(L \rightarrow L)}$ be such that $f \sqsubseteq h$. Then $h \sqsubseteq h$ so that $f \circ h \sqsubseteq h \circ h$ and by monotony $f \circ f \sqsubseteq f \circ h$ hence $f \circ f \sqsubseteq h \circ h$ proving that $\lambda g. g \circ g$ is a monotone operator on $\overline{M(L \rightarrow L)}$. Consequently the set of all retractions on L is $fp(\lambda g. g \circ g)$ which according to 1.18 is a complete lattice $fp(\lambda g. g \circ g)(\sqsubseteq, luis(\lambda g. g \circ g)(\perp) = \perp, llis(\lambda g. g \circ g)(\top) = \top, \lambda S. luis(\lambda g. g \circ g)(\sqcup S), \lambda S. llis(\lambda g. g \circ g)(\sqcap S))$ which is the image of $\overline{M(L \rightarrow L)}$ by the upper preclosure operator \bar{J} and the image of $\underline{M(L \rightarrow L)}$ by the lower preclosure operator \underline{J} hence the image of $(L \rightarrow L)$ by \bar{R} and \underline{R} .

$\forall f \in (L \rightarrow L)$, $\bar{J} \circ \bar{M}(f) \in \overline{M(L \rightarrow L)}$ hence $\bar{J} \circ \bar{M} = \bar{M} \circ \bar{J} \circ \bar{M}$ (Th. 2.3.) Also $\bar{M} \circ \bar{J} \circ \bar{M}(f)$ is idempotent so that it is a fixed point of $\lambda g. g \circ g$ and consequently a fixed point of \bar{J} proving that $\bar{J} \circ \bar{M} \circ \bar{J} \circ \bar{M}(f) = \bar{M} \circ \bar{J} \circ \bar{M}(f) = \bar{J} \circ \bar{M}(f)$. $\bar{J} \circ \bar{M}$ is idempotent but also monotone since \bar{M} is a monotone operator on $(L \rightarrow L)$ and \bar{J} a monotone operator on $\overline{M(L \rightarrow L)}$, hence \bar{R} and by duality \underline{R} are retractions on $(L \rightarrow L)$.

Let $f \in (L \rightarrow L)$ and $\rho \in \overline{R}(L \rightarrow L)$ be such that $\rho \sqsubseteq f$. Then $\rho = \overline{R}(\rho) \sqsubseteq \overline{R}(f)$ since ρ is a retraction on L and \overline{R} is monotone.

By duality if ρ is a retraction on L greater than or equal to f then $\underline{R}(f)$ is less than or equal to ρ . In general f and $\overline{R}(f)$ on one hand and f and $\underline{R}(f)$ on the other hand are not comparable. However $\underline{R}(f) \sqsubseteq \overline{R}(f)$ (1.18). *End of proof.*

6. The lattice of all upper preclosure operators on a complete lattice

DEFINITION 6.1

$$\overline{A} \in (\overline{R}(L \rightarrow L) \rightarrow \overline{R}(L \rightarrow L))$$

$$\overline{A} = \lambda f.luis(\lambda g.g \circ \overline{E}(g))(f) = \lambda f.luis(\lambda g.g \sqcup g \circ \overline{E}(g))(f)$$

$$\underline{P}, \underline{P} \in ((L \rightarrow L) \rightarrow (L \rightarrow L))$$

$$\overline{P} = \overline{A} \circ \overline{R}$$

$$\underline{P} = \underline{A} \circ \underline{R}$$

THEOREM 6.2 *The set of all upper preclosure operators on a complete lattice $L(\sqsubseteq, \perp, \top, \sqcup, \sqcap)$ is a complete lattice $\overline{P}(L \rightarrow L)$ $(\sqsubseteq, \perp, \top, \lambda S.luis(\lambda g.g \circ \overline{E}(g)) \circ luis(\lambda g.g \circ g)(\sqcup S), \lambda S.lis(\lambda g.g \circ g)(\sqcap S))$ which is the image of $(L \rightarrow L)$ by the retractions \underline{P} and \overline{P} satisfying $\underline{P} \sqsubseteq \overline{P}$.*

$\forall f \in (L \rightarrow L)$ if $\overline{\rho}$ is an upper preclosure operator on L greater than (less than) or equal to f then $\overline{P}(f)$ ($\underline{P}(f)$) is less than (greater than) or equal to $\overline{\rho}$.

