A Tight Approximation Bound for the Stable Marriage Problem with Restricted Ties

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Abstract

The problem of finding a maximum cardinality stable matching in the presence of ties and unacceptable partners, called MAX SMTI, is a well-studied NP-hard problem. The MAX SMTI is NP-hard even for highly restricted instances where (i) ties appear only in women's preference lists and (ii) each tie appears at the end of each woman's preference list. The current best lower bounds on the approximation ratio for this variant are 1.1052 unless P=NP and 1.25 under the unique games conjecture, while the current best upper bound is 1.4616. In this paper, we improve the upper bound to 1.25, which matches the lower bound under the unique games conjecture. Note that this is the first special case of the MAX SMTI where the tight approximation bound is obtained. The improved ratio is achieved via a new analysis technique, which avoids the complicated case-by-case analysis used in earlier studies. As a by-product of our analysis, we show that the integrality gap of natural IP and LP formulations for this variant is 1.25. We also show that the unrestricted MAX SMTI cannot be approximated with less than 1.5 unless the approximation ratio of a certain special case of the minimum maximal matching problem can be improved.

1998 ACM Subject Classification G.2.1 Combinatorics

Keywords and phrases stable marriage with ties and incomplete lists, approximation algorithm, integer program, linear program relaxation, integrality gap

Digital Object Identifier 10.4230/LIPIcs.xxx.yyy.p

1 Introduction

The stable marriage problem [12, 27] is a classical combinatorial problem introduced by Gale and Shapley in their celebrated seminal paper [9]. An input of this problem includes two sets; a set of men and a set of women. Each man submits a preference list that orders women according to his preference, and similarly each woman submits her preference list. Given these lists, the problem is to find a stable matching, a matching without any *blocking pairs*,

Conference title on which this volume is based on.
Editors: Billy Editor and Bill Editors; pp. 1–20
Leibniz International Proceedings in Informatics
LIPICS Schloss Dagstuhl – Leibniz-Zentrum für Informatik, Dagstuhl Publishing, Germany

Table 1 Four problem settings of MAX SMTI

Two-sided-ties case (2T)		One-sided-ties case (1T)	
$m_1 : w_2\ w_1$	$w_1 : m_1\ m_2$	$m_1 : w_2\ w_1$	$w_1 : m_1$
$m_2 : (w_1\ w_2)\ w_3$	$w_2 : m_1\ m_2$	$m_2 : w_2\ w_3$	$w_2 : (m_1\ m_2)\ m_3$
$m_3 : w_3$	$w_3 : (m_2\ m_3)$	$m_3 : w_3\ w_2$	$w_3 : m_2\ m_3$
Restricted two-sided-ties case (R2T)		**Restricted one-sided-ties case (R1T)**	
$m_1 : w_2\ w_1$	$w_1 : m_1\ m_2$	$m_1 : w_1\ w_3$	$w_1 : m_1\ m_3$
$m_2 : w_1\ (w_2\ w_3)$	$w_2 : m_1\ m_2$	$m_2 : w_2\ w_3$	$w_2 : m_2$
$m_3 : w_3$	$w_3 : (m_2\ m_3)$	$m_3 : w_3\ w_1$	$w_3 : m_2\ (m_1\ m_3)$

where a blocking pair is that of a man and a woman who might elope (a formal definition is given in Sec. 2). There are several variants of the problem in terms of the format of preference lists. In its most general setting, preference lists may contain ties (i.e., two or more men indifferent to woman w may be included in a tie of w's list), and may be incomplete (i.e., a man who is not acceptable to w is missing in her list). It is known that there exists at least one stable matching in any instance, but there may exist stable matchings of different sizes. The problem of finding a maximum cardinality stable matching in this setting (called the Maximum Stable Marriage problem with Ties and Incomplete lists, or MAX SMTI for short) is NP-hard [18, 28]; therefore, the approximability of the MAX SMTI has been intensively studied.

The MAX SMTI has been studied for various settings (see Table 1). The most general setting is the two-sided-ties problem (2T), where ties may appear in both men's and women's lists. The second setting is the one-sided-ties problem (1T), where ties can appear only in women's lists (i.e., men's preference lists are strictly ordered and may not include ties). The third setting is the restricted one-sided-ties problem (R1T), where, in addition to the second setting, ties can appear only at the end of women's preference lists. The R1T was first studied by Irving and Manlove [17], inspired by an actual application for the Scottish Foundation Allocation Scheme (SFAS) [16]. The SFAS is designed to assign residents to hospitals under the condition that a resident (a man in our marriage case) submits a strictly-ordered preference list while a hospital (a woman) submits a preference list that may contain one tie of arbitrary length at the end of the list. In this paper, we also consider another natural setting, the restricted two-sided ties problem (R2T), which was not studied before. In R2T, ties can appear in both sexes' lists, but the position must be the end of the lists.

Let us review previous results on the approximability of the MAX SMTI for 2T, 1T, and R1T. It is easy to see that any algorithm that produces a stable matching is a 2-approximation algorithm, but the existence of a $(2-\epsilon)$-approximation algorithm for the MAX SMTI is nontrivial. For 2T, after several attempts to obtain $(2-o(1))$-approximation algorithms [19, 20], a 1.875-approximation algorithm was first presented in [21]. Later, Király [25] improved it to 1.6667 and McDermid [30] to 1.5, which is the current best approximation ratio. Recently, Király [26] and Paluch [31] presented simpler linear time algorithms with the same approximation ratio of 1.5. On the negative side, it is known that 1.1379-approximation is NP-hard and 1.3333-approximation is UG-hard via a reduction from the vertex cover problem [35]. (Here "UG-hard" means that there is no better approximation algorithm if the unique games conjecture [24] is true.) Also, an integrality gap for 2T is shown to be at least $1.5-o(1)$ [22] for a natural integer programming formulation, which rules out the possibility of using some current techniques (e.g., rounding and primal-dual

Table 2 Upper and lower bounds on approximation ratio and lower bounds on integrality gap (new results discussed in this paper are in bold)

	2T	R2T	1T	R1T
Upper bounds on approximation ratio	1.5 [30]	1.5 [30]	1.4616 [5]	1.4616 [5] $\rightarrow$ **1.25**
Lower bounds on integrality gap	$1.5 - o(1)$ [22]	– $\rightarrow$ **1.3333**	1.3678 [22] $\rightarrow$ $\mathbf{1.5 - o(1)}$	– $\rightarrow$ **1.25**
UG lower bounds on approximation ratio	1.3333 [35]	– $\rightarrow$ **1.3333**	1.25 [13, 35]	1.25 [13, 35]
Lower bounds assuming MMM-Bi-APM is hard to approximate	– $\rightarrow$ **1.5**	–	–	–

algorithms) to show $(1.5 - \epsilon)$-approximation.

It is still open whether there exists a $(1.5 - \epsilon)$-approximation algorithm for 2T and R2T, but this problem is resolved for 1T and R1T by providing a 1.4706-approximation algorithm that uses a linear programming (LP) relaxation [22]. Huang and Kavitha [15] improved the approximation ratio to 1.4667 by developing a linear time algorithm, and Radnai [32] tightened the analysis of this algorithm to show 1.4643-approximation. Very recently, Dean and Jalasutram [5] showed that the algorithm presented in [22] provides the current best approximation ratio 1.4616 through an analysis using the idea of factor-revealing LP. On the negative side of 1T and R1T, it is known that 1.1052-approximation is NP-hard and $(1.25 - \epsilon)$-approximation is UG-hard, via a reduction from the vertex cover problem [13, 35]. The integrality gap of the natural IP formulation is known to be at least 1.3678 for 1T [22], but there is no known lower bound on the integrality gap for R1T.

Our contributions.

Our contributions (and previous results) are summarized in Table 2.

Our first contribution is to provide a tight upper bound of 1.25 for R1T. This is the first upper bound result for the MAX SMTI that matches a UG lower bound. Our algorithm is LP-based, which is almost the same as that in [22], but in this paper we introduce a novel analysis, which not only avoids the tedious case-analysis used in earlier studies, but also significantly improves the approximation ratio. In all previous analyses of the current algorithms, we create a bipartite graph G that is the union of two matchings M^* and M, where M^* is a largest stable matching and M is a stable matching obtained from an approximation algorithm (see Fig. 1). It is easy to see that the number of short paths in this graph is directly related to the approximation factor. Indeed, Király [25] showed that his algorithm does not create length-three paths for 1T, and McDermid [30] did the same for 2T, both achieving a 1.5 upper bound. The analyses of the algorithms discussed in [15, 22] bounded the number of length-five paths for 1T, which led to breaking the 1.5 barrier. Unfortunately, natural extension of this approach to longer paths seems to have a quick limit since there is no obvious way of getting rid of complicated case analysis. In fact, the current best bound for 1T [5], resulting in an improvement from 1.4643 to 1.4616, requires a computer-assisted proof to bound the numbers of length-seven and length-nine paths.

