# Robust Sparsification for Matroid Intersection with Applications

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**Abstract**

Matroid intersection is a classical optimization problem where, given two matroids over the same ground set, the goal is to find the largest common independent set. In this paper, we show that there exists a certain "sparsifer": a subset of elements, of size $O(|S^{opt}| \cdot 1/\varepsilon)$, where $S^{opt}$ denotes the optimal solution, that is guaranteed to contain a $3/2 + \varepsilon$ approximation, while guaranteeing certain robustness properties. We call such a small subset a *Density Constrained Subset* (DCS), which is inspired by the *Edge-Degree Constrained Subgraph* (EDCS) [Bernstein and Stein, 2015], originally designed for the maximum cardinality matching problem in a graph. Our proof is constructive and hinges on a greedy decomposition of matroids, which we call the *density-based decomposition*. We show that this sparsifier has certain robustness properties that can be used in one-way communication and random-order streaming models.

Specifically, we use the DCS to design a one-way communication protocol for matroid intersection and obtain a $3/2 + \varepsilon$ approximation, using a message of size $O(|S^{opt}| \cdot 1/\varepsilon)$. This matches the best achievable ratio for the one-way communication bipartite matching [Goel, Kapralov, and Khanna, 2012].

Moreover, the DCS can be used to design a streaming algorithm in the random-order streaming model requiring the space of $O(|S^{opt}| \cdot poly(\log(n), 1/\varepsilon))$, where $n$ is the size of the stream (the ground set of the matroids). Our algorithm guarantees a $3/2+\varepsilon$ approximation *in expectation* and, when the size of $S^{opt}$ is not too small, *with high probability*. Prior to our work, the best approximation ratio of a streaming algorithm in the random-order streaming model was an expected $2 - \delta$ for some small constant $\delta > 0$ [Guruganesh and Singla, 2017].

## 1 Introduction

The matroid intersection problem is a fundamental problem in combinatorial optimization. In this problem we are given two matroids $\mathcal{M}_1 = (V, \mathcal{I}_1)$ and $\mathcal{M}_2 = (V, \mathcal{I}_2)$, and the goal is to find the largest common independent set in both matroids, *i.e.*, $\arg\max_{S \in \mathcal{I}_1 \cap \mathcal{I}_2} |S|$. This problem was introduced and solved by Edmonds [Edm70, Edm71, Edm79] in the 70s. The importance of matroid intersection stems from the large variety of combinatorial optimization problems it captures; well-known examples in computer science include bipartite matching and packing of spanning trees/arborescences.

In this paper we introduce a "sparsifer" for the matroid intersection problem and use it to design algorithms for two problems closely related to streaming: a one-way communication protocol and a streaming algorithm in the random-order streaming model.

**Structural Result for a Matroid Intersection Sparsifier** Our starting point is the maximum matching problem. To deal with massive graphs, a common tool is sparsification, *i.e.*, the extraction of a subgraph having fewer edges but preserving some desired property. Various graph sparsifiers have been introduced to maintain a large matching, *e.g.*, see [BS15, BHI18, RSW22, GP13] and the references therein. The particular sparsifier that has inspired our work is the *Edge-Degree Constrained Subgraph* (EDCS) introduced by Bernstein and Stein [BS15].

**Definition 1** (from [BS15])**.** *Let $G = (V, E)$ be a graph, and $H$ a subgraph of $G$. Given any integer parameters $\beta \geq 2$ and $\beta^- \leq \beta - 1$, we say that a subgraph $H = (V, E_H)$ is a $(\beta, \beta^-)$-*EDCS *of $G$ if $H$ satisfies the following properties (for $v \in V$, $\deg_H(v)$ denotes the degree of $v$ in $H$):*

*(i) For any edge $(u, v) \in H$,* $\quad \deg_H(u) + \deg_H(v) \leq \beta$*;*

*(ii) For any edge $(u, v) \in G \backslash H$,* $\deg_H(u) + \deg_H(v) \geq \beta^-$*.*

The size of an EDCS is easily controlled by the parameter $\beta$ as it is $O(\beta \cdot |M_G|)$, where $M_G$ is the maximum matching. The key property of EDCSes is that, by choosing some $\beta$ and $\beta^-$ in the order of $O(poly(1/\varepsilon))$, an EDCS is guaranteed to contain a $3/2 + \varepsilon$ approximation of the maximum matching [AB19, BS16]. As a result, EDCSes have been used to approximate maximum matching in the dynamic, random-order streaming, communication, and sublinear settings with success, for instance see [AB23, BK22, BKS23, BRR23, Ber20, BS15, BS16, GSSU22, Kis22].

As bipartite graph matching is a special case of matroid intersection, the special case when both matroids are partition matroids, one is naturally prompted to ask: is there an analogue of EDCS for general matroid intersection? However, even very slight generalizations of partition matroids, such as laminar matroids (*i.e.*, adding nested cardinality constraints on groups of vertices on each side of the bipartite graph), it is already unclear how to properly define the equivalent of EDCSes. In fact, to the best of our knowledge, in this setting of laminar matroids, nothing is known about getting approximation ratios comparable to those for simple matching in random streams [Ber20] or in communication complexity [AB19].

To properly generalize EDCS, the first question would be: what could be the equivalent of a vertex degree in a graph, in the context of a matroid? To answer this question, we make use of the notion of *density* of a subset in a matroid and introduce the *density-based decomposition.*[1] In the following discussion, we assume that readers are familiar with matroids. All the formal definitions can be found in Section 2.

**Definition 2.** *Let $\mathcal{M} = (V, \mathcal{I})$ be a matroid. The* density *of a subset $U \subseteq V$ in $\mathcal{M}$ is defined as*

$$\rho_{\mathcal{M}}(U) = \frac{|U|}{\mathrm{rank}_{\mathcal{M}}(U)}.$$

*By convention, the density of an empty set is 0, and the density of a non-empty set of rank 0 is $+\infty$.*

We now explain, at a high level, how densities are used. Let $V' \subseteq V$ be a subset of elements ($V'$ is meant to be our "sparsifier") and consider the matroid $\mathcal{M}'$, which is the original matroid $\mathcal{M}$ restricted to $V'$. Then we apply the following greedy procedure: find the densest set $U_1 \subseteq V'$ and then contract $\mathcal{M}'$ by $U_1$; next find the densest set $U_2 \subseteq V' \backslash U_1$ in the contracted matroid $\mathcal{M}'/U_1$ and again contract $\mathcal{M}'/U_1$ by $U_2$, and so on (for more details about this method and the contraction of a matroid, we refer the reader to Section 2). This greedy procedure induces a *density-based decomposition* of $V' = U_1 \cup \cdots \cup U_k$, where $k$ is the rank of the original matroid $\mathcal{M}$ (note that some of the last $U_i$s could be empty; to give a better intuition about this decomposition an example is provided in Figure 2). As a result, each element of $V'$ can be assigned a density based on this decomposition, namely, the density of the set $U_i$ where it appears in, computed with respect to the contracted matroid that was used for the construction of that $U_i$. Each element $v \in V \backslash V'$ can also be assigned a density, namely, the density of the elements in the first set $U_i$ such that $v$ is spanned by $U_1 \cup \cdots \cup U_i$ in the matroid $\mathcal{M}$.

Therefore using this notion of *density-based decomposition* of $V'$ in the restricted matroid $\mathcal{M}'$ we can define for every element $v \in V$ an *associated density* $\tilde{\rho}_{\mathcal{M}}(v)$ with respect to $V'$ (a formal definition is

[1] This decomposition is closely related to the notion of *principal sequence* [Fuj08]; this aspect will be discussed later.

provided in Definition 19). This associated density plays the role analogous to the vertex degree in a graph. With the associated densities of the elements, we can define a *Density-Constrained Subset* (DCS) for matroid intersection:

**Definition 3.** *Let $\mathcal{M}_1 = (V, \mathcal{I}_1)$ and $\mathcal{M}_2 = (V, \mathcal{I}_2)$ be two matroids. Let $\beta$, $\beta^-$ be two integers such that $\beta \geq \beta^- + 7$. A subset $V' \subseteq V$ is called a $(\beta, \beta^-)$-*DCS *if it satisfies the following properties:*

*(i) For any $v \in V'$, $\quad \tilde{\rho}_{\mathcal{M}_1}(v) + \tilde{\rho}_{\mathcal{M}_2}(v) \leq \beta$;*

*(ii) For any $v \in V \backslash V'$, $\tilde{\rho}_{\mathcal{M}_1}(v) + \tilde{\rho}_{\mathcal{M}_2}(v) \geq \beta^-$.*

By a constructive proof, we show that such $(\beta, \beta^-)$-DCSes always exist (Theorem 23). This proof is based on a local search argument similar to that of [AB19] but here it requires to understand how the density-based decomposition of $V'$ is affected when an element is added or removed from $V'$ — hence we need the two important "modification lemmas", namely, Lemmas 20 and 21. We also prove that DCSes are compact, in the sense that their size is up to $\beta$ times the cardinality of the optimal solution (Proposition 22). Moreover, DCSes always contain a good approximation of the optimal solution:

**Theorem 4.** *Let $\varepsilon > 0$. For integers $\beta$, $\beta^-$ such that $\beta \geq \beta^- + 7$ and $(\beta^- - 4) \cdot (1 + \varepsilon) \geq \beta$, any $(\beta, \beta^-)$-DCS $V'$ contains a $3/2 + \varepsilon$ approximation of the maximum cardinality common independent set.*

Theorem 4 can be compared to the result for EDCSes in bipartite graphs:

**Theorem 5** (from [AB19])**.** *Let $\varepsilon > 0$. For integers $\beta$, $\beta^-$ such that $\beta \geq \beta^- + 1$ and $\beta^- \geq \beta \cdot (1 - \varepsilon/4)$, any $(\beta, \beta^-)$-EDCS $H$ of a bipartite graph $G$ contains a $3/2 + \varepsilon$ approximation of the maximum matching.*

The proof of Theorem 4 is the most crucial part of our work. In the following we briefly discuss our methodology and highlight the important ideas in our proof.