PROOF: The set of all upper preclosure operators on L is the set of monotone and idempotent operators on L (i. e. elements ρ of $\overline{R}(L \rightarrow L) = \underline{R}(L \rightarrow L)$) satisfying the upper connectivity axiom (i. e. $\rho = \lambda x.(\rho(x) \sqcup \rho(x))$) that is to say which are fixed points of $F = \lambda g.(\lambda x.(g(x) \sqcup g(x))) = \lambda g.g \circ \overline{E}(g)$. Let ρ be an element of $\overline{R}(L \rightarrow L)$. Then $F(\rho)$ is monotone. Also $F(\rho) \circ F(\rho) = \lambda x.(\rho(x) \sqcup \rho(x)) \circ \lambda x.(\rho(x) \sqcup \rho(x)) = \lambda x.(\rho(\rho(x) \sqcup \rho(x)) \sqcup \rho(\rho(x) \sqcup \rho(x))) = \rho(\rho(x) \sqcup \rho(x)) = \rho(x) \sqcup \rho(x) = F(\rho)$ since ρ is idempotent and proving that $F(\rho)$ is idempotent. Therefore F is an operator on $\overline{R}(L \rightarrow L)$. Let ρ and η be elements of $\overline{R}(L \rightarrow L)$ such that $\rho \sqsubseteq \eta$. Then $\overline{E}(\rho) \sqsubseteq \overline{E}(\eta)$ and by monotony $F(\rho) = \rho \circ \overline{E}(\rho) \sqsubseteq \rho \circ \overline{E}(\eta) \sqsubseteq \eta \circ \overline{E}(\eta) = F(\eta)$ proving that F is a monotone operator on

$\overline{R}(L \rightarrow L)$. $\forall \rho \in \overline{R}(L \rightarrow L)$ $\rho = F(\rho)$ implies $F(\rho) \sqsubseteq \rho$. Reciprocally if $F(\rho) \sqsubseteq \rho$ then $\lambda x.x \sqsubseteq \overline{E}(\rho)$ implies by monotony $\rho \sqsubseteq \rho \circ \overline{E}(\rho) = F(\rho)$ and by antisymmetry $\rho = F(\rho)$ proving that $\overline{fp}(F) = \overline{postfp}(F)$. It follows from 5.2 and 1.16 that the set of all upper preclosure operators on L is a complete lattice $\overline{postfp}(F)(\sqsubseteq, \text{luis}(F)(\perp) = \perp, \top, \lambda S.\text{luis}(F) \circ \text{luis}(\lambda g.g \circ g)(\sqcup S), \lambda S.\text{llis}(\lambda g.g \circ g)(\cap S))$ which is the image of $\overline{R}(L \rightarrow L) = \underline{R}(L \rightarrow L)$ by the upper closure operator $\overline{A} = \lambda f.\text{luis}(F)(f)$ hence the image of $((L \rightarrow L)$ by \overline{P} and \underline{P} .

\underline{P} and \overline{P} are monotone since \overline{R} and \underline{R} are monotone and \overline{A} is an upper closure whence monotone operator on $\overline{R}(L \rightarrow L) = \underline{R}(L \rightarrow L)$. Also $\forall f \in (L \rightarrow L), \overline{A} \circ \overline{R}(f)$ is monotone and idempotent so that $\overline{A} \circ \overline{R}(f) = \overline{R} \circ \overline{A} \circ \overline{R}(f)$. $\overline{A} \circ \overline{R}(f)$ satisfies the upper connectivity axiom and therefore it is a fixed point of F and consequently a fixed point of \overline{A} so that $\overline{A} \circ \overline{R}(f) = \overline{A} \circ \overline{A} \circ \overline{R}(f) = \overline{A} \circ \overline{R} \circ \overline{A} \circ \overline{R}(f)$ proving that \overline{P} is idempotent. The same way \underline{P} is a retraction on $(L \rightarrow L)$. Since $\underline{R} = \overline{R}$ and \overline{A} is monotone we have $\underline{P} = \overline{A} \circ \underline{R} \sqsubseteq \overline{A} \circ \overline{R} = \overline{P}$.