In this paper, we come back to a more direct and standard approach in the analysis

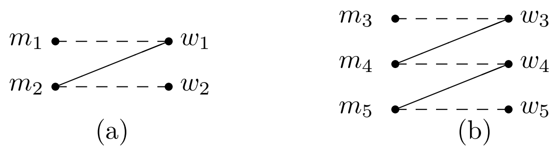

Figure 1 Illustrations of (a) a length-three path and (b) a length-five path in the union graph G, where solid edges represent pairs from M and dashed edges represent pairs from M^*.

of an LP-based approximation algorithm. That is, we just apply a formula relating the LP-relaxed (optimum) value and the optimum integral value. A notable difference between our new analysis and the old one [22] is that we partition M into three sets of pairs based on M and M^* in the old analysis, whereas we do so based on M only in the new analysis. This difference allows us to avoid handling long paths (such as length-seven and length-nine paths). One might be curious about an extension of this approach to 1T or even 2T, which is our obvious future goal.

Our second contribution is to give a 1.3333-UG lower bound for R2T, which is obtained by modifying the reduction used to show the same UG lower bound for 2T [35]. This result implies that R2T is strictly harder than R1T under the unique games conjecture, while such a separation is unknown for 2T and 1T. For R2T, we also show that the integrality gap of the natural IP formulation is at least 1.3333.

Our third contribution is to show a new lower bound on the integrality gap for 1T. Specifically, we construct an instance of 1T whose integrality gap is at least $1.5 - o(1)$. This result suggests that the integrality gap of 1.5 for 2T is no longer a convincing bad sign for improving the current 1.5 upper bound because we already have a better approximation ratio for 1T (1.4616) than the integrality gap. Note that our new integrality gap of $1.5 - o(1)$ does not contradict the upper bounds of less than 1.5 [5, 22] since the technique used in the algorithms of these studies is not a simple LP rounding.

Our final contribution is to give support to the inapproximability of 2T by relating 2T to the minimum maximal matching problem (MMM). The MMM is a classical optimization problem, which asks us to find a maximal matching with minimum cardinality in a given undirected graph. This problem is known to be NP-hard even for a very restricted class of graphs (including bipartite graphs) [10, 14, 36]. It is also known that the MMM is equivalent to the minimum edge dominating set problem (MEDS) with respect to approximability [36], that is, there exists an α-approximation algorithm for the MMM if and only if there exists an α-approximation algorithm for the MEDS. So far, the approximability of the MMM (and equivalently that of the MEDS) has been extensively studied [2, 3, 8, 11, 29], but none achieved a $(2 - \epsilon)$-approximation for any constant $\epsilon > 0$ even on bipartite graphs. Regarding the inapproximability, the current best lower bound under P$\neq$NP is 7/6 for general graphs [4]. Based on the reduction in [18], we show that a $(1.5 - \epsilon)$-approximation algorithm for 2T for a constant $\epsilon > 0$ implies a $(2 - \epsilon')$-approximation algorithm for the MMM on bipartite graphs with an almost-perfect matching (which we call the MMM-Bi-APM) for a constant $\epsilon' > 0$. Note that there is no known $(2 - \epsilon')$-approximation algorithm, even for the MMM-Bi-APM.

2 Preliminaries

We now give notations, most of which are taken from [22]. An instance I of the MAX SMTI is composed of n men, n women, and each person's preference list that may be incomplete and may include ties. If a person p includes a person q (of the opposite sex) in p's preference list, we say that q is *acceptable* to p. Without loss of generality, we assume that a man m is acceptable to w if and only if w is acceptable to m. A matching M is a set of pairs (m, w) such that m is acceptable to w and vice versa, and each person appears at most once in M. If $(m, w) \in M$, we say that m (w) is *matched in* M, and write $M(m) = w$ and $M(w) = m$. If p does not appear in M, we say that p is *single in* M. If m strictly prefers w_i to w_j, we write $w_i \succ_m w_j$. If w_i and w_j are tied in m's list (including the case in which $w_i = w_j$), we write $w_i =_m w_j$. The statement $w_i \succeq_m w_j$ is true if and only if $w_i \succ_m w_j$ or $w_i =_m w_j$. We use similar notations for women's preference lists. We say that m and w form a *blocking pair* for a matching M (or simply, (m, w) *blocks* M) if the following three conditions are met: (i) $M(m) \neq w$ but m and w are acceptable to each other, (ii) $w \succ_m M(m)$ or m is single in M, and (iii) $m \succ_w M(w)$ or w is single in M. A matching M is called *stable* if there is no blocking pair for M.

The MAX SMTI is the problem of finding the largest stable matching. The following IP formulation of MAX SMTI instance I, denoted as $IP(I)$, is a generalization of the one for the original stable marriage problem given in [33, 34]. For each pair (m, w), we introduce a binary variable $x_{m,w}$.

$$
\begin{aligned}
&\text{Maximize:} && \sum_i \sum_j x_{i,j} && \\
&\text{Subject to:} && \sum_i x_{i,w} \leq 1 && \forall w \quad (1)\\
& && \sum_j x_{m,j} \leq 1 && \forall m \quad (2)\\
& \sum_{j \succeq_m w} x_{m,j} + \sum_{i \succeq_w m} x_{i,w} - x_{m,w} \geq 1 && && \forall (m,w) \in A \quad (3)\\
& && x_{m,w} = 0 && \forall (m,w) \notin A \quad (4)\\
& && x_{m,w} \in \{0,1\} && \forall (m,w) \quad (5)
\end{aligned}
$$

Here, A is the set of mutually acceptable pairs, that is, $(m, w) \in A$ if and only if m and w are acceptable to each other. In this formulation, "$x_{m,w} = 1$" is interpreted as "m and w are matched," and "$x_{m,w} = 0$" otherwise. Thus the objective function is equal to the size of a matching. Note that Constraint (3) ensures that (m, w) is not a blocking pair. When $x_{m,w} = 1$, all three terms of the left-hand side are 1; hence, Constraint (3) is satisfied. When $x_{m,w} = 0$, either the first or the second term of the left-hand side must be 1, which implies that m (respectively w) must be matched with a partner as good as w (respectively m). The notation $LP(I)$ denotes the linear program relaxation of $IP(I)$ in which Constraint (5) is replaced with "$0 \leq x_{m,w} \leq 1$."

3 Approximation Algorithm for R1T

3.1 Algorithm GSA-LP

We now describe our approximation algorithm GSA-LP for instance I in which the men's lists are strict and the women's lists may contain ties. This algorithm is a simpler version of the algorithm given in [22], whose pseudo-code is given in Algorithm 1. In this algorithm, we maintain a variable p_m (initially set to one), which stores the current position for m in his preference list, and another priority value f_m (initially set to zero) for each m.

The GSA-LP algorithm consists of a sequence of proposals from men to women, as the standard Gale-Shapley algorithm. When a woman receives proposals from two men, she keeps the better one and rejects the other. If two men are in the same tie, the woman chooses

Algorithm 1 GSA-LP (Gale-Shapley Algorithm with LP solution)

Input: An SMTI instance I

Output: A matching M

1: Formulate the given instance I as an integer program $IP(I)$
2: Solve its LP relaxation $LP(I)$ and obtain an optimal solution $x^*(= \{x^*_{i,j}\})$
3: Let $M := \emptyset$
4: Set $f_m := 0$ and $p_m := 1$ for each man m
5: **while** there exists an m such that (m is single in M) and ($f_m \leq 3$) **do**
6: Let m be an arbitrary such man
7: **if** p_m is no larger than the length of m's preference list **then**
8: Let w be the p_m-th woman of m's preference list
9: **if** m has not proposed to w yet **then**
10: Set $f_m := f_m + x^*_{m,w}$ and $p_m := 1$
11: **else**
12: Set $p_m := p_m + 1$
13: **end if**
14: // Let m propose to w
15: **if** w is single in M **then**
16: Set $M := M \cup \{(m, w)\}$
17: **else if** $m \succ_w M(w)$ or ($m =_w M(w)$ and $f_m > f_{M(w)}$) **then**
18: Set $M := M \cup \{(m, w)\} \setminus \{(M(w), w)\}$
19: **end if**
20: **else**
21: Set $f_m := f_m + 2$ and $p_m := 1$ // m goes to the second round
22: **end if**
23: **end while**
24: **return** M

the man with the larger priority value f_m (if two values are the same, she keeps the current partner).

Intuitively, there are two rounds of proposals for each m. In the first round, whenever m sends a proposal to w for the first time, the priority value f_m is increased by $x^*_{m,w}$ (Lines 9–10). When he is rejected by w (either immediately or later after once accepted), he restarts his sequence of proposals from the top of his list. Note that in the restarted sequence of proposals, women he proposes to are not new until w. Up to that woman, f_m does not change and the restart does not happen. If m has proposed to all of the women in his list and he is still single, then f_m increases by 2 and m goes to his second round (Lines 20–21). In the second round, m sends a sequence of proposals from the top of his list again. Meanwhile, f_m does not change and restart never happens (that is, m sends at most one proposal to each woman in the second round). It is not hard to see that this algorithm runs in polynomial time and outputs a stable matching based on a similar argument in [22].