Many algorithms for optimization problems are analyzed based on primal-duality of linear programs. Even though the convex hull of common independent sets can be described by a linear program [Sch03], we choose not to use its dual program. Instead we use the simpler mini-max theorem of Edmonds [Edm70].

**Theorem 6** (Matroid intersection theorem [Edm70])**.** *Given two matroids $\mathcal{M}_1 = (V, \mathcal{I}_1)$ and $\mathcal{M}_2 = (V, \mathcal{I}_2)$, the maximum size of a set in $\mathcal{I}_1 \cap \mathcal{I}_2$ is*

$$\min_{U \subseteq V} (\mathrm{rank}_{\mathcal{M}_1}(U) + \mathrm{rank}_{\mathcal{M}_2}(V \backslash U)).$$

The minimizers $U$ and $V \backslash U$ in the formula will serve as the "dual" to bound the size of the optimal solution. In particular, in our proof, we consider the two matroids $\mathcal{M}'_1$ and $\mathcal{M}'_2$ derived from original matroids $\mathcal{M}_1$ and $\mathcal{M}_2$ restricted to $V'$. Edmonds' theorem states that we can find $C_1$ and $C_2$ such that $C_1 \cup C_2 = V'$, $C_1 \cap C_2 = \emptyset$, and $\mathrm{rank}_{\mathcal{M}'_1}(C_1) + \mathrm{rank}_{\mathcal{M}'_2}(C_2)$ is equal to the size of the maximum common independent set in $V'$, denoted as $\mu(V')$. The question then boils down to compare the size of an optimal solution with $\mathrm{rank}_{\mathcal{M}'_1}(C_1) + \mathrm{rank}_{\mathcal{M}'_2}(C_2)$.

To achieve this, we will use a certain greedy procedure to choose a subset $S$ of the optimal solution so that in the contracted matroids $\mathcal{M}_1/S$ and $\mathcal{M}_2/S$, all the remaining elements of the optimal solution not in $S$ are spanned in at least one of these contracted matroids (either by $C_1$ in $\mathcal{M}_1/S$ or by $C_2$ in $\mathcal{M}_2/S$). As these elements are of size at most $\mathrm{rank}_{\mathcal{M}_1/S}(C_1) + \mathrm{rank}_{\mathcal{M}_2/S}(C_2) \leq \mathrm{rank}_{\mathcal{M}'_1}(C_1) + \mathrm{rank}_{\mathcal{M}'_2}(C_2) = \mu(V')$ (by Edmonds' theorem), we just need to bound the size of $S$. We will use a strategy to bound the size of $S$ by $(1/2 + \varepsilon) \cdot (\mathrm{rank}_{\mathcal{M}'_1}(C_1) + \mathrm{rank}_{\mathcal{M}'_2}(C_2))$. That part of the proof hinges on the construction of well-chosen subsets of $C_1$ and $C_2$ (see Lemmas 24 and 25), and on the properties of the density-based decomposition. In fact, in the case of graph matching (two partition matroids), the proof for EDCSes can be done by an edge counting argument [AB19], whereas here we need a more sophisticated proof strategy— how the density decomposition is useful and exploited is fully displayed in the proofs of Lemmas 24 and 25.

**Remark 7.** *When the two matroids are partition matroids of the same rank, the definition of associated density for an element matches the notion of degree for the endpoint of an edge: hence in that case our DCS definition corresponds to that of EDCS in a bipartite graph.*

**Application to One-Way Communication** We consider the following one-way communication problem [Kus97]: Alice is given some part $V_A$ of the common ground set $V$, while Bob holds the other part $V_B$. The goal for Alice is to send a single message to Bob so that Bob outputs an approximate maximum common independent set. If Alice sends her whole ground set $V_A$ to Bob, then the latter will be able to recover the exact solution. However in this game, we assume that communication is costly, so we would like to do as best as possible while restricting ourselves to use a message of size $O(\mu(V))$, where $\mu(V)$ denotes the size of the optimal solution. For instance, if Alice sends only a maximum intersection in $V_A$ then Bob is able to complete it to make it a maximal set (a set such as no element can be added to it without creating a circuit in one of the two matroids), and we then obtain a 2 approximation protocol. The interest in studying one-way communication problems lies in their connection with the single-pass streaming model [GKK12] and other computational models, as they, in a certain way, capture the essence of trade-offs regarding message sizes.

Our problem is a natural generalization of the one-way communication problem for matchings, which has been studied in [AB19, GKK12]: the edges of the graph are splitted by some adversary between Alice and Bob, and Alice has to send a small message to Bob so he can recover some good matching. In particular, when both matroids are partition matroids, our problem is equivalent to the one-way communication in a bipartite graph. Protocols have been provided for the one-way communication matching problem to get a 3/2 approximation, see [AB19, GKK12]. Moreover, we know that for bipartite graphs with $k$ vertices on each side, any protocol providing an approximation guarantee better than 3/2 requires a message of size at least $k^{1+\Omega(1/\log\log k)}$ [GKK12]. Therefore in our general case of matroid intersection one cannot expect to beat the 3/2 approximation ratio using a message of size $O(\mu(V))$.

Assadi and Bernstein [AB19] used the EDCS sparsifier to get the optimal 3/2 approximation ratio. In Section 4 we show that our DCS sparsifier has the same robustness property: if Alice builds some DCS and sends it to Bob, Bob will be able to get an approximate solution with a ratio close to 3/2. Proving this requires only a slight adaptation of the proof of Theorem 4.

**Theorem 8.** *There exists a one-way communication protocol that, given any $\varepsilon > 0$, computes a $3/2 + \varepsilon$ approximation to maximum matroid intersection using a message of size $O(\mu(V)/\varepsilon)$ from Alice to Bob, where $\mu(V)$ denotes the size of the optimal solution of the matroid intersection problem.*

Hence our result closes the gap between matching and matroid intersection, and matches the 3/2 bound for bipartite matching. It shows that matroid intersection and matching problems have similar one-way communication limitations, despite the more complex structure of matroids.

**Application to Random-Order Streams** The *streaming* model of computation [FKM$^+$05] has been motivated by the recent rise of massive datasets, where we cannot afford to store the entire input in memory. Given that the ground set is made of $|V| = n$ elements, in the streaming model $V$ is presented to the algorithm as a stream of elements $v_1, \ldots, v_n$. The algorithm is allowed to make a single pass over that stream and, ideally, uses a memory roughly proportional to the output size (up to a poly-logarithmic factor): therefore the main challenge in this model is that we have to discard many elements through the execution of the algorithm.

We note that, in the most general model where an adversary decides the order of the elements, it has been a long-standing open question whether the maximum matching in bipartite graphs (a very simple case of matroid intersection) can be approximated within a factor better than 2, *i.e.*, the ratio achievable by the simple greedy algorithm.

Our focus here is on the *random-order* streaming model, where the permutation of the elements of $V$ in the stream is chosen uniformly at random. This is a natural assumption as real-world data has little reason of being ordered in an adversarial way (even though the distribution may not be entirely random either). In fact, as mentioned in [KMM12], the random-order streaming model might better explain why certain algorithms perform better in practice than their theoretical bounds under an adversary model. It is noteworthy that under the random-order streaming model, for the maximum matching, quite a few recent papers have shown that the approximation factor of 2 can be beaten [KMM12, GKMS19, Kon18, FHM$^+$20, Ber20, HS22]. In addition, in the adversary model, Kapralov [Kap21] shows that to get an approximation factor better than $1+\ln 2 \approx 1.69$, one needs $k^{1+\Omega(1/\log\log k)}$ space, even in bipartite graphs (here $k$ denotes the number of vertices on each side). The paper of Bernstein [Ber20] proves that it is possible to beat this adversarial-order lower bound in the random-order model, by achieving a $3/2+\varepsilon$ approximation while using only $O(k \cdot poly(\log(k), 1/\varepsilon))$ space, thus demonstrating a separation between the adversary model and the random-order model.