$\forall f \in (L \rightarrow L)$, let $\overline{\rho} \in \overline{P}(L \rightarrow L)$ be such that $f \sqsubseteq \overline{\rho}$. Then $\overline{\rho} \in \overline{R}(L \rightarrow L)$ and 5.2 implies $\overline{R}(f) \sqsubseteq \overline{\rho}$. Since $\overline{\rho}$ is closed under \overline{A} we have $\overline{P}(f) = \overline{A}(\overline{R}(f)) \sqsubseteq \overline{A}(\overline{\rho}) = \overline{\rho}$. Similarly $\overline{\rho} \sqsubseteq f$ implies $\overline{\rho} = \overline{A}(\overline{\rho}) \sqsubseteq \overline{A}(\overline{R}(f)) = \overline{P}(f)$. End of proof.

7. The lattice of all comparing operators on a complete lattice

DEFINITION 7.1

$$\overline{K}, \underline{K} \in ((L \rightarrow L) \rightarrow (L \rightarrow L))$$

$$\overline{K} = \lambda f.(\lambda x.(if \underline{f}(x) \sqsubseteq x \text{ then } f(x) \text{ else } x \sqcup f(x)))$$

$$\underline{K} = \lambda f.(\lambda x.(if \underline{x} \sqsubseteq f(x) \text{ then } f(x) \text{ else } x \cap f(x)))$$

LEMMA 7.2 If $f \in (L \rightarrow L)$ then $\overline{K}(f)$ (respectively $\underline{K}(f)$) is the least (greatest) comparing operator on L greater than (less than) or equal to f .

COROLLARY 7.3 \overline{K} is an upper closure operator on $(L \rightarrow L)$, \underline{K} is a lower closure operator on $(L \rightarrow L)$. The set of all comparing operators on the complete lattice L is a complete sublattice $\overline{K}(L \rightarrow L) = \underline{K}(L \rightarrow L)(\sqsubseteq, \perp, \top, \sqcup, \cap)$ of $(L \rightarrow L)$.

LEMMA 7.4

- $\overline{K \circ M}$ is an upper-closure operator on $(L \rightarrow L)$
- $\underline{K \circ M}$ is a lower closure operator on $(L \rightarrow L)$
- $(\overline{M}(L \rightarrow L) \cap \overline{K}(L \rightarrow L)) = (\underline{M}(L \rightarrow L) \cap \underline{K}(L \rightarrow L)) = \overline{K \circ M}(L \rightarrow L) = \underline{K \circ M}(L \rightarrow L)$ is a complete sublattice $(\sqsubseteq, \perp, \top, \sqcup, \sqcap)$ of $(L \rightarrow L)$.

PROOF: $\overline{K \circ M}$ is monotone and extensive since it is the composition of upper closure operators. $\forall f \in (L \rightarrow L)$, $\overline{K}(\overline{M}(f))$ is either equal to $\overline{M}(f)$ or to $\lambda x.(x \sqcup \overline{M}(f)(x))$ proving that $\overline{K}(\overline{M}(f))$ is monotone and therefore equal to $\overline{M}(\overline{K}(\overline{M}(f)))$. Hence $\overline{K}(\overline{M}(f)) = \overline{K}(\overline{K}(\overline{M}(f))) = \overline{K}(\overline{M}(\overline{K}(\overline{M}(f))))$ proving that $\overline{K \circ M}$ is an upper closure operator and by duality $\underline{K \circ M}$ is a lower closure operator on $(L \rightarrow L)$. Also $\overline{K \circ M}(f)$ (respectively $\underline{K \circ M}(f)$) is the least (greatest) monotone and comparing operator on \overline{L} greater than (less than) or equal to f . Hence according to 4.11, $(\overline{M}(L \rightarrow L) \cap \overline{K}(L \rightarrow L)) = (\underline{M}(L \rightarrow L) \cap \underline{K}(L \rightarrow L))$ is the complete sublattice $\overline{K \circ M}(L \rightarrow L) = \underline{K \circ M}(L \rightarrow L)$ $(\sqsubseteq, \perp, \top, \sqcup, \sqcap)$ of $(L \rightarrow L)$. *End of proof.*