This algorithm is designed so that, if w is eventually matched to m at the termination of the algorithm, and there is another man m' who proposed to w at least once and is tied with m in her preference list (i.e., $m =_w m'$), then we have the following inequality

$$f_{m'} = \sum_{j' \succ_{m'} w, j' \notin S} x^*_{m',j'} \leq \sum_{j \succeq_m l(m), j \notin S} x^*_{m,j} = f_m, \tag{6}$$

where $l(m)$ denotes the woman least preferred by m among those who have received a proposal from m during the algorithm. Here, f_m and $f_{m'}$ refer to the values at the termination of the algorithm. Also note that, when m proposes to the woman at the end of his list for the first time, $f_m \leq 1$. If m is single in M at the termination of the algorithm, he has proposed to all the women in his list, and $f_m > 1$ holds.

3.2 Analysis of Approximation Ratio

3.2.1 Overview of Analysis

Our analysis is similar to the LP-based analysis used in [22], but the major difference between these two approaches is that our analysis does not use the bipartite graph on which older analyses (e.g., [15, 22, 25, 30]) heavily rely. Let us fix an instance I, and let M be the stable matching output from GSA-LP. We partition M into P, R, and T. Specifically, P is the set of pairs $(m, w) \in M$ such that $f_m > 1$ at the end of the algorithm, T is the set of pairs $(m, w) \in M$ such that $f_m \leq 1$, w has a tie, and m is contained in her tie, and $R = M \setminus (P \cup T)$. Let S be the set of men and women who are single in M.

Now we analyze the approximation ratio of GSA-LP under the assumption that ties can appear only at the end of women's preference lists. Recall that $x^*_{i,j}$ is the value of $x_{i,j}$ for the optimum solution x^* of $LP(I)$. Note that if $x^*_{m,w} > 0$ for $m, w \in S$, then $(m, w) \in A$ by Constraint (4) of $LP(I)$, so (m, w) is a blocking pair for M. This contradicts the stability of M; hence, $\sum_{i,j \in S} x^*_{i,j} = 0$. Now, let us define the value $x^*(X)$ for a subset $X \subseteq M$ as:

$$x^*(X) = \sum_{(m,w) \in X} \left(\sum_j x^*_{m,j} + \sum_i x^*_{i,w} + \sum_{j \in S} x^*_{m,j} + \sum_{i \in S} x^*_{i,w} \right).$$

It is not difficult to see that $x^*(P) + x^*(R) + x^*(T) = 2 \sum_i \sum_j x^*_{i,j}$, since $\sum_{i,j \in S} x^*_{i,j} = 0$. Note that $|M^*|$ and $\sum_i \sum_j x^*_{i,j}$ are the optimal values for $IP(I)$ and $LP(I)$ respectively, where M^* is an optimal solution of I (that is, one of the maximum stable matchings of I). Hence we have that $|M^*| \leq \sum_i \sum_j x^*_{i,j} = (x^*(P) + x^*(R) + x^*(T))/2$. We later prove the following key lemma.

▶ **Lemma 1.** $x^*(P) + x^*(R) + x^*(T) \leq \frac{5}{2}(|P| + |R| + |T|)$.

From this, we have that $|M^*| \leq (x^*(P) + x^*(R) + x^*(T))/2 \leq \frac{5}{4}(|P| + |R| + |T|) = \frac{5}{4}|M|$, and Theorem 2 follows:

▶ **Theorem 2.** *The approximation ratio of GSA-LP is at most* $5/4$ *for R1T.*

Remarks on Integrality Gap.

The proof of Theorem 2 implies

$$\sum_i \sum_j x^*_{i,j} \leq \frac{5}{4}|M| \leq \frac{5}{4}|M^*|,$$

and this means that the integrality gap of $IP(I)$ is at most $5/4$. This result is tight because the integrality gap of $IP(I)$ is at least $5/4$, which is shown in Threorem 17.

3.2.2 Proof Sketch of Lemma 1

For readability, we first give a simpler proof of Lemma 1 for a special case in which two conditions (which we explain shortly) hold. The full proof (without conditions) is included in Sec. 3.2.3. We first define the following symbols (see also Fig. 2).

$$m_p : \underbrace{\cdots w_p \cdots}_{\pi} \qquad\qquad w_p : \underbrace{\cdots m_p \cdots}_{p}$$

$$m_r : \underbrace{\cdots}_{\rho}\underbrace{w_r \cdots}_{\overline{\rho}} \qquad\qquad w_r : \underbrace{\cdots m_r}_{r}\underbrace{\cdots}_{\overline{r}}$$

$$m_t : \underbrace{\cdots w_t \cdots l(m_t)}_{\tau}\underbrace{\cdots}_{\overline{\tau}} \qquad\qquad w_t : \underbrace{\cdots\cdots(\cdots m_t \cdots)}_{t}$$

Figure 2 Illustrations of symbols π, ρ, $\overline{\rho}$, τ, $\overline{\tau}$, p, r, $\overline{r}$, and t for pairs $(m_p, w_p) \in P$, $(m_r, w_r) \in R$, and $(m_t, w_t) \in T$.

$\pi = \{(m, j) \in A \mid (m, w) \in P, j \notin S\}$,
$p = \{(i, w) \in A \mid (m, w) \in P, i \notin S\}$,
$\rho = \{(m, j) \in A \mid (m, w) \in R, j \succ_m w, j \notin S\}$,
$\overline{\rho} = \{(m, j) \in A \mid (m, w) \in R, w \succeq_m j, j \notin S\}$,
$r = \{(i, w) \in A \mid (m, w) \in R, i \succeq_w m, i \notin S\}$,
$\overline{r} = \{(i, w) \in A \mid (m, w) \in R, m \succ_w i, i \notin S\}$,
$\tau = \{(m, j) \in A \mid (m, w) \in T, j \succeq_m l(m), j \notin S\}$,
$\overline{\tau} = \{(m, j) \in A \mid (m, w) \in T, l(m) \succ_m j, j \notin S\}$, and
$t = \{(i, w) \in A \mid (m, w) \in T, i \notin S\}$.

For $X \in \{\pi, \rho, \overline{\rho}, \tau, \overline{\tau}, p, r, \overline{r}, t\}$, $Y \in \{\pi, \rho, \overline{\rho}, \tau, \overline{\tau}\}$, and $Z \in \{p, r, \overline{r}, t\}$, we define $\sigma(X)$ and $\sigma(Y, Z)$ as

$$\sigma(X) = \sum_{(m,w)\in X} x^*_{m,w} \quad\text{and}\quad \sigma(Y, Z) = \sum_{(m,w)\in Y\cap Z} x^*_{m,w}.$$

The two conditions we use in this section are (I) $P = \emptyset$ and (II) $\sigma(\rho)/|R| \le \sigma(\tau)/|T|$. Condition (II) is introduced to avoid using Inequality (6) in the analysis, which can make the proof significantly simpler. Inequality (6) implies that we have $f_{m'} \le f_m$ for any $(m, w) \in T$ and $(m', w') \in R$ such that both m and m' have proposed to w during the course of GSA-LP. The intuitive meaning of Condition (II) is that $f_{m'} \le f_m$ holds *on average* if we choose $(m, w) \in T$ and $(m', w') \in R$ uniformly at random. Note that $f_m \approx \sigma(\tau)/|T|$ and $f_{m'} \approx \sigma(\rho)/|R|$ for m and m' selected in this manner. Note also that Condition (II) does not hold in general because, in its interpretation, we do not guarantee that m' proposes to w (which is the case for Inequality (6)).

First, we prove several useful lemmas. Note that the claims similar to these lemmas are already proven in [22]. Lemma 3 is immediate from the definition since $\sigma(\pi) = \sigma(p) = 0$ from Condition (I). Lemma 4 also holds under Condition (I), but Lemmas 5–7 hold without any condition. Among these lemmas, Lemma 7 plays a key role in the analysis for R1T because it uses the restriction that each woman can contain a tie at the end of her preference list.

▶ **Lemma 3.** *(Under Condition (I))* $\sigma(\rho) + \sigma(\overline{\rho}) + \sigma(\tau) + \sigma(\overline{\tau}) = \sigma(r) + \sigma(\overline{r}) + \sigma(t)$.

▶ **Lemma 4.** *(Under Condition (I))* $\sigma(r) = \sigma(\overline{\rho}, r) + \sigma(\overline{\tau}, r)$.