For our main topic, matroid intersection, a simple greedy algorithm gives again an approximation ratio of 2. Guruganesh and Singla [GS17] have shown that it is possible to obtain the factor of $2-\delta$ in expectation, for some small $\delta > 0$.[2] We show that this factor can be significantly improved. In fact, in Section 5, we use our DCS construction in the context of random-order streams to design an algorithm. The framework developed in Section 5 is a slight modification of that of [Ber20, HS22].

**Theorem 9.** *Let $1/4 > \varepsilon > 0$. One can extract from a randomly-ordered stream of elements a common independent subset in two matroids with an approximation ratio of $3/2+\varepsilon$ in expectation, using $O(\mu(V) \cdot \log(n) \cdot \log(k) \cdot (1/\varepsilon)^3)$ memory, where $\mu(V)$ denotes the size of the optimal solution, and $k$ is the smaller rank of the two given matroids. Moreover the approximation ratio is worse than $3/2+\varepsilon$ only with probability at most $\exp(-1/32 \cdot \varepsilon^2 \cdot \mu(V)) + n^{-3}$.*

Thus, not only do we improve upon the factor $2-\delta$ [GS17], but also we demonstrate that it is possible to beat the adversarial-order lower bound of $1+\ln 2 \approx 1.69$ of [Kap21] for the matroid intersection problem as well in the random order model (assuming that $n$ is polynomial in $k$).

**Remark 10.** *When the size the the optimal solution $\mu(V)$ is $\Omega(\log(n)/\varepsilon^2)$, we obtain a good approximation ratio with high probability, as the probability of failure will be $n^{-O(1)}$ (and $n$ is assumed to be very big as we are in the streaming setting). Unlike in [Ber20, HS22], we cannot guarantee with high probability a good approximation ratio when the solution is small: in fact, when a matching is relatively small we can prove that the graph has a limited number of edges (so we can afford to store all of them), but for the matroid intersection problem, a small maximum intersection of two matroids does not imply that the ground set is small as well.*

**Density-Based Decomposition and Principal Partitions** The notion of densest subsets and density-based decompositions is closely related to the theory of *principal partitions*. The latter indeed comes from a long line of research in various domains, ranging from graphs, matrices, matroids, to submodular systems. We refer to readers to a survey of Fujishige [Fuj08]. Below we give a quick outline.

Let $V'$ be the ground set of a matroid $\mathcal{M}$. By the theory of principal partitions, there exist a sequence of nested sets, called *principal sequence*, $F_1 \subset F_2 \subset \cdots \subset F_k = V'$, and a sequence of critical values $\lambda_1 > \lambda_2 > \cdots > \lambda_k$, so that the matroid obtained by contracting $F_{i-1}$ and restricted to $F_i$, is "uniformly dense" (*i.e*, no set has a larger density than the ground set itself), with density $\lambda_i$. In our context, recall that $V'$ is decomposed into $U_1, U_2, \ldots, U_k$ by a greedy procedure. Then it can be seen that $F_1 = U_1, F_2 = U_1 \cup U_2, \ldots, F_k = U_1 \cup \cdots \cup U_k$. In this sense, our density-based decomposition can be regarded as a rewriting of the principal sequence, and some basic results stated in Section 2 are already known in the context of principal partitions. However, we adopt this term and this way of decomposing the elements to better emphasize the "greedy" nature of our approach and to facilitate our presentation.

The most important consequence of the theory of principal partitions for us is that the densest sets $U_1, \ldots, U_k$ in our greedy procedure can be computed in polynomial time by using submodular function minimization [Fuj08]. We briefly explain how it can be done. For any density $\rho$, we can find in polynomial time the largest set $U_\rho$ minimizing the submodular function $f_\rho(U) = \rho \cdot \text{rank}_{\mathcal{M}}(U) - |U|$ (*e.g.*, see [Sch03]). Hence we can find the largest density $\rho^*$ and the associated largest densest subset in polynomial time using binary search: for some value $\rho$, if $U_\rho = \emptyset$ then it means that $\rho^* < \rho$, and if $U_\rho \neq \emptyset$ it means that $\rho^* \geq \rho$. The exact value of $\rho^*$ can be found as densities can only be rational numbers with denominators bounded by the rank $k$ of the matroid. For the largest densest subset $U_{\rho^*}$, we have $f_{\rho^*}(U_{\rho^*}) = 0$, and when $\rho < \rho^*$ we have $f_\rho(U_\rho) \leq f_\rho(U_{\rho^*}) < 0$.

Although the above procedure can be costly in running time, for some simple matroids that may be of more practical importance, such as laminar or transversal matroids, it should be possible to compute the

[2]It should be emphasized that Guruganesh and Singla consider the more stringent "online" model.

density-based decomposition faster, because of their particular structures. Moreover, in our algorithms, as we frequently update the ground set on which we compute the decomposition by adding or removing one element, there may be room to improve our time complexity: we leave as an open question whether updating a density-based decomposition when performing these kinds of operations can be done more efficiently, without re-computing the whole decomposition each time.

Analysing more carefully how the density-based decomposition and the DCS could be updated efficiently may also lead to an application of DCSes to dynamic matroid intersection (note that the EDCS was originally proposed for dynamic graph matching [BS15]). In that setting, elements are added into or removed from the ground set and the objective is to maintain an approximate maximum matroid intersection, while guaranteeing a small update time.

**Related Work** Matroid intersection is an ubiquitous subject in theoretical computer science. We refer the reader to the comprehensive book of Schrijver [Sch03]. Although in the traditional offline setting we know since the 70s that the problem can be solved in polynomial time [Edm70, Edm71, Edm79], improving the running time of matroid intersection is still a very active area [BMNT23, Bli21, CLS$^+$19].

The importance of matroid intersection comes from the large variety of combinatorial optimization problems it captures, the most well-known being bipartite matching and packing of spanning trees/arborescences. Moreover, other applications can be found in electric circuit theory [Mur99, Rec89], rigidity theory [Rec89], and network coding [DFZ11]. In general, matroids generalize numerous combinatorial constraints; as a result matroid intersection can appear in very diverse contexts. For instance, a recent trend in machine learning is the "fairness" constraints (*e.g.*, see [CKLV19] and references therein), which can be encoded by partition or laminar matroids (for nested constraints). Machine scheduling constraints is another example of matroid application, in that case using transversal matroids, see [GT84, XG94].

For the one-way communication problem [Kus97], the case of maximum matching has been studied in [AB19, GKK12], for which a $3/2 + \varepsilon$ approximation is obtained. We are not aware of any previous result for the matroid intersection problem in that model. In general, one-way communication is often used to get a better understanding of streaming problems, see [FNSZ20, GKK12].

In the *adversarial* streaming, the trivial greedy algorithm building a maximal independent set (an independent set that cannot be extended) achieves a 2 approximation [CCPV07, Mes06]. Improving that approximation ratio is a major open question in the field of streaming algorithms, even for the simple case of bipartite matching (an intersection of two partition matroids). On the hardness side, we know that an approximation ratio better than $1 + \ln 2 \approx 1.69$ cannot be achieved [Kap21] (previously, an inapproximability of $1 + 1/(e-1) \approx 1.58$ had been established in [Kap13]) for the maximum bipartite matching — hence for the matroid intersection as well. Note that matroid intersection has been studied in the streaming setting under the adversarial model (in the more general case of weighted/submodular optimisation), for instance see [CGQ15, FKK18, GJS21].

In comparison with the adversarial model, for the *random-order* streaming, Guruganesh and Singla have obtained a $2 - \delta$ approximation ratio (for some small $\delta > 0$) for matroid intersection [GS17]. To our knowledge, it is the only result beating the factor of 2 for the general matroid intersection problem. In the maximum matching problem (not necessarily in bipartite graphs), a pioneering result was first obtained by Konrad, Magniez, and Mathieu [KMM12] with an approximation ratio strictly below 2 for simple matchings. The approximation ratio was later improved in a sequence of papers [GKMS19, Kon18, FHM$^+$20, Ber20]. Currently the best result for matchings is due to Assadi and Behnezhad [AB21], who obtained the ratio of $3/2 - \delta$ for some small constant $\delta \sim 10^{-14}$.

# 2 Density-Based Decomposition

Let $\mathcal{M} = (V, \mathcal{I})$ be a matroid on the ground set $V$. Recall that a pair $\mathcal{M} = (V, \mathcal{I})$ is a matroid if the following three conditions hold: (1) $\emptyset \in \mathcal{I}$, (2) if $X \subseteq Y \in \mathcal{I}$, then $X \in \mathcal{I}$, and (3) if $X, Y \in \mathcal{I}, |Y| > |X|$, there exists an element $e \in Y \backslash X$ such that $X \cup \{e\} \in \mathcal{I}$. The sets in $\mathcal{I} \subseteq \mathcal{P}(V)$ are the *independent sets*. The *rank* of a subset $X \subseteq V$ is $\text{rank}_{\mathcal{M}}(X) = \max_{Y \subseteq X,\, Y \in \mathcal{I}} |Y|$. The rank of a matroid is $\text{rank}_{\mathcal{M}}(V)$. Observe that this notion generalizes that of linear independence in vector spaces.