8. The lattice of all quasi-closure operators on a complete lattice

DEFINITION 8.1

$$\begin{aligned} \overline{B}, B &\in (\overline{K \circ M}(L \rightarrow L) \rightarrow \overline{K \circ M}(L \rightarrow L)) \\ \underline{B} &= \text{llis}(\lambda g.g \circ \overline{E}(g)) \circ \text{llis}(\lambda g.g \sqcap g \circ \underline{E}(g)) \\ \overline{B} &= \text{llis}(\lambda g.g \circ \underline{E}(g)) \circ \text{llis}(\lambda g.g \sqcup g \circ \overline{E}(g)) \\ \underline{Q}, \overline{Q} &\in ((L \rightarrow L) \rightarrow (L \rightarrow L)) \\ \underline{Q} &= \underline{B \circ K \circ M} \\ \overline{Q} &= \overline{B \circ K \circ M} \end{aligned}$$

Bernard [1,Th.V.II.4]'s theorem states that the set Γ of all quasi-closure operators on a complete lattice $L(\sqsubseteq, \perp, \top, \sqcup, \sqcap)$ is a complete lattice $\Gamma(\sqsubseteq, \perp, \top, \lambda S.(\sqcup E(S)) \circ (\sqcup \overline{E}(S)), \lambda S.(\sqcap \overline{E}(S)) \circ (\sqcap E(S))$) where (\sqcup, \sqcap) and (\sqcup, \sqcap) respectively denote the union and intersection in the complete lattices of upper and lower closure operators on L . We now give a constructive and simpler version of this theorem:

THEOREM 8.2 *The set of all quasi-closure operators on a complete lattice $L(\sqsubseteq, \perp, \top, \sqcup, \sqcap)$ is a complete lattice $\overline{Q}(L \rightarrow L)$ ($\sqsubseteq, \perp, \top, \lambda S.luis(\lambda g.g \circ \overline{E}(g))(\sqcup S), \lambda S.llis(\lambda g.g \circ \overline{E}(g))(\sqcap S)$) which is the image of $(L \rightarrow L)$ by the retractions \underline{Q} and \overline{Q} satisfying $\underline{Q} \sqsubseteq \overline{Q}$.*

$\forall f \in (L \rightarrow L)$, if ρ is a quasi-closure operator on L greater than (respectively less than) or equal to f then $\overline{Q}(f)$ ($\underline{Q}(f)$) is less than (greater than) or equal to ρ .

PROOF: The quasi-closure operators on L are the monotone and comparing operators on L (i. e. $\rho \in \overline{K \circ M}(L \rightarrow L)$) satisfying the connectivity axiom $\rho = \lambda x.\rho(x \sqcap \rho(x)) = \lambda x.\rho(x \sqcup \rho(x))$ that is the elements of $\overline{K \circ M}(L \rightarrow L)$ which are common fixed points of the functionals $\underline{F} = \lambda g.g \circ \overline{E}(g) = \lambda g.(\lambda x.(g(x \sqcap g(x))))$ and $\overline{F} = \lambda g.g \circ \underline{E}(g) = \lambda g.(\lambda x.(g(x \sqcup g(x))))$.

$\forall \rho \in \overline{K \circ M}(L \rightarrow L)$, ρ is monotone and so are $\underline{F}(\rho)$ and $\overline{F}(\rho)$. Also ρ is comparing so that given $x \in L$ either $x \sqsubseteq \rho(x)$ in which case $x \sqsubseteq \rho(x) = \rho(x \sqcap \rho(x)) = \underline{F}(\rho)(x)$ or $\rho(x) \sqsubseteq x$ in which case $\underline{F}(\rho)(x) = \rho(x \sqcap \rho(x)) = \rho(\rho(x))$ and by monotony $\rho(\rho(x)) \sqsubseteq \rho(x) \sqsubseteq x$. Hence $\underline{F}(\rho)$ is comparing proving that \underline{F} (and by duality \overline{F}) is an operator on $\overline{K \circ M}(L \rightarrow L)$.