Proof. By definition and Condition (I), we have $\sigma(r) = \sigma(\rho, r) + \sigma(\overline{\rho}, r) + \sigma(\tau, r) + \sigma(\overline{\tau}, r)$. To prove this lemma, we show that $\sigma(\rho, r) + \sigma(\tau, r) = 0$. If this does not hold, then there is a pair $(i, j) \in (\rho \cup \tau) \cap r$. Such (i, j) satisfies $M(j) \ne i$ by the definitions of ρ, τ, and r. Also, $i =_j M(j)$ holds because $(i, j) \in r$ means that $i \succeq_j M(j)$ and $(i, j) \in \rho \cup \tau$ means that i must have proposed to j and woman j rejected the proposal, which implies $M(j) \succeq_j i$. However, $(M(j), j) \in R$ means that $M(j)$ is not included in j's tie; a contradiction. ◀

▶ **Lemma 5.** *For any* $(m,w) \in M$,

$$\sum_{j \succ_m w, j \in S} x^*_{m,j} = 0 \quad \textit{and} \quad \sum_{i \succ_w m, i \in S} x^*_{i,w} = 0.$$

Proof. For the left equation, suppose that there is a woman j such that $j \in S$ and $j \succ_m w$. Then j must include m in her list. Hence, (m,j) blocks M, which contradicts the stability of M. Therefore, no such j exists; hence, $\sum_{j \succ_m w, j \in S} x^*_{m,j}$ actually sums up an empty set of variables. The right equation can also be validated in a similar way. ◀

▶ **Lemma 6.** $\sigma(\rho) + \sigma(r) \geq |R|$.

Proof. For each $(m,w) \in R$, we have

$$\sum_{j \succ_m w, j \notin S} x^*_{m,j} + \sum_{i \succeq_w m, i \notin S} x^*_{i,w} = \sum_{j \succ_m w} x^*_{m,j} + \sum_{i \succeq_w m} x^*_{i,w} \geq 1.$$

The equality comes from Lemma 5 and the fact that m is not in a tie of w's list because $(m,w) \in R$. The inequality comes from Constraint (3) of LP formulation. By adding this inequality for all $(m,w) \in R$, we have $\sigma(\rho) + \sigma(r) \geq |R|$. ◀

▶ **Lemma 7.** *For any* $(m,w) \in T$,

$$\sum_{i \in S} x^*_{i,w} = 0.$$

Proof. Since $(m,w) \in T$, there is no man i such that $m \succ_w i$ because m is included in w's tie, which is located at the end of the list. If $i \in S$ and $i \succ_w m$, then $x^*_{i,w} = 0$ by Lemma 5. We show that there is no i such that $i \in S$ and $i =_w m$. Suppose on the contrary that $i (\in S)$ and m are tied in w's list. Since i is single in M, i must have proposed to w with $f_i > 1$. By the definition of T, $f_m \leq 1$ when m proposed to w. Therefore, it is impossible that m, rather than i, is matched with w in M; a contradiction. This completes the proof. ◀

Now we are ready to give the proof of Lemma 1. Recall that $P = \emptyset$ by Condition (I); hence, $x^*(P) = 0$. Our goal here is $x^*(R) + x^*(T) \leq \frac{5}{2}(|R| + |T|)$. If $\sigma(\rho) > |R||T|/(|R|+|T|)$, then we have

$$\begin{aligned}
x^*(R) + x^*(T) &= \sum_{(m,w) \in R} \left(2\sum_j x^*_{m,j} + 2\sum_i x^*_{i,w} - \sum_{j \notin S} x^*_{m,j} - \sum_{i \notin S} x^*_{i,w} \right) \\
&\quad + \sum_{(m,w) \in T} \left(2\sum_j x^*_{m,j} + 2\sum_i x^*_{i,w} - \sum_{j \notin S} x^*_{m,j} - \sum_{i \notin S} x^*_{i,w} \right) \\
&\leq 4|R| + 4|T| - \sigma(\rho) - \sigma(\overline{\rho}) - \sigma(\tau) - \sigma(\overline{\tau}) - \sigma(r) - \sigma(\overline{r}) - \sigma(t) \\
&= 4|R| + 4|T| - 2\sigma(\rho) - 2\sigma(\overline{\rho}) - 2\sigma(\tau) - 2\sigma(\overline{\tau}) && \text{(by Lemma 3)} \\
&\leq 4|R| + 4|T| - 2\sigma(\rho) - 2\sigma(\overline{\rho}, r) - 2\sigma(\tau) - 2\sigma(\overline{\tau}, r) \\
&= 4|R| + 4|T| - 2\sigma(\rho) - 2\sigma(\tau) - 2\sigma(r) && \text{(by Lemma 4)} \\
&\leq 2|R| + 4|T| - 2\sigma(\tau) && \text{(by Lemma 6)} \\
&\leq 2|R| + 4|T| - 2\frac{|T|}{|R|}\sigma(\rho) && \text{(by Condition (II))} \\
&< 2|R| + 4|T| - \frac{2|T|^2}{|R| + |T|} && \text{(since } \sigma(\rho) > |R||T|/(|R|+|T|)) \\
&\leq \frac{5}{2}(|R| + |T|).
\end{aligned}$$

Otherwise (i.e., $\sigma(\rho) \le |R||T|/(|R|+|T|)$), then we have

$$\begin{aligned}
x^*(R)+x^*(T) &= \sum_{(m,w)\in R}\left(2\sum_j x^*_{m,j} + 2\sum_i x^*_{i,w} - \sum_{j\notin S} x^*_{m,j} - \sum_{i\notin S} x^*_{i,w}\right) \\
&\quad + \sum_{(m,w)\in T}\left(2\sum_j x^*_{m,j} + \sum_{i\notin S} x^*_{i,w} - \sum_{j\notin S} x^*_{m,j}\right) && \text{(by Lemma 7)} \\
&\le 4|R| - \sigma(\rho) - \sigma(\overline{\rho}) - \sigma(r) - \sigma(\overline{r}) + 2|T| + \sigma(t) - \sigma(\tau) - \sigma(\overline{\tau}) \\
&= 4|R| + 2|T| - 2\sigma(r) - 2\sigma(\overline{r}) && \text{(by Lemma 3)} \\
&\le 4|R| + 2|T| - 2(|R| - \sigma(\rho)) && \text{(by Lemma 6)} \\
&\le 2|R| + 2|T| + \frac{2|R||T|}{|R|+|T|} && \text{(since } \sigma(\rho) \le |R||T|/(|R|+|T|)) \\
&\le \frac{5}{2}(|R|+|T|).
\end{aligned}$$

◀

3.2.3 Full Proof of Lemma 1

In this section, we give a full proof of Lemma 1 (without Conditions (I) and (II)). Recall that in Sec. 3.2.2, we defined nine symbols such as π and p. For the full proof, we need three more symbols: For each $(m,w) \in T$, let

$\tau_m = \{(m,j) \in A \mid j \succeq_m l(m), j \notin S\}$,
$\overline{\tau_m} = \{(m,j) \in A \mid l(m) \succ_m j, j \notin S\}$, and
$t_w = \{(i,w) \in A \mid i \notin S\}$.

The following Lemmas 8–10 are unconditional counterparts of Lemmas 3 and 4 in Sec. 3.2.2.

▶ **Lemma 8.** *(i) For any w, $\sigma(t_w) = \sigma(\pi,t_w) + \sigma(\rho,t_w) + \sigma(\overline{\rho},t_w) + \sigma(\tau,t_w) + \sigma(\overline{\tau},t_w)$ and (ii) $\sigma(t) = \sigma(\pi,t) + \sigma(\rho,t) + \sigma(\overline{\rho},t) + \sigma(\tau,t) + \sigma(\overline{\tau},t)$.*

Proof. By definition, $t_w = (\pi \cap t_w) \cup (\rho \cap t_w) \cup (\overline{\rho} \cap t_w) \cup (\tau \cap t_w) \cup (\overline{\tau}, t_w)$, and the five intersections in the right-hand side are mutually disjoint. This proves (i). (ii) can be proved similarly. ◀

▶ **Lemma 9.** *(i) $\sigma(\overline{r}) \ge \sigma(\pi,\overline{r}) + \sigma(\rho,\overline{r}) + \sigma(\tau,\overline{r})$. (ii) $\sigma(\overline{\rho}) \ge \sigma(\overline{\rho},r) + \sigma(\overline{\rho},t)$. (iii) $\sigma(\overline{\tau_m}) \ge \sigma(\overline{\tau_m},r)$. (iv) $\sigma(p) \ge \sigma(\pi,p) + \sigma(\rho,p) + \sigma(\tau,p)$. (v) $\sigma(\tau_m) \ge \sigma(\tau_m,t)$. (vi) $\sigma(\overline{\tau_m}) \ge \sigma(\overline{\tau_m},r) + \sigma(\overline{\tau_m},t)$.*