A subset $C \subseteq V$ is a *circuit* if $C$ is a minimal non-independent set, *i.e.*, for every $v \in C$, $C \backslash \{v\} \in \mathcal{I}$. We will assume that no element in $V$ is a circuit by itself (called "loop" in the literature) throughout the paper. The *span* of a subset $X \subseteq V$ in the matroid $\mathcal{M}$ is defined as $\text{span}_{\mathcal{M}}(X) = \{x \in V, \text{rank}_{\mathcal{M}}(X \cup \{x\}) = \text{rank}_{\mathcal{M}}(X)\}$, these elements are called spanned by $X$ in $\mathcal{M}$. For more details about matroids, we refer the reader to [Sch03].

The *restriction* and *contraction* of a matroid results in another matroid.

**Definition 11** (Restriction)**.** *Let $\mathcal{M} = (V, \mathcal{I})$ be a matroid, and let $V' \subseteq V$ be a subset. Then we define the restriction of $\mathcal{M}$ to $V'$ as $\mathcal{M}' = \mathcal{M}|V' = (V', \mathcal{I}')$ where $\mathcal{I}' = \{S \subseteq V' : S \in \mathcal{I}\}$.*

**Definition 12** (Contraction)**.** *Let $\mathcal{M} = (V, \mathcal{I})$ be a matroid, and let $U \subseteq V$. Then we define the contracted matroid $\mathcal{M}/U = (V \backslash U, \mathcal{I}_U)$ taking an arbitrary base $\mathcal{B}_U$ of $U$ and setting $\mathcal{I}_U = \{S \subseteq V \backslash U : S \cup \mathcal{B}_U \in \mathcal{I}\}$.*

It is well-known that any choice of $\mathcal{B}_U$ produces the same $\mathcal{I}_U$, as a result the definition of contraction is unambiguous. The following proposition comes directly from the definition.

**Proposition 13.** *Let $\mathcal{M} = (V, \mathcal{I})$ be a matroid and let $A \subseteq B \subseteq V$. Then we have* $\text{rank}_{\mathcal{M}/A}(B \backslash A) = \text{rank}_{\mathcal{M}}(B) - \text{rank}_{\mathcal{M}}(A)$.

Here we recall the definition of density that we will use in the following.

**Definition 2.** *Let $\mathcal{M} = (V, \mathcal{I})$ be a matroid. The* density *of a subset $U \subseteq V$ in $\mathcal{M}$ is defined as*

$$\rho_{\mathcal{M}}(U) = \frac{|U|}{\text{rank}_{\mathcal{M}}(U)}.$$

*By convention, the density of an empty set is 0, and the density of a non-empty set of rank 0 is $+\infty$.*

The following proposition, which we will use frequently, states how the density is changed after a matroid is contracted.

**Proposition 14.** *Let $\mathcal{M} = (V, \mathcal{I})$ be a matroid. If $A \subseteq B \subseteq V$ and $U \subseteq V \backslash B$ we have the following inequality:*

$$\rho_{\mathcal{M}/A}(U) \leq \rho_{\mathcal{M}/B}(U),$$

*assuming that $\rho_{\mathcal{M}/A}(U) < +\infty$.*

*Proof.* In fact, $\text{rank}_{\mathcal{M}/A}(U) \geq \text{rank}_{\mathcal{M}/B}(U)$, while the cardinality $|U|$ remains obviously the same. ☐

The notions of density and matroid contraction allow to define the *density-based decomposition* $U_1, \ldots, U_k$ of a subset $V' \subseteq V$ as follows. First, consider the matroid $\mathcal{M}'$ defined as matroid $\mathcal{M} = (V, \mathcal{I})$ restricted to the subset $V' \subseteq V$. Then select from $V'$ the set $U_1$ of largest density in $\mathcal{M}'$ (in case several sets have the same largest density, choose the one with the largest cardinality). Then again choose the set $U_2$ of largest density (again choose the one with the largest cardinality) in $\mathcal{M}'/U_1$ and so on (see a formal description in Algorithm 1). As the rank of the matroid is $k$, after at most $k$ steps in the loop the set $\bigcup_{i=1}^{k} U_i$ is equal to $V'$. Observe that some latter sets of the decomposition may be empty. Moreover, this decomposition is unique as the choice of maximum cardinality densest subset at each step is unique (see Proposition 16). We note that as we assume that no element is a circuit by itself, our construction guarantees that no set $U_i$ has infinite density.

**Algorithm 1** Algorithm for building a density-based decomposition of a set $V'$ in $\mathcal{M}' = (V', \mathcal{I}')$

1: $\forall 1 \leq i \leq k, U_i \leftarrow \emptyset$
2: **for** $j = 1 \ldots k$ **do**
3: $U_j \leftarrow$ the densest subset of largest cardinality in $\mathcal{M}'/(\bigcup_{i=1}^{j-1} U_i)$

To give some intuition about this decomposition, we provide an example for a laminar matroid, that is represented in Figure 1 and decomposed in Figure 2.

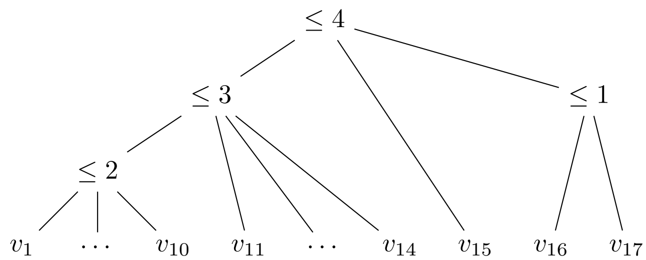


Figure 1: Representation of a laminar matroid $\mathcal{M} = (V, \mathcal{I})$ on a ground set $V = \{v_1, \ldots, v_{17}\}$. The leaves represent elements of the ground set, and the inner nodes represent cardinality constraints on the elements in their associated subtree (*e.g.*, if $S \in \mathcal{I}$, then $|S \cap \{v_1, \ldots, v_{14}\}| \leq 3$).

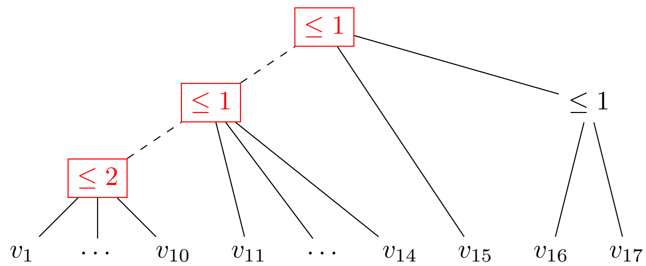


Figure 2: Density-based decomposition of the laminar matroid $\mathcal{M}$ represented in Figure 1. We have the densest subset $U_1 = \{v_1, \ldots, v_{10}\}$, then the second densest subset $U_2 = \{v_{11}, \ldots, v_{14}\}$ and finally $U_3 = \{v_{15}, v_{16}, v_{17}\}$. Their densities are respectively 5, 4, and 3. Note that here $k = 4$ so we have an additional set $U_4 = \emptyset$ of density zero.

**Proposition 15.** *Let $\mathcal{M} = (V, \mathcal{I})$ be a matroid and let $B$ be the subset that reaches the maximum density $\rho^* < +\infty$. Then given any $A \subsetneq B$, $\rho_{\mathcal{M}/A}(B \backslash A) \geq \rho^*$.*

*Proof.* If $\text{rank}_{\mathcal{M}/A}(B \backslash A) = 0$ then $\rho_{\mathcal{M}/A}(B \backslash A) = +\infty$ and we are done; otherwise, by Proposition 13:

$$\rho_{\mathcal{M}}(B) = \frac{\text{rank}_{\mathcal{M}}(A) \cdot \rho_{\mathcal{M}}(A) + \text{rank}_{\mathcal{M}/A}(B \backslash A) \cdot \rho_{\mathcal{M}/A}(B \backslash A)}{\text{rank}_{\mathcal{M}}(A) + \text{rank}_{\mathcal{M}/A}(B \backslash A)},$$

hence $\rho_{\mathcal{M}}(B)$ is a weighted average of $\rho_{\mathcal{M}}(A)$ and $\rho_{\mathcal{M}/A}(B \backslash A)$. As $\rho_{\mathcal{M}}(A) \leq \rho^*$ (by definition of $\rho^*$), it implies that $\rho_{\mathcal{M}/A}(B \backslash A) \geq \rho^*$. □

The following proposition states that the densest sets are closed under union, hence we have the uniqueness of the maximum cardinality densest subset.