\underline{F} and \overline{F} are monotone operators on $\overline{K \circ M}(L \rightarrow L)$ and $\underline{F} \sqsubseteq \overline{F}$. Let us show that $\overline{F \circ \underline{F}} \sqsubseteq \underline{F \circ \overline{F}}$ that is $\forall \rho \in \overline{K \circ M}(L \rightarrow L), \forall x \in L, \overline{F}(\underline{F}(\rho))(x) = \rho((x \sqcup \rho(x \sqcap \rho(x))) \sqcap \rho(x \sqcup \rho(x \sqcap \rho(x)))) \sqsubseteq \rho((x \sqcap \rho(x \sqcup \rho(x \sqcap \rho(x)))) \sqcup \rho(x \sqcap \rho(x \sqcup \rho(x \sqcap \rho(x)))))) = \underline{F}(\overline{F}(\rho))(x)$. If $x \sqsubseteq \rho(x)$ then $(x \sqcup \rho(x \sqcap \rho(x))) = \rho(x)$ and $(x \sqcap \rho(x \sqcup \rho(x))) = x$ so that we must show that $\rho(\rho(x) \sqcap \rho(\rho(x))) \sqsubseteq \rho(x \sqcap \rho(x))$ which is true since $(\rho(x) \sqcap \rho(\rho(x))) = \rho(x)$ and $(x \sqcup \rho(x)) = \rho(x)$. Else $\rho(x) \sqsubseteq x$ and then $(x \sqcup \rho(x \sqcap \rho(x))) = x$ and $(x \sqcap \rho(x \sqcup \rho(x))) = \rho(x)$ so that we must show that $\rho(x \sqcap \rho(x)) \sqsubseteq \rho(\rho(x) \sqcup \rho(\rho(x)))$ which is true since $(x \sqcap \rho(x)) = \rho(x)$ and $(\rho(x) \sqcup \rho(\rho(x))) = \rho(x)$. Applying 1.19 and using 7.4 we see that the set $fp(\underline{F}, \overline{F})$ of common fixed points of \underline{F} and \overline{F} is a complete lattice $fp(\underline{F}, \overline{F})(\sqsubseteq, luis(\overline{F})(\perp) = \perp, llis(\underline{F})(\top) = \top, \lambda S.luis(\overline{F})(\sqcup S), \lambda S.llis(\underline{F})(\sqcap S))$ which is the image of $\overline{K \circ M}(L \rightarrow L) = \underline{K \circ M}(L \rightarrow L)$ by the lower preclosure operator \underline{B} and the upper preclosure operator \overline{B} satisfying $\underline{B} \sqsubseteq \overline{B}$ hence the image of $(L \rightarrow L)$ by \underline{Q} and \overline{Q} .

Since $\underline{M} \sqsubseteq \overline{M}$, $\underline{K} \sqsubseteq \overline{K}$, $\underline{B} \sqsubseteq \overline{B}$ and \underline{K} , \underline{B} are monotone we have $\underline{Q} \sqsubseteq \overline{Q}$. \underline{Q} and \overline{Q} are composition of monotone operators and therefore monotone. Also \overline{Q} (and dually \underline{Q}) is idempotent since $\forall f \in (L \rightarrow L)$, $\overline{Q}(f)$ is monotone (hence $\overline{M} \circ \overline{Q}(f) = \overline{Q}(f)$), comparing (hence $\overline{K} \circ \overline{Q}(f) = \overline{Q}(f)$) and satisfies the connectivity axiom (hence is a common fixed point of \underline{F} and \overline{F} therefore fixed point of \overline{B}) so that $\overline{B} \circ \overline{K} \circ \overline{M} \circ \overline{Q}(f) = \overline{Q}(f)$.

$\forall f \in (L \rightarrow L)$, $\forall \rho \in \overline{Q}(L \rightarrow L)$, if $f \sqsubseteq \rho$ then $\overline{Q}(f) \sqsubseteq \overline{Q}(\rho) = \rho$ and if $\rho \sqsubseteq f$ then $\rho = \underline{Q}(\rho) \sqsubseteq \underline{Q}(f)$. *End of proof.*

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