Proof. (i) By definition, $\sigma(\overline{r}) = \sigma(\pi,\overline{r}) + \sigma(\rho,\overline{r}) + \sigma(\overline{\rho},\overline{r}) + \sigma(\tau,\overline{r}) + \sigma(\overline{\tau},\overline{r}) \ge \sigma(\pi,\overline{r}) + \sigma(\rho,\overline{r}) + \sigma(\tau,\overline{r})$. (ii)–(vi) can be proved similarly. ◀

▶ **Lemma 10.** *(i) $\sigma(\rho) = \sigma(\rho,p) + \sigma(\rho,\overline{r}) + \sigma(\rho,t)$. (ii) $\sigma(\pi) = \sigma(\pi,p) + \sigma(\pi,\overline{r}) + \sigma(\pi,t)$. (iii) $\sigma(\tau) = \sigma(\tau,p) + \sigma(\tau,\overline{r}) + \sigma(\tau,t)$. (iv) $\sigma(r) = \sigma(\overline{\rho},r) + \sigma(\overline{\tau},r)$.*

Proof. (i) By definition, $\sigma(\rho) = \sigma(\rho,p) + \sigma(\rho,r) + \sigma(\rho,\overline{r}) + \sigma(\rho,t)$. Now we show $\sigma(\rho,r) = 0$. If this does not hold, then there is a pair $(i,j) \in \rho \cap r$. Pair $(i,j) \in \rho$ implies that $(i,M(i)) \in R$ and $j \succ_i M(i)$, and $(i,j) \in r$ implies that $(M(j),j) \in R$ and $i \succeq_j M(j)$. Since $(M(j),j) \in R$, i and $M(j)$ are not tied in j's preference list; hence, $i \succ_j M(j)$. Since $j \succ_i M(i)$, i proposed to j during the algorithm. Hence, j must be matched with i or more a preferable man, which contradicts $i \succ_j M(j)$. Therefore, no such (i,j) exists and $\sigma(\rho,r) = 0$.

(ii) By definition, $\sigma(\pi) = \sigma(\pi, p) + \sigma(\pi, r) + \sigma(\pi, \overline{r}) + \sigma(\pi, t)$. We show that $\sigma(\pi, r) = 0$. If not, there are a man i and a woman j such that $(i, M(i)) \in P$, $(M(j), j) \in R$, and $i \succeq_j M(j)$. It is impossible that $i =_j M(j)$ because $i \neq M(j)$ since $(i, M(i)) \in P$ and $(M(j), j) \in R$, and i and $M(j)$ are not tied in j's preference list by the definition of R. Therefore $i \succ_j M(j)$. Also, by the definition of P, $f_i > 1$ at the end of the algorithm; hence, i must have proposed to all of the women in his list, especially, to j. Using a similar argument to the proof of part (i), we have a contradiction, implying that $\sigma(\pi, r) = 0$.

(iii) By definition, $\sigma(\tau) = \sigma(\tau, p) + \sigma(\tau, r) + \sigma(\tau, \overline{r}) + \sigma(\tau, t)$. By noting that m proposes to all the women from the top of the list to $l(m)$, we can show that $\sigma(\tau, r) = 0$ using a similar argument as the proofs of parts (i) and (ii).

(iv) By definition, $\sigma(r) = \sigma(\pi, r) + \sigma(\rho, r) + \sigma(\overline{\rho}, r) + \sigma(\tau, r) + \sigma(\overline{\tau}, r)$. We already proved that $\sigma(\pi, r) = 0$, $\sigma(\rho, r) = 0$, and $\sigma(\tau, r) = 0$. ◀

The following Lemma 11 is an elaborate version of Lemma 7.

▶ **Lemma 11.** *(i) For any $(m, w) \in P$, $\sum_{j \in S} x^*_{m,j} = 0$. (ii) For any $(m, w) \in T$, $\sum_{i \in S} x^*_{i,w} = 0$.*

Proof. (i) $(m, w) \in P$ implies that m has proposed to all the women in his list; hence, they are matched in M. Therefore, there is no (m, j) such that $(m, w) \in P$ and $j \in S$.

(ii) Since $(m, w) \in T$, there is no man i such that $m \succ_w i$ because w includes m in her tie at the end of her preference list. If $i \in S$ and $i \succ_w m$, then $x^*_{i,w} = 0$ by Lemma 5. We show that there is no i such that $i \in S$ and $i =_w m$. Suppose on the contrary that $i (\in S)$ and m are tied in w's list. Since i is single in M, i must have proposed to w with $f_i > 1$. By the definition of T, $f_m \leq 1$ when m proposed to w. Therefore, it is impossible that m, rather than i, is matched with w in M; a contradiction. This completes the proof. ◀

In the subsequent three lemmas, we bound $x^*(P)$, $x^*(R)$, and $x^*(T)$, which will lead to the proof of Lemma 15.

▶ **Lemma 12.** $x^*(P) \leq 2|P| + \sigma(\pi, \overline{r}) + \sigma(\pi, t) - \sigma(\rho, p) - \sigma(\tau, p)$.

Proof. We have

$$
\begin{aligned}
x^*(P) &= \sum_{(m,w)\in P} \left(\sum_j x^*_{m,j} + \sum_i x^*_{i,w} + \sum_{j \in S} x^*_{m,j} + \sum_{i \in S} x^*_{i,w} \right) \\
&= \sum_{(m,w)\in P} \left(\sum_{j \notin S} x^*_{m,j} + 2 \sum_i x^*_{i,w} - \sum_{i \notin S} x^*_{i,w} \right) \\
&\leq \sigma(\pi) + 2|P| - \sigma(p) \\
&\leq \sigma(\pi) + 2|P| - \sigma(\pi, p) - \sigma(\rho, p) - \sigma(\tau, p) \\
&= 2|P| + \sigma(\pi, \overline{r}) + \sigma(\pi, t) - \sigma(\rho, p) - \sigma(\tau, p).
\end{aligned}
$$

The first equality is the definition of $x^*(P)$, and the second equality is from Lemma 11(i). The first inequality is from the definitions of $\sigma(\pi)$ and $\sigma(p)$, and the fact that $\sum_{(m,w)\in P} \sum_i x^*_{i,w} \leq |P|$ by Constraint (1) of LP formulation. The second inequality is from Lemma 9(iv). The last equality is from Lemma 10(ii). ◀

▶ **Lemma 13.** $x^*(R) \leq 2|R| + \sigma(\rho) - \sigma(\overline{\rho}, t) + \sigma(\overline{\tau}, r) - \sigma(\overline{r})$.

Proof. We have

$$\begin{aligned}
x^*(R) &= \sum_{(m,w)\in R}\left(\sum_{j} x^*_{m,j} + \sum_{i} x^*_{i,w} + \sum_{j\in S} x^*_{m,j} + \sum_{i\in S} x^*_{i,w}\right)\\
&= \sum_{(m,w)\in R}\left(2\sum_{j} x^*_{m,j} + 2\sum_{i} x^*_{i,w} - \sum_{j\notin S} x^*_{m,j} - \sum_{i\notin S} x^*_{i,w}\right)\\
&\leq 2|R| + 2|R| - (\sigma(\rho)+\sigma(\overline{\rho})) - (\sigma(r)+\sigma(\overline{r}))\\
&\leq 2|R| + \sigma(\rho) - \sigma(\overline{\rho}, r) - \sigma(\overline{\rho}, t) + \sigma(r) - \sigma(\overline{r})\\
&= 2|R| + \sigma(\rho) - \sigma(\overline{\rho}, t) + \sigma(\overline{\tau}, r) - \sigma(\overline{r}).
\end{aligned}$$

The first equality is the definition of $x^*(R)$. The first inequality is from Constraints (1) and (2) of LP formulation and definitions of $\sigma(\rho)$, $\sigma(\overline{\rho})$, $\sigma(r)$, and $\sigma(\overline{r})$. The last inequality is from Lemmas 9(ii) and 6. The last equality is from Lemma 10(iv). ◀

▶ **Lemma 14.**

$$x^*(T) \leq 2|T| - \sigma(\overline{\tau}, r) - \sigma(t) + \sum_{(m,w)\in T} \min\{2 - \sigma(\tau_m), 2\sigma(t_w) - (\sigma(\tau_m, t) + \sigma(\overline{\tau_m}, t))\}.$$

Proof. By definition,

$$x^*(T) = \sum_{(m,w)\in T}\left(\sum_{j} x^*_{m,j} + \sum_{i} x^*_{i,w} + \sum_{j\in S} x^*_{m,j} + \sum_{i\in S} x^*_{i,w}\right).$$

We will bound the quantity inside the parentheses in two ways. First, for each $(m, w) \in T$, we have

$$\begin{aligned}
&\sum_{j} x^*_{m,j} + \sum_{i} x^*_{i,w} + \sum_{j\in S} x^*_{m,j} + \sum_{i\in S} x^*_{i,w}\\
&\quad\leq 1 + 1 + (1 - \sigma(\tau_m) - \sigma(\overline{\tau_m})) + (1 - \sigma(t_w))\\
&\quad\leq 4 - (\sigma(\tau_m) + \sigma(\overline{\tau_m}, r)) - \sigma(t_w).
\end{aligned} \tag{7}$$

The first inequality is from Constraints (1) and (2) of LP formulation and definitions of $\sigma(\tau_m)$, $\sigma(\overline{\tau_m})$, and $\sigma(t_w)$. The last inequality is from Lemma 9(iii).