**Proposition 16.** *Let $\mathcal{M} = (V, \mathcal{I})$ be a matroid. Let $\rho^* = \max_{U \subseteq V} \rho_{\mathcal{M}}(U) < +\infty$. Then given any two sets $W_1$, $W_2$ of density $\rho^*$, $\rho_{\mathcal{M}}(W_1 \cup W_2) = \rho^*$.*

*Proof.* If $W_1 \subseteq W_2$, then the proposition is trivially true. So assume that $W_1 \backslash W_2 \neq \emptyset$, and we can observe that

$$\rho^* \leq \rho_{\mathcal{M}/(W_1 \cap W_2)}(W_1 \backslash (W_1 \cap W_2)) \leq \rho_{\mathcal{M}/W_2}(W_1 \backslash (W_1 \cap W_2)),$$

where the first inequality uses Proposition 15 and the second uses Proposition 14. As a result, by the facts that $\rho_{\mathcal{M}}(W_2) = \rho^*$ and that $\rho_{\mathcal{M}/W_2}(W_1 \backslash (W_1 \cap W_2)) \geq \rho^*$, we obtain $\rho_{\mathcal{M}}(W_1 \cup W_2) \geq \rho^*$. Hence we have $\rho_{\mathcal{M}}(W_1 \cup W_2) = \rho^*$. □

Here is a first proposition about density-based decompositions, stating that the densities decrease (as we could observe in the example of Figure 2).

**Proposition 17.** *For all $1 \leq j \leq k-1$, $\rho_{\mathcal{M}'/(\bigcup_{i=1}^{j-1} U_i)}(U_j) \geq \rho_{\mathcal{M}'/(\bigcup_{i=1}^{j} U_i)}(U_{j+1})$. Moreover, if we have $\rho_{\mathcal{M}'/(\bigcup_{i=1}^{j-1} U_i)}(U_j) > 0$, then $\rho_{\mathcal{M}'/(\bigcup_{i=1}^{j-1} U_i)}(U_j) > \rho_{\mathcal{M}'/(\bigcup_{i=1}^{j} U_i)}(U_{j+1})$.*

*Proof.* We proceed by contradiction. Suppose that $\rho_{\mathcal{M}'/(\bigcup_{i=1}^{j-1} U_i)}(U_j) < \rho_{\mathcal{M}'/(\bigcup_{i=1}^{j} U_i)}(U_{j+1})$. Then it implies that $\rho_{\mathcal{M}'/(\bigcup_{i=1}^{j-1} U_i)}(U_j \cup U_{j+1}) > \rho_{\mathcal{M}'/(\bigcup_{i=1}^{j-1} U_i)}(U_j)$. Specifically, denoting $k_j = \text{rank}_{\mathcal{M}'/(\bigcup_{i=1}^{j-1} U_i)}(U_j)$ and $k_{j+1} = \text{rank}_{\mathcal{M}'/(\bigcup_{i=1}^{j} U_i)}(U_{j+1})$, we have $\rho_{\mathcal{M}'/(\bigcup_{i=1}^{j-1} U_i)}(U_j \cup U_{j+1}) = \rho_{\mathcal{M}'/(\bigcup_{i=1}^{j-1} U_i)}(U_j) \cdot \frac{k_j}{k_j+k_{j+1}} + \rho_{\mathcal{M}'/(\bigcup_{i=1}^{j} U_i)}(U_{j+1}) \cdot \frac{k_{j+1}}{k_j+k_{j+1}} > \rho_{\mathcal{M}'/(\bigcup_{i=1}^{j-1} U_i)}(U_j)$, contradicting the hypothesis that $U_j$ was the densest set in $\mathcal{M}'/(\bigcup_{i=1}^{j-1} U_i)$.

For the second part of the proposition, suppose that $\rho_{\mathcal{M}'/(\bigcup_{i=1}^{j-1} U_i)}(U_j) > 0$ and $\rho_{\mathcal{M}'/(\bigcup_{i=1}^{j-1} U_i)}(U_j) = \rho_{\mathcal{M}'/(\bigcup_{i=1}^{j} U_i)}(U_{j+1})$. Then it implies that $\rho_{\mathcal{M}'/(\bigcup_{i=1}^{j-1} U_i)}(U_j \cup U_{j+1}) = \rho_{\mathcal{M}'/(\bigcup_{i=1}^{j-1} U_i)}(U_j)$, contradicting the supposition that $U_j$ was the maximum cardinality densest set. □

The following proposition comes straightforwardly from the definition of the densities:

**Proposition 18.** *We always have $\sum_{j=1}^{k} \text{rank}_{\mathcal{M}'/(\bigcup_{i=1}^{j-1} U_i)}(U_j) \cdot \rho_{\mathcal{M}'/(\bigcup_{i=1}^{j-1} U_i)}(U_j) = |V'|$.*

Now we define the *associated density* of a given element $v \in V$ with respect to the decomposition of $V'$.

**Definition 19.** *Let $U_1, \ldots, U_k$ to the density-based decomposition of $V'$. Then, given an element $v \in V$, its* associated density *with respect to the decomposition of $V'$ is defined as*

$$\tilde{\rho}_{\mathcal{M}}(v) = \begin{cases} \rho_{\mathcal{M}'/(\bigcup_{i=1}^{j-1} U_i)}(U_j) \text{ for } j = \min\{j \in [\![1,k]\!] : v \in \text{span}_{\mathcal{M}}(\bigcup_{i=1}^{j} U_i)\} & \text{if } v \in \text{span}_{\mathcal{M}}(V') \\ 0 & \text{otherwise} \end{cases}$$

We emphasize that the associated density $\tilde{\rho}_{\mathcal{M}}$ is defined for *all* elements in $V$, not just the elements of $V'$ (this is why we use the subscript $\mathcal{M}$ instead of $\mathcal{M}'$). We also emphasize that here the associated density is dependent on $V'$, even though that dependence is not displayed in our notation: we will just write $\tilde{\rho}_{\mathcal{M}}$, instead of the more cumbersome $\tilde{\rho}_{\mathcal{M},V'}$. For elements $v \in V'$, note that if $v \in U_j$ then we have necessarily $\tilde{\rho}_{\mathcal{M}}(v) = \rho_{\mathcal{M}'/(\bigcup_{i=1}^{j-1} U_i)}(U_j)$; in fact, if $v$ is spanned by $\bigcup_{i=1}^{j_0} U_i$ for some $j_0 < j$, then we could have increased the density of $U_{j_0}$ by adding $v$ into $U_{j_0}$, contradicting the assumption that $U_{j_0}$ was the densest subset when it was selected.

We now explain how such a decomposition behaves when an element is added to or deleted from the set $V'$. These two following lemmas are crucial in the existence proof of DCSes. Their statements are quite natural (for instance, stating that adding an element does not cause a diminution of the density associated with any other element, and cannot increase the density of that new element by more than one), however their proofs are rather technical and in fact proving these lemmas is the most difficult step to show the existence of DCSes. From now on, we will use the exponents $^{\text{old}}$ and $^{\text{new}}$ to denote the states before and after the insertion/deletion operation.

The proofs of the following lemmas can be found in Appendix A.

**Lemma 20.** *Suppose a new element $u^{\text{new}} \in V \backslash V'$ is added to $V'$. Then we have the following properties:*

*(i) For all $j \in [\![1,k]\!]$, for all $v \in U_j^{\text{old}}$, $\tilde{\rho}_{\mathcal{M}}^{\text{new}}(v) \geq \rho_{\mathcal{M}'^{\text{old}}/(\bigcup_{i=1}^{j-1} U_i^{\text{old}})}(U_j^{\text{old}})$.*

*(ii) For all $v \in V$, $\tilde{\rho}_{\mathcal{M}}^{\text{new}}(v) \geq \tilde{\rho}_{\mathcal{M}}^{\text{old}}(v)$.*

*(iii) We have the inequality $\tilde{\rho}_{\mathcal{M}}^{\text{old}}(u^{\text{new}}) \leq \tilde{\rho}_{\mathcal{M}}^{\text{new}}(u^{\text{new}}) \leq \tilde{\rho}_{\mathcal{M}}^{\text{old}}(u^{\text{new}}) + 1$.*

*(iv) For all $v \in V'$ such that $\tilde{\rho}_{\mathcal{M}}^{\text{old}}(v) < \tilde{\rho}_{\mathcal{M}}^{\text{old}}(u^{\text{new}})$ or $\tilde{\rho}_{\mathcal{M}}^{\text{old}}(v) > \tilde{\rho}_{\mathcal{M}}^{\text{old}}(u^{\text{new}}) + 1$, we have the equality $\tilde{\rho}_{\mathcal{M}}^{\text{old}}(v) = \tilde{\rho}_{\mathcal{M}}^{\text{new}}(v)$.*

**Lemma 21.** *Suppose an old element $u^{\text{old}} \in V'$ is deleted from $V'$. Then we have the following properties:*

*(i) For all $j \in [\![1,k]\!]$, for all $v \in U_j^{\text{old}}$, $\tilde{\rho}_{\mathcal{M}}^{\text{new}}(v) \leq \rho_{\mathcal{M}'^{\text{old}}/(\bigcup_{i=1}^{j-1} U_i^{\text{old}})}(U_j^{\text{old}})$.*

*(ii)* *For all* $v \in V$, $\tilde{\rho}^{\text{new}}_{\mathcal{M}}(v) \leq \tilde{\rho}^{\text{old}}_{\mathcal{M}}(v)$.