Next, for each $(m, w) \in T$, we have

$$\begin{aligned}
&\sum_{j} x^*_{m,j} + \sum_{i} x^*_{i,w} + \sum_{j\in S} x^*_{m,j} + \sum_{i\in S} x^*_{i,w}\\
&\quad= 2\sum_{j} x^*_{m,j} + \sum_{i\notin S} x^*_{i,w} - \sum_{j\notin S} x^*_{m,j}\\
&\quad\leq 2 + \sigma(t_w) - (\sigma(\tau_m) + \sigma(\overline{\tau_m}))\\
&\quad\leq 2 + \sigma(t_w) - (\sigma(\tau_m, t) + \sigma(\overline{\tau_m}, r) + \sigma(\overline{\tau_m}, t)).
\end{aligned} \tag{8}$$

The equality comes from Lemma 11(ii) and the last inequality is from (v) and (vi) of Lemma 9.

Combining Inequalities (7) and (8), we have

$$\begin{aligned}
x^*(T) &= \sum_{(m,w)\in T}\left(\sum_j x^*_{m,j} + \sum_i x^*_{i,w} + \sum_{j\in S} x^*_{m,j} + \sum_{i\in S} x^*_{i,w}\right)\\
&\le \sum_{(m,w)\in T} \min\{4-(\sigma(\tau_m)+\sigma(\overline{\tau_m},r))-\sigma(t_w),\\
&\qquad\qquad 2+\sigma(t_w)-(\sigma(\tau_m,t)+\sigma(\overline{\tau_m},r)+\sigma(\overline{\tau_m},t))\}\\
&= 2|T|-\sigma(\overline{\tau},r)-\sigma(t)\\
&\quad + \sum_{(m,w)\in T}\min\{2-\sigma(\tau_m), 2\sigma(t_w)-(\sigma(\tau_m,t)+\sigma(\overline{\tau_m},t))\}.
\end{aligned}$$

◀

To simplify notation, let $\delta_{m,w} = 2(\sigma(\tau,t_w)+\sigma(\overline{\tau},t_w)-\sigma(\tau_m,t)-\sigma(\overline{\tau_m},t))$ for each $(m,w)\in T$.

▶ **Lemma 15.** $x^*(P)+x^*(R)+x^*(T) \le$

$$2|P|+2|R|+2|T|+\sum_{(m,w)\in T}\min\{2-2\sigma(\tau_m), 2\sigma(\pi,t_w)+2\sigma(\rho,t_w)+\delta_{m,w}\}.$$

Proof. Starting from Lemmas 12, 13, and 14, we have the sequence of deformations of the formula. To help following the deformations, we give underlines to the terms that are used for the deformation.

$$\begin{aligned}
&x^*(P)+x^*(R)+x^*(T)\\
&\le 2|P|+2|R|+2|T|+(\sigma(\pi,\overline{r})+\sigma(\pi,t)-\sigma(\rho,p)-\sigma(\tau,p))\\
&\quad +(\sigma(\rho)-\sigma(\overline{\rho},t)\underline{+\sigma(\overline{\tau},r)}-\sigma(\overline{r}))\\
&\quad \underline{-\sigma(\overline{\tau},r)}-\sigma(t)+\sum_{(m,w)\in T}\min\{2-\sigma(\tau_m),2\sigma(t_w)-(\sigma(\tau_m,t)+\sigma(\overline{\tau_m},t))\}\\
&= 2|P|+2|R|+2|T|+(\sigma(\pi,\overline{r})+\sigma(\pi,t)-\sigma(\rho,p)-\sigma(\tau,p))\\
&\quad +(\sigma(\rho)\underline{-\sigma(\overline{\rho},t)}-\sigma(\overline{r}))\\
&\quad -\sigma(t)+\sum_{(m,w)\in T}\min\{2-\sigma(\tau_m),2\sigma(t_w)-(\sigma(\tau_m,t)+\sigma(\overline{\tau_m},t))\}\\
&\le 2|P|+2|R|+2|T|+(\underline{\sigma(\pi,\overline{r})+\sigma(\pi,t)}-\sigma(\rho,p)-\sigma(\tau,p))+(\sigma(\rho)\underline{-\sigma(\overline{r})})\\
&\quad \underline{-\sigma(t)}+\sum_{(m,w)\in T}\min\{2-\sigma(\tau_m),2\sigma(t_w)-(\sigma(\overline{\rho},t_w)+\sigma(\tau_m,t)+\sigma(\overline{\tau_m},t))\}\\
&\qquad\qquad\qquad\qquad\qquad\qquad \text{(Use Lemmas 8(ii) and 9(i).)}\\
&\le 2|P|+2|R|+2|T|-\underline{\sigma(\rho,p)}-\sigma(\tau,p)+(\underline{\sigma(\rho)-\sigma(\rho,\overline{r})}-\sigma(\tau,\overline{r}))\\
&\quad -\sigma(\overline{\rho},t)\underline{-\sigma(\rho,t)}-\sigma(\overline{\tau},t)-\sigma(\tau,t)\\
&\quad +\sum_{(m,w)\in T}\min\{2-\sigma(\tau_m),2\sigma(t_w)-(\sigma(\overline{\rho},t_w)+\sigma(\tau_m,t)+\sigma(\overline{\tau_m},t))\}\\
&\qquad\qquad\qquad\qquad\qquad\qquad \text{(Use Lemma 10(i).)}
\end{aligned}$$

$$
\begin{aligned}
&= 2|P| + 2|R| + 2|T| \underline{-\sigma(\tau,p) - \sigma(\tau,\overline{r}) - \sigma(\overline{\rho},t) - \sigma(\overline{\tau},t) - \sigma(\tau,t)} \\
&\quad + \sum_{(m,w)\in T} \min\{2-\sigma(\tau_m), 2\sigma(t_w) - (\sigma(\overline{\rho},t_w) + \sigma(\tau_m,t) + \sigma(\overline{\tau_m},t))\} \\
&\qquad\qquad\qquad\qquad\qquad\qquad \text{(Use Lemma 10(iii).)} \\
&\le 2|P| + 2|R| + 2|T| \\
&\quad + \sum_{(m,w)\in T} \min\{2-2\sigma(\tau_m), \underline{2\sigma(t_w)} - (\underline{2\sigma(\overline{\rho},t_w)} + 2\sigma(\tau_m,t) + 2\sigma(\overline{\tau_m},t))\} \\
&\qquad\qquad\qquad\qquad\qquad\qquad \text{(Use Lemma 8(i).)} \\
&= 2|P| + 2|R| + 2|T| + \sum_{(m,w)\in T} \min\{2-2\sigma(\tau_m), \\
&\qquad 2\sigma(\pi,t_w) + 2\sigma(\rho,t_w) + 2(\sigma(\tau,t_w) + \sigma(\overline{\tau},t_w) - \sigma(\tau_m,t) - \sigma(\overline{\tau_m},t))\}.
\end{aligned}
$$

For the last inequality, we also used the inequality $\min\{a,b\} - c \le \min\{a-x, b-y\}$, where $x \le c$ and $y \le c$. ◀

▶ **Lemma 16.**

$$\sum_{(m,w)\in T} \min\{2-2\sigma(\tau_m), 2\sigma(\pi,t_w) + 2\sigma(\rho,t_w) + \delta_{m,w}\} \le \frac{1}{2}(|P| + |R| + |T|).$$

Proof. For each pair $(m,w) \in T$, let $P(w) = \{i \mid (i, M(i)) \in P, i =_w m\}$, $R(w) = \{i \mid (i, M(i)) \in R, i =_w m, w \succ_i M(i)\}$, and $PR(w) = P(w) \cup R(w)$. Then it is not difficult to see that

$$
\begin{aligned}
\sigma(\pi,t_w) + \sigma(\rho,t_w) &= \sum_{(i,M(i))\in P} x^*_{i,w} + \sum_{(i,M(i))\in R, w\succ_i M(i)} x^*_{i,w} \\
&= \sum_{(i,M(i))\in P, i=_w m} x^*_{i,w} + \sum_{(i,M(i))\in R, w\succ_i M(i), i=_w m} x^*_{i,w} \\
&= \sum_{i\in PR(w)} x^*_{i,w}. \qquad (9)
\end{aligned}
$$

The first equality is from the definitions of $\sigma(\pi, t_w)$ and $\sigma(\rho, t_w)$ for $(m,w) \in T$. For the second equality, first note that there is no man i such that $m \succ_w i$ because $(m,w) \in T$. Also, note that any i considered in the summation has proposed to w during the execution of the algorithm; hence, there is no i such that $i \succ_w m$ and $x^*_{i,w} > 0$. Therefore, considering only i such that $i =_w m$ suffices. The last equality is from the definition of $PR(w)$.

For w such that $(m,w) \in T$ and a man $i \in PR(w)$, we define $\pi_i = \{(i,j) \in A\}$ if $i \in P(w)$, $\rho_i = \{(i,j) \in A \mid j \succ_i M(i)\}$ if $i \in R(w)$, and $\pi\rho_i = \pi_i \cup \rho_i$ for $i \in PR(w)$. Then, define $\nu_{i,w} = x^*_{i,w}/\sigma(\pi\rho_i)$ and

$$\nu_w = \sum_{i\in PR(w)} \nu_{i,w}.$$

Now, for each $(m,w) \in T$, we have

$$\sigma(\tau_m) = \sum_{j\succeq_m l(m)} x^*_{m,j} \ge \max_{i\in PR(w)} \sigma(\pi,\rho_i) \ge \sum_{i\in PR(w)} \frac{\nu_{i,w}}{\nu_w}\sigma(\pi,\rho_i) = \frac{1}{\nu_w}\sum_{i\in PR(w)} x^*_{i,w}. \quad (10)$$

The first equality is due to the definition of τ_m and the fact that m has proposed to any woman j such that $j \succeq_m l(m)$. For the first inequality, we used Inequality (6). More specifically, we used the fact that each man $i \in PR(w)$ must have proposed to w with the f-value of at least

$$\sigma(\pi, \rho_i) = \sum_{(i,j)\in\pi\rho_i} x^*_{i,j},$$

but m, who proposed to w with the f-value

$$\sum_{j \succeq_m l(m)} x^*_{m,j},$$

is eventually matched to w. For the second inequality, we used the fact that a (weighted) average of $\sigma(\pi, \rho_i)$ over $i \in PR(w)$ is no more than the maximum over $i \in PR(w)$.

For $(m, w) \in T$ such that

$$\sum_{i\in PR(w)} x^*_{i,w} \le \frac{\nu_w}{2(1+\nu_w)}(2 - \delta_{m,w}),$$

we have

$$2\sigma(\pi, t_w) + 2\sigma(\rho, t_w) + \delta_{m,w} = 2\sum_{i\in PR(w)} x^*_{i,w} + \delta_{m,w} \le \frac{2\nu_w + \delta_{m,w}}{1+\nu_w} \le \frac{1+\nu_w}{2} + \delta_{m,w}. \quad (11)$$

The first equality comes from Equation (9). For the last inequality, we used $\frac{\delta_{m,w}}{1+\nu_w} \le \delta_{m,w}$, and the fact that $2x/(1+x) \le (1+x)/2$ holds for any $x \ge 0$.

For $(m, w) \in T$ such that

$$\sum_{i\in PR(w)} x^*_{i,w} \ge \frac{\nu_w}{2(1+\nu_w)}(2 - \delta_{m,w}),$$

we have, from Inequality (10)

$$2 - 2\sigma(\tau_m) \le 2 - \frac{2}{\nu_w}\sum_{i\in PR(w)} x^*_{i,w} \le 2 - \frac{2-\delta_{m,w}}{1+\nu_w} = \frac{2\nu_w + \delta_{m,w}}{1+\nu_w} \le \frac{1+\nu_w}{2} + \delta_{m,w}. \quad (12)$$

Therefore, we have

$$\begin{aligned}
&\sum_{(m,w)\in T} \min\{2 - 2\sigma(\tau_m), 2\sigma(\pi, t_w) + 2\sigma(\rho, t_w) + \delta_{m,w}\} \\
&\le \sum_{(m,w)\in T} \left(\frac{1+\nu_w}{2} + \delta_{m,w}\right) \\
&= \sum_{(m,w)\in T} \frac{1+\nu_w}{2} + 2\sum_{(m,w)\in T} (\sigma(\tau, t_w) + \sigma(\overline{\tau}, t_w) - \sigma(\tau_m, t) - \sigma(\overline{\tau_m}, t)) \\
&= \sum_{(m,w)\in T} \frac{1+\nu_w}{2} \\
&= \frac{1}{2}\sum_{(m,w)\in T}\left(1 + \sum_{i\in PR(w)} \nu_{i,w}\right) \\
&= \frac{|T|}{2} + \frac{1}{2}\sum_{(i,M(i))\in P\cup R}\ \sum_{(m,w)\in T} \nu_{i,w} \\
&\le \frac{|P| + |R| + |T|}{2}.
\end{aligned}$$

The first inequality comes from Inequalities (11) and (12). For the second equality, note that

$$\sum_{(m,w)\in T} (\sigma(\tau, t_w) + \sigma(\overline{\tau}, t_w)) = \sigma(\tau, t) + \sigma(\overline{\tau}, t) = \sum_{(m,w)\in T} (\sigma(\tau_m, t) + \sigma(\overline{\tau_m}, t))$$

by the definitions of $\sigma(\tau, t_w)$, $\sigma(\overline{\tau}, t_w)$, $\sigma(\tau_m, t)$, and $\sigma(\overline{\tau_m}, t)$. For the last equality, we exchanged the order of summation. The last inequality is due to the fact that for each $(i, M(i)) \in P \cup R$,

$$\sum_{(m,w)\in T} \nu_{i,w} \leq 1.$$

◀

Combining Lemmas 15 and 16, we can easily obtain Lemma 1. ◀

4 Lower Bounds

In this section, we show several results related to the inapproximability of the MAX SMTI. We first show three lower bounds on the integrality gap of the IP formulation given in Sec. 2, though the proof of Theorem 18 is omitted due to limitations of space.

▶ **Theorem 17.** *The integrality gap of the IP formulation given in Sec. 2 is at least* 1.25 *for R1T.*

Proof. We show an R1T instance I_1 whose integrality gap is (at least) 1.25.

m_1:	w_1	w_1:	$m_2\ m_3\ m_1$
m_2:	$w_2\ w_1$	w_2:	$(m_2\ m_3)$
m_3:	$w_2\ w_1\ w_3$	w_3:	m_3

One of the largest stable matchings for I_1 is $M^* = \{(m_2, w_1), (m_3, w_2)\}$. There is a feasible fractional solution x for $LP(I_1)$ such that $x_{m_1,w_1} = x_{m_2,w_1} = x_{m_2,w_2} = x_{m_3,w_2} = x_{m_3,w_3} = 0.5$. Hence, the integrality gap is at least $(5 \times 0.5)/|M^*| = 1.25$. ◀

▶ **Theorem 18.** *The integrality gap of the IP formulation given in Sec. 2 is at least* 1.3333 *for R2T.*

▶ **Theorem 19.** *The integrality gap of the IP formulation given in Sec. 2 is at least* $1.5 - o(1)$ *for 1T.*

Proof. We show a 1T instance I_3 whose integrality gap is $1.5 - o(1)$.

m_1:	$w_1\ w_1'$	w_1:	$(m_1\ m_2\ m_3\ \cdots\ m_k)\ m_1'$
m_2:	$w_1\ w_2\ w_2'$	w_2:	$(m_2\ m_3\ \cdots\ m_k)\ m_1'$
m_3:	$w_1\ w_2\ w_3\ w_3'$	w_3:	$(m_3\ \cdots\ m_k)\ m_3'$
$\vdots$			
m_{k-1}:	$w_1\ w_2\ w_3\ \cdots\ w_{k-1}\ w_{k-1}'$	w_{k-1}:	$(m_{k-1}\ m_k)\ m_{k-1}'$
m_k:	$w_1\ w_2\ w_3\ \cdots\ w_{k-1}\ w_k\ w_k'$	w_k:	$m_k\ m_k'$
m_1':	w_1	w_1':	m_1
$\vdots$			
m_k':	w_k	w_k':	m_k

The largest stable matching M^* for this instance is $\{(m_1, w_1), (m_2, w_2), \ldots, (m_k, w_k)\}$. There is a feasible fractional solution x for $LP(I_3)$ such that $x_{m_i,w_j} = 1/k$, $x_{m_i,w'_i} = 1 - i/k$, and $x_{m'_j,w_j} = (j-1)/k$ for all of the pairs $(i,j) \in \{1, 2, \ldots, k\}^2$. Hence, the integrality gap is at least

$$LP(I_3)/|M^*| = \left(k + \sum_{j=1}^{k} \frac{j-1}{k}\right)/k = 3/2 - o(1).$$

◀

The lower bound of Theorem 19 is an improvement over the previous bound of 1.3678 [22]. This result rules out some current techniques to obtain an approximation algorithm with a factor of $1.5 - \epsilon$ for 1T.