*(iii)* *We have the inequality* $\tilde{\rho}^{\text{old}}_{\mathcal{M}}(u^{\text{old}}) \geq \tilde{\rho}^{\text{new}}_{\mathcal{M}}(u^{\text{old}}) \geq \tilde{\rho}^{\text{old}}_{\mathcal{M}}(u^{\text{old}}) - 1$.

*(iv)* *For all* $v \in V'$ *such that* $\tilde{\rho}^{\text{old}}_{\mathcal{M}}(v) > \tilde{\rho}^{\text{old}}_{\mathcal{M}}(u^{\text{old}})$ *or* $\tilde{\rho}^{\text{old}}_{\mathcal{M}}(v) < \tilde{\rho}^{\text{old}}_{\mathcal{M}}(u^{\text{old}}) - 1$, *we have the equality* $\tilde{\rho}^{\text{old}}_{\mathcal{M}}(v) = \tilde{\rho}^{\text{new}}_{\mathcal{M}}(v)$.

# 3 Density-Constrained Subsets for Matroid Intersection

Consider two matroids $\mathcal{M}_1 = (V, \mathcal{I}_1)$ and $\mathcal{M}_2 = (V, \mathcal{I}_2)$, both of rank $k$ (if the matroids have different ranks, we can truncate the rank of the matroid of larger rank without changing the solution of the matroid intersection problem). We recall the definition of a *Density-Constrained Subset* (DCS).

**Definition 3.** *Let* $\mathcal{M}_1 = (V, \mathcal{I}_1)$ *and* $\mathcal{M}_2 = (V, \mathcal{I}_2)$ *be two matroids. Let* $\beta$, $\beta^-$ *be two integers such that* $\beta \geq \beta^- + 7$. *A subset* $V' \subseteq V$ *is called a* $(\beta, \beta^-)$-DCS *if it satisfies the following properties:*

*(i)* *For any* $v \in V'$, $\quad \tilde{\rho}_{\mathcal{M}_1}(v) + \tilde{\rho}_{\mathcal{M}_2}(v) \leq \beta$;

*(ii)* *For any* $v \in V \backslash V'$, $\tilde{\rho}_{\mathcal{M}_1}(v) + \tilde{\rho}_{\mathcal{M}_2}(v) \geq \beta^-$.

Here is a simple bound on the size of a DCS.

**Proposition 22.** *For any set* $V' \subseteq V$ *satisfying Property (i) of Definition 3,* $|V'| \leq \beta \cdot \mu(V)$, *where* $\mu(V)$ *denotes the maximum cardinality common independent subset in* $V$.

*Proof.* We proceed by contradiction. By Theorem 6, we know that there exists a set $S \subseteq V$ such that $\text{rank}_{\mathcal{M}_1}(S) + \text{rank}_{\mathcal{M}_2}(V \backslash S) = \mu(V)$. If $|V'| > \beta \cdot \mu(V)$, then it means that either $|V' \cap S| > \beta \cdot \text{rank}_{\mathcal{M}_1}(S)$ or that $|V' \cap (V \backslash S)| > \beta \cdot \text{rank}_{\mathcal{M}_2}(V \backslash S)$. In both cases, we have a densest subset, either in $\mathcal{M}'_1$ or $\mathcal{M}'_2$, that has a density larger than $\beta$, contradicting Property (i) of Definition 3. ☐

We show the existence of $(\beta, \beta^-)$-DCSes by construction, using a local search algorithm inspired by the one used in [AB19]. In our proof we introduce a new potential function and we use Lemmas 20 and 21 to generalize their procedure; details of the proof can be found in Appendix A.

**Theorem 23.** *For any two matroids* $\mathcal{M}_1 = (V, \mathcal{I}_1)$ *and* $\mathcal{M}_2 = (V, \mathcal{I}_2)$ *of rank* $k$, *and for any integer parameters* $\beta \geq \beta^- + 7$, *a* $(\beta, \beta^-)$*-DCS can be computed using at most* $2 \cdot \beta^2 \cdot \mu(V)$ *local improvement steps.*

The main interest of DCS is that they always contain a relatively good approximation of the maximum cardinality matroid intersection.

**Theorem 4.** *Let* $\varepsilon > 0$. *For integers* $\beta$, $\beta^-$ *such that* $\beta \geq \beta^- + 7$ *and* $(\beta^- - 4) \cdot (1 + \varepsilon) \geq \beta$, *any* $(\beta, \beta^-)$*-DCS* $V'$ *contains a* $3/2 + \varepsilon$ *approximation of the maximum cardinality common independent set.*

*Proof.* Let $V'$ be a $(\beta, \beta^-)$-DCS, and let $C_1$ and $C_2$ be sets such that $C_1 \cup C_2 = V'$, $C_1 \cap C_2 = \emptyset$ and minimizing the sum $\text{rank}_{\mathcal{M}'_1}(C_1) + \text{rank}_{\mathcal{M}'_2}(C_2) = \text{rank}_{\mathcal{M}_1}(C_1) + \text{rank}_{\mathcal{M}_2}(C_2)$; by Theorem 6 we know that $\text{rank}_{\mathcal{M}'_1}(C_1) + \text{rank}_{\mathcal{M}'_2}(C_2) = \mu(V')$, the size of the maximum common independent set in $V'$.

Now consider the optimal common independent set $O$ in $V$. Our objective is to bound both $|O \backslash S|$ and $|S|$ for some well-chosen subset $S \subseteq O$ to get an upper bound of $|O|$. We will build that auxiliary set $S$ as follows, starting with $S = \emptyset$. If there exists an element $o_1 \in O$ such that $o_1 \notin \text{span}_{\mathcal{M}_1}(C_1) \cup \text{span}_{\mathcal{M}_2}(C_2)$, then we add $o_1$ into $S$ and we now consider the contracted matroids $\mathcal{M}_1/S$ and $\mathcal{M}_2/S$. We keep the same sets $C_1$ and $C_2$ and we try again to find an element $o_2 \in O \backslash S$ such that $o_2 \notin \text{span}_{\mathcal{M}_1/S}(C_1) \cup \text{span}_{\mathcal{M}_2/S}(C_2)$, and we add $o_2$ to $S$. We repeat this operation until it is no longer possible to add into $S$ any other element of $O$. The idea behind this greedy procedure to build $S$ is that, if we instead define $S$ naively as the set of elements in $O$ that are not in $\text{span}_{\mathcal{M}_1}(C_1) \cup \text{span}_{\mathcal{M}_2}(C_2)$ (which would be a simpler way to get a set $S$ satisfying inequality (1) below), then this may yield a much bigger set $S$ for which we could not get a proper bound, whereas here the greedy procedure gives us a tool to bound $|S|$ as it will allow us to prove the crucial inequality (2) later.

By the above greedy procedure, $O\backslash S$ is a common independent subset in $\mathcal{M}_1/S$ and $\mathcal{M}_2/S$ restricted to $V' \cup O\backslash S$, and $\text{span}_{\mathcal{M}_1/S}(C_1) \cup \text{span}_{\mathcal{M}_2/S}(C_2) \supseteq V' \cup O\backslash S$. We now observe that

$$\begin{aligned}
|O\backslash S| &\leq \min_{U\subseteq V'\cup O\backslash S}(\text{rank}_{\mathcal{M}_1/S}(U) + \text{rank}_{\mathcal{M}_2/S}((V' \cup O\backslash S)\backslash U)) \\
&\leq \text{rank}_{\mathcal{M}_1/S}(\text{span}_{\mathcal{M}_1/S}(C_1)) + \text{rank}_{\mathcal{M}_2/S}((V' \cup O\backslash S)\backslash(\text{span}_{\mathcal{M}_1/S}(C_1))) \\
&\leq \text{rank}_{\mathcal{M}_1/S}(\text{span}_{\mathcal{M}_1/S}(C_1)) + \text{rank}_{\mathcal{M}_2/S}(\text{span}_{\mathcal{M}_2/S}(C_2)) \\
&= \text{rank}_{\mathcal{M}_1/S}(C_1) + \text{rank}_{\mathcal{M}_2/S}(C_2) \\
&\leq \text{rank}_{\mathcal{M}_1}(C_1) + \text{rank}_{\mathcal{M}_2}(C_2) \\
&= \mu(V'),
\end{aligned}$$

where in the first inequality we use Theorem 6, in the second inequality we consider $U = \text{span}_{\mathcal{M}_1/S}(C_1)$, in the third inequality we use that $(V'\cup O\backslash S)\backslash(\text{span}_{\mathcal{M}_1/S}(C_1)) \subseteq \text{span}_{\mathcal{M}_2/S}(C_2)$ and in the last inequality we use that the rank function in a contracted matroid is always smaller than the rank function in the original matroid. Therefore we obtain