Second, we show a relation between the general 2T case of the MAX SMTI and a special case of the minimum maximal matching problem (MMM-Bi-APM(ϵ)) with respect to inapproximability, which is formally written as Theorem 21.

▶ **Definition 20.** The MMM-Bi-APM(ϵ) (for given ϵ such that $0 < \epsilon < 1/2$) is the problem to find a minimum maximal matching on a given balanced bipartite graph $G = (U, V, E)$ ($|U| = |V|$) that contains a matching M of size at least $(1-\epsilon)|U|$ $(= (1-\epsilon)|V|)$.

▶ **Theorem 21.** *If the MMM-Bi-APM(ϵ) is* NP*-hard to approximate to within a factor of* $2 - \epsilon$*, then the MAX SMTI with two-sided ties (2T) is* NP*-hard to approximate to within a factor of* $3/2 - O(\epsilon)$.

Proof. We show that, if there is an approximation algorithm with approximation ratio $\alpha = 3/(2 + \epsilon + 2\epsilon(1-\epsilon)) = 3/2 - O(\epsilon)$ for the MAX SMTI with two-sided ties (2T), then there is a $(2-\epsilon)$-approximation algorithm for the MMM-Bi-APM(ϵ).

To show this, let $G = (U, V, E)$ such that $|U| = |V| = n$ be a balanced bipartite graph, an input of the MMM-Bi-APM(ϵ). Let $U = \{u_1, \ldots, u_n\}$ and $V = \{v_1, \ldots, v_n\}$. We construct an instance I_G of MAX SMTI as follows. Let $k = \lfloor (1+\epsilon)n/2 \rfloor$. I_G consists of $n + k$ men $u_i (1 \le i \le n)$ and $x_i (1 \le i \le k)$, and $n + k$ women $v_i (1 \le i \le n)$ and $w_i (1 \le i \le k)$. Each man u_i corresponds to a vertex u_i of U, and each woman v_i corresponds to a vertex v_i of V. Hereafter, we do not distinguish between the names of these persons and vertices. The preference lists are given in the following. For a vertex $v \in U \cup V$, $N(v)$ denotes the set of vertices incident to v, and $[N(v)]$ denotes an arbitrary ordering of vertices in $N(v)$.

u_1:	$([N(u_1)])\ w_1\ \cdots\ w_k$	v_1:	$([N(v_1)])\ x_1\ \cdots\ x_k$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
u_n:	$([N(u_n)])\ w_1\ \cdots\ w_k$	v_n:	$([N(v_n)])\ x_1\ \cdots\ x_k$
x_1:	$v_1\ \cdots\ v_n$	w_1:	$u_1\ \cdots\ u_n$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
x_k:	$v_1\ \cdots\ v_n$	w_k:	$u_1\ \cdots\ u_n$

Let M^* be a minimum maximal matching of $G = (U, V, E)$. Then $|M^*| \ge (1-\epsilon)n/2$ by the assumption that G contains a matching of size at least $(1-\epsilon)n$. Next, it is easy to see that the above MAX SMTI instance I_G has a stable matching of size $|M^*| + (2n - 2|M^*|) = 2n - |M^*|$. (If $(u_i, v_j) \in M^*$ then include the pair (u_i, v_j). If u_i is unmatched in M^*, then include $(u_i, w_{i'})$ for some i', and similarly if v_j is unmatched in M^*, include $(x_{j'}, v_j)$ for some j'.) Therefore, if we have an α-approximation algorithm for the MAX SMTI, then this algorithm produces a matching M of size at least $(2n - |M^*|)/\alpha$. Let T be the set of pairs in M between people in U and V. First, it is easy to see that T is a maximal matching of G since if G is not maximal then M contains a blocking pair. Next, note that the size of M is exactly $2n - |T|$, which implies that $2n - |T| \geq (2n - |M^*|)/\alpha$. Hence we have $|T|/|M^*| \leq 2(1-1/\alpha)n/|M^*| + 1/\alpha \leq 4(1-1/\alpha)/(1-\epsilon) + 1/\alpha$, where we use $|M^*| \geq (1-\epsilon)n/2$ for the last inequality. Therefore, if there is an algorithm with approximation ratio $\alpha = 3/(2+\epsilon+2\epsilon(1-\epsilon))$ for the MAX SMTI, then T is an approximate solution for the MMM-Bi-APM(ϵ) with an approximation ratio at most

$$\begin{aligned}
|T|/|M^*| &\leq 4(1-1/\alpha)/(1-\epsilon) + 1/\alpha \\
&= 4(1-(2+\epsilon+2\epsilon(1-\epsilon))/3)/(1-\epsilon) + (2+\epsilon+2\epsilon(1-\epsilon))/3 \\
&= \frac{4}{3} - \frac{8}{3}\epsilon + \frac{1}{3}(2+\epsilon+2\epsilon(1-\epsilon)) \\
&= 2 - \frac{5}{3}\epsilon - \frac{2}{3}\epsilon^2 \\
&< 2 - \epsilon.
\end{aligned}$$

◀

The (in)approximability of the MMM-Bi-APM(ϵ) is unknown, but we informally discuss it. It would be easy to construct a $(2-\epsilon)$-approximation algorithm for the MMM-Bi-APM(ϵ) if we had an approximation algorithm with a constant approximation ratio for *the maximum balanced independent set problem on bipartite graphs*, which asks us to find a largest independent set $U' \cup V'$ such that $U' \subseteq U$, $V' \subseteq V$, and $|U'| = |V'|$ in a given bipartite graph $G = (U, V, E)$. However, this problem is known to be NP-hard and is hard to approximate (does not allow any constant approximation algorithm) under plausible assumptions [1, 6, 7, 23]. Although these results do not immediately rule out the existence of the $(2-\epsilon)$-approximation algorithm for the MMM-Bi-APM(ϵ), they imply some difficulty of this problem.

Finally, we also show another inapproximability result for R2T, which is formally written as Theorem 22. This result can be obtained by slightly modifying the inapproximability proof for 2T given in [35].

▶ **Theorem 22.** *The MAX SMTI problem in which each person is allowed to include a tie only at the end of the preference list (R2T) is* NP*-hard to approximate with any factor smaller than* 33/29 *and is* UG*-hard to approximate with any factor smaller than* 4/3.

Proof. Yanagisawa [35] used a reduction from the minimum vertex cover problem with a perfect matching (which is UG hard to $(2-\epsilon)$-approximate) to the 2T problem. In the reduction, he used the following gadget for each edge in a perfect matching.

$$\begin{array}{llcll}
v_j^A: & v_j^b & \quad & v_i^a: & v_i^B \\
v_i^B: & (e_{ij}^c\ v_j^b)\ v_{i_1}^b\ \cdots\ v_{i_{di}}^b\ v_i^a & & v_j^b: & e_{ij}^C\ v_i^B\ v_{j_1}^B\ \cdots\ v_{j_{dj}}^B\ v_j^A \\
e_{ij}^C: & e_{ij}^c\ (v_i^b\ v_j^b) & & e_{ij}^c: & (v_j^B\ v_i^B)\ e_{ij}^C \\
v_j^B: & (e_{ij}^c\ v_i^b)\ v_{j_1}^b\ \cdots\ v_{j_{dj}}^b\ v_j^a & & v_i^b: & e_{ij}^C\ v_j^B\ v_{i_1}^B\ \cdots\ v_{i_{di}}^B\ v_i^A \\
v_i^A: & v_i^b & & v_j^a: & v_j^B
\end{array}$$

By modifying this gadget to the following one, we can satisfy the restriction that all the ties appear at the end of preference lists.

$$\begin{array}{llcll}
v_j^A: & v_j^b & \quad & v_i^a: & v_i^B \\
v_i^B: & e_{ij}^c\ v_j^b\ v_{i_1}^b\ \cdots\ v_{i_{di}}^b\ v_i^a & & v_j^b: & e_{ij}^C\ v_i^B\ v_{j_1}^B\ \cdots\ v_{j_{dj}}^B\ v_j^A \\
e_{ij}^C: & e_{ij}^c\ (v_i^b\ v_j^b) & & e_{ij}^c: & (v_j^B\ v_i^B\ e_{ij}^C) \\
v_j^B: & e_{ij}^c\ v_i^b\ v_{j_1}^b\ \cdots\ v_{j_{dj}}^b\ v_j^a & & v_i^b: & e_{ij}^C\ v_j^B\ v_{i_1}^B\ \cdots\ v_{i_{di}}^B\ v_i^A \\
v_i^A: & v_i^b & & v_j^a: & v_j^B
\end{array}$$

It is easy to see that a matching M is stable for a MAX SMTI instance obtained from the original gadget if and only if M is stable for the MAX SMTI instance obtained from the modified gadget. Hence, the inapproximability result for 2T carries over to R2T. ◀

Acknowledgements The authors would like to thank the anonymous reviewers for their valuable comments. This work was supported by JSPS KAKENHI Grant Numbers 24500013 and 25240002.

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