$$|O\backslash S| \leq \mu(V'). \tag{1}$$

Hence we need to upper-bound the value of $|S|$. Some carefully chosen subsets $R_{l,i}$ and $Q_{l,i}$ will allow us to get that upper-bound, and their construction is displayed in the following lemmas — it is in the proof of these lemmas that the DCS structure is fully exploited. Observing that $\beta^- \cdot |S|$ is bounded by the sum of the $\tilde{\rho}_{\mathcal{M}_l}(o_i)$ (as for each $o_i \in S$, we have $\beta^- \leq \tilde{\rho}_{\mathcal{M}_1}(o_i) + \tilde{\rho}_{\mathcal{M}_2}(o_i)$, because of Property (ii) of the DCS), we will build disjoint subsets $R_{l,i}$ of $V'$ (Lemma 24) to bound each $\tilde{\rho}_{\mathcal{M}_l}(o_j)$ with $|R_{l,j}|$ (in particular, see Lemma 24 (iv)). We will then use an auxiliary partition $Q_{l,j}$ of the union of the $R_{l,j}$s (Lemma 25) to bound the total size of the $R_{l,j}$s, using the properties of the DCS and the properties of those sets. By wrapping-up everything in the end, we will be able to get an upper bound on the size of $S$, in a way somehow similar to [AB19].

We recall that the sets $U_{l,i}$ refer to the density-based decomposition of $V'$ in the matroid $\mathcal{M}_l$.

**Lemma 24.** *For $l \in \{1, 2\}$, we can build sets $R_{l,1}, \ldots, R_{l,|S|}$ satisfying the following properties:*

*(i) the $R_{l,i}$ are disjoint;*

*(ii) for all $j \in [\![1, |S|]\!]$ we have $R_{l,j} \subseteq V'\backslash C_l$;*

*(iii) for all $j \in [\![1, |S|]\!]$, for all $v \in R_{l,j}$, $|R_{l,j}| = \lfloor\tilde{\rho}_{\mathcal{M}_l}(v)\rfloor - 1$;*

*(iv) for all $j \in [\![1, |S|]\!]$, $|R_{l,j}| \geq \lfloor\tilde{\rho}_{\mathcal{M}_l}(o_j)\rfloor - 1$.*

*Proof.* Fix an $l$. We divide $S$ into two groups: those that are spanned by $\bigcup_{i=1}^k U_{l,i}$ and those that are not. Precisely, $S_U = S \cap \text{span}_{\mathcal{M}_l}(\bigcup_{i=1}^k U_{l,i})$ and $S_{\overline{U}} = S\backslash S_U$.

We will extract from $U_{l,1}, \ldots, U_{l,k}$ subsets $R_{l,x}$ for each $o_x \in S_U$. For the other elements $o_y \in S_{\overline{U}}$, we create $R_{l,y} = \emptyset$ and associate $o_y$ with $R_{l,y}$. It is easy to verify that Properties (ii)-(iv) hold in the latter case (for Property (iv), recall that by Definition 19, $\tilde{\rho}_{\mathcal{M}}(o_y) = 0$). We next explain how to construct $R_{l,x}$ for $o_x \in S_U$.

For $j = 1$ to $k$, we split a *subset* of $U_{l,j}\backslash C_l$ into

$$r_{l,j} = \max\left(0, \text{rank}_{\mathcal{M}'_l/(\bigcup_{i=1}^{j-1} U_{l,i})}(U_{l,j}) - \text{rank}_{\mathcal{M}'_l/(\bigcup_{i=1}^{j-1} U_{l,i}\cap C_l)}(U_{l,j} \cap C_l)\right)$$

sets of size $\lfloor\rho_{\mathcal{M}'_l/(\bigcup_{i=1}^{j-1} U_{l,i})}(U_{l,j})\rfloor - 1$. It is always possible as we have, when $r_{l,j} > 0$,

$$\left\lfloor\frac{|U_{l,j}\backslash C_l|}{r_{l,j}}\right\rfloor = \left\lfloor\frac{|U_{l,j}\backslash C_l|}{\text{rank}_{\mathcal{M}'_l/(\bigcup_{i=1}^{j-1} U_{l,i})}(U_{l,j}) - \text{rank}_{\mathcal{M}'_l/(\bigcup_{i=1}^{j-1} U_{l,i}\cap C_l)}(U_{l,j} \cap C_l)}\right\rfloor$$

$$\geq \left\lfloor \frac{|U_{l,j} \backslash C_l|}{\text{rank}_{\mathcal{M}'_l/(\bigcup_{i=1}^{j-1} U_{l,i})}(U_{l,j}) - \text{rank}_{\mathcal{M}'_l/(\bigcup_{i=1}^{j-1} U_{l,i})}(U_{l,j} \cap C_l)} \right\rfloor$$

$$= \lfloor \rho_{\mathcal{M}'_l/(\bigcup_{i=1}^{j-1} U_{l,i} \cup (C_l \cap U_{l,j}))}(U_{l,j} \backslash C_l) \rfloor \qquad \text{by Proposition 13}$$

$$\geq \lfloor \rho_{\mathcal{M}'_l/(\bigcup_{i=1}^{j-1} U_{l,i})}(U_{l,j}) \rfloor. \qquad \text{by Proposition 15}$$

where in the first inequality we use that $\text{rank}_{\mathcal{M}'_l/(\bigcup_{i=1}^{j-1} U_{l,i})}(U_{l,j} \cap C_l) \leq \text{rank}_{\mathcal{M}'_l/(\bigcup_{i=1}^{j-1} U_{l,i} \cap C_l)}(U_{l,j} \cap C_l)$.

Then the $R_{l,x}$s, for $o_x \in S_U$ are decided by a greedy procedure. Let $x_1, \ldots, x_{|S_U|}$ be the indices of the elements of $S_U$, ordered so that $\tilde{\rho}_{\mathcal{M}_l}(o_{x_1}) \geq \cdots \geq \tilde{\rho}_{\mathcal{M}_l}(o_{x_{|S_u|}})$. The first $r_{l,1}$ subsets drawn from $U_{l,1} \backslash C_l$ are assigned to be $R_{l,x_1}, \ldots R_{l,x_{r_{l,1}}}$; the following $r_{l,2}$ subsets drawn from $U_{l,2} \backslash C_2$ are assigned to be $R_{l,x_{r_{l,1}+1}}, \ldots, R_{l,x_{r_{l,1}+r_{l,2}}}$, and so on.

Notice that by this procedure, properties (ii) and (iii) hold easily for $R_{l,x}$, $o_x \in S_U$. To prove property (iv), we will prove the following inequality for all $j$:

$$\sum_{i=1}^{j} r_{l,i} \geq \left| S \cap \text{span}_{\mathcal{M}_l} \left( \bigcup_{i=1}^{j} U_{l,i} \right) \right|. \tag{2}$$

To see why inequality (2) implies (iv), for $j = 1$ to $k$, let us define the set $S_j$ of elements with "density level" $j$, *i.e.*, $S_j = S \cap (\text{span}_{\mathcal{M}_l}(\bigcup_{i=1}^{j} U_{l,i}) \backslash \text{span}_{\mathcal{M}_l}(\bigcup_{i=1}^{j-1} U_{l,i}))$. If $o_{x_t} \in S_j$, by Definition 19, we need to associate $o_{x_t}$ with a set $R_{l,x_t}$ drawn from one of $U_{l,1}, \ldots, U_{l,j}$, as such a set $R_{l,x_t}$ will have a size larger than or equal to $\lfloor \rho_{\mathcal{M}'_l/(\bigcup_{i=1}^{j-1} U_{l,i})}(U_{l,j}) \rfloor - 1 = \lfloor \tilde{\rho}_{\mathcal{M}_l}(o_{x_t}) \rfloor - 1$. The majorization in (2) shows that our greedy procedure will guarantee that a large enough set is assigned to $o_{x_t}$, as the inequality (2) implies that $t \leq \sum_{i=1}^{j} r_{l,i}$, hence Property (iv) would follow.

To prove (2), we begin by observing that our greedy procedure in constructing $S$ ensures that

$$\text{rank}_{\mathcal{M}_l}(C_l \cup S) = \text{rank}_{\mathcal{M}_l}(C_l) + |S|,$$

implying that no circuit in $\mathcal{M}_l$ involves a non-empty subset of $S$ and a non-empty subset of a base in $C_l$. Therefore, given any $\hat{C}_l \subseteq C_l$ and $\hat{S} \subseteq S$,

$$\text{rank}_{\mathcal{M}_l}(\hat{C}_l \cup \hat{S}) = \text{rank}_{\mathcal{M}_l}(\hat{C}_l) + |\hat{S}|.$$

With this observation, we can derive

$$\begin{aligned}
\text{rank}_{\mathcal{M}_l}\left(\bigcup_{i=1}^{j} U_{l,i}\right) &= \text{rank}_{\mathcal{M}_l}\left(\text{span}_{\mathcal{M}_l}\left(\bigcup_{i=1}^{j} U_{l,i}\right)\right) \\
&\geq \text{rank}_{\mathcal{M}_l}\left((C_l \cup S) \cap \text{span}_{\mathcal{M}_l}\left(\bigcup_{i=1}^{j} U_{l,i}\right)\right) \\
&= \text{rank}_{\mathcal{M}_l}\left(C_l \cap \text{span}_{\mathcal{M}_l}\left(\bigcup_{i=1}^{j} U_{l,i}\right)\right) + \left|S \cap \text{span}_{\mathcal{M}_l}\left(\bigcup_{i=1}^{j} U_{l,i}\right)\right| \\
&\geq \text{rank}_{\mathcal{M}_l}\left(C_l \cap \bigcup_{i=1}^{j} U_{l,i}\right) + \left|S \cap \text{span}_{\mathcal{M}_l}\left(\bigcup_{i=1}^{j} U_{l,i}\right)\right|,
\end{aligned}$$

where the last inequality is actually an equality, as we have $C_l \cap \bigcup_{i=1}^{j} U_{l,i} = C_l \cap \text{span}_{\mathcal{M}_l}\left(\bigcup_{i=1}^{j} U_{l,i}\right)$ here.[3]

We now finish the proof of inequality (2) by observing that

[3] However, for Lemmas 27 and 31 in the next two sections, this will be in fact an inequality, as in the proof of those two lemmas, $C_l$ may contain elements not in $V'$.

$$\text{rank}_{\mathcal{M}_l}\left(\bigcup_{i=1}^{j} U_{l,i}\right) - \text{rank}_{\mathcal{M}_l}\left(C_l \cap \bigcup_{i=1}^{j} U_{l,i}\right)$$
$$= \sum_{i=1}^{j}\Big(\text{rank}_{\mathcal{M}'_l/(\bigcup_{i=1}^{j-1} U_{l,i})}(U_{l,j}) - \text{rank}_{\mathcal{M}'_l/(\bigcup_{i=1}^{j-1} U_{l,i}\cap C_l)}(U_{l,j}\cap C_l)\Big) \le \sum_{i=1}^{j} r_{l,i},$$

where the equality comes from applying Proposition 13 recursively.

We have by now shown that Properties (ii)-(iv) hold in general. Property (i) holds trivially by our construction. Thus the proof is complete. □

We denote $R_l = \bigcup_{i=1}^{|S|} R_{l,i}$ and $R = \bigcup_{l\in\{1,2\}} R_l$. Note that $R_l \subseteq V'\backslash C_l$ and $R \subseteq V'$.

**Lemma 25.** *For $l \in \{1, 2\}$, we can build sets $Q_{l,1}, \ldots, Q_{l,\text{rank}_{\mathcal{M}_l}(R_{3-l})}$ satisfying the following properties:*

*(i) the $Q_{l,j}$ are disjoint;*

*(ii) $\bigcup_{i=1}^{\text{rank}_{\mathcal{M}'_l}(R_{3-l})} Q_{l,i} = R_{3-l}$;*

*(iii) for all $v \in Q_{l,i}$, $|Q_{l,i}| \le \tilde{\rho}_{\mathcal{M}_l}(v) + 1$.*

*Proof.* Fix an $l$. For $j = 1$ to $k$, we split the set $U_{l,j} \cap R_{3-l}$ into $\text{rank}_{\mathcal{M}'_l/(\bigcup_{i=1}^{j-1} U_{l,i}\cap R_{3-l})}(U_{l,j} \cap R_{3-l})$ sets of size at most

$$\left\lceil \frac{|U_{l,j} \cap R_{3-l}|}{\text{rank}_{\mathcal{M}'_l/(\bigcup_{i=1}^{j-1} U_{l,i}\cap R_{3-l})}(U_{l,j} \cap R_{3-l})} \right\rceil \le \rho_{\mathcal{M}'_l/(\bigcup_{i=1}^{j-1} U_{l,i}\cap R_{3-l})}(U_{l,j} \cap R_{3-l}) + 1$$
$$\le \rho_{\mathcal{M}'_l/(\bigcup_{i=1}^{j-1} U_{l,i})}(U_{l,j} \cap R_{3-l}) + 1 \qquad \text{by Proposition 14}$$
$$\le \rho_{\mathcal{M}'_l/(\bigcup_{i=1}^{j-1} U_{l,i})}(U_{l,j}) + 1. \qquad \text{by construction of } U_{l,j}$$

These are the aforementioned sets $Q_{l,x}$. It is clear that those $Q_{l,x}$ will be disjoint, and that for all $v \in Q_{l,x} \subseteq U_{l,j}$, we have

$$\tilde{\rho}_{\mathcal{M}_l}(v) + 1 = \rho_{\mathcal{M}'_l/(\bigcup_{i=1}^{j-1} U_{l,i})}(U_{l,j}) + 1 \ge |Q_{l,x}|.$$

Observe that by induction, for any $1 \le r \le k$, we have $\sum_{j=1}^{r} \text{rank}_{\mathcal{M}'_l/(\bigcup_{i=1}^{j-1} U_{l,i}\cap R_{3-l})}(U_{l,j} \cap R_{3-l}) = \text{rank}_{\mathcal{M}'_l}(R_{3-l} \cap (\bigcup_{i=1}^{r} U_{l,i}))$ (using Proposition 13) and hence for $r = k$ we get

$$\sum_{j=1}^{k} \text{rank}_{\mathcal{M}'_l/(\bigcup_{i=1}^{j-1} U_{l,i}\cap R_{3-l})}(U_{l,j} \cap R_{3-l}) = \text{rank}_{\mathcal{M}'_l}(R_{3-l}),$$

therefore the number of sets $Q_{l,x}$ built that way is exactly $\text{rank}_{\mathcal{M}'_l}(R_{3-l})$, as desired. □

We now continue the proof of Theorem 4. For all $v \in R \subseteq V'$, we know by Property (i) of Definition 3 that:

$$\tilde{\rho}_{\mathcal{M}_1}(v) + \tilde{\rho}_{\mathcal{M}_2}(v) \le \beta.$$

Hence summing over all the elements of $R$:

$$\beta \cdot |R| \ge \sum_{v\in R} \tilde{\rho}_{\mathcal{M}_1}(v) + \tilde{\rho}_{\mathcal{M}_2}(v)$$
$$= \sum_{\substack{l\in\{1,2\}\\ i\in\{1,\ldots|S|\}}} \sum_{v\in R_{l,i}} \tilde{\rho}_{\mathcal{M}_l}(v) + \sum_{\substack{l\in\{1,2\}\\ i\in\{1,\ldots\text{rank}_{\mathcal{M}_l}(R_{3-l})\}}} \sum_{v\in Q_{l,i}} \tilde{\rho}_{\mathcal{M}_l}(v)$$
$$\ge \sum_{\substack{l\in\{1,2\}\\ i\in\{1,\ldots|S|\}}} |R_{l,i}| \cdot (|R_{l,i}| + 1) + \sum_{\substack{l\in\{1,2\}\\ i\in\{1,\ldots\text{rank}_{\mathcal{M}_l}(R_{3-l})\}}} |Q_{l,i}| \cdot (|Q_{l,i}| - 1)$$

$$= \sum_{\substack{l\in\{1,2\}\\ i\in\{1,\dots|S|\}}} |R_{l,i}|^2 + \sum_{\substack{l\in\{1,2\}\\ i\in\{1,\dots \mathrm{rank}_{\mathcal{M}_l}(R_{3-l})\}}} |Q_{l,i}|^2$$
$$\geq \frac{|R|^2}{2\cdot|S|} + \frac{|R|^2}{\mathrm{rank}_{\mathcal{M}_1}(R_2)+\mathrm{rank}_{\mathcal{M}_2}(R_1)}.$$

The first equality holds because of Lemma 25 (ii). The second inequality uses Lemmas 24 (iii) and 25 (iii). The second last inequality holds because $\sum_{l,i}|R_{l,i}| = \sum_{l,i}|Q_{l,i}| = |R|$. Finally, the last inequality comes from the minimization of the function under the constraint $\sum_{l,i}|R_{l,i}| = \sum_{l,i}|Q_{l,i}| = |R|$.

Hence we get

$$\beta \geq \frac{|R|}{2\cdot|S|} + \frac{|R|}{\mathrm{rank}_{\mathcal{M}_1}(R_2)+\mathrm{rank}_{\mathcal{M}_2}(R_1)}.$$

As the elements of $S$ satisfy Property (ii) of Definition 3, and because of Property (iv) in Lemma 24, we know that for all $o_i \in S$,

$$\beta^- \leq \tilde{\rho}_{\mathcal{M}_1}(o_i) + \tilde{\rho}_{\mathcal{M}_2}(o_i) \leq |R_{1,i}| + |R_{2,i}| + 4,$$

so by averaging over all the elements of $S$ we get

$$\beta^- \leq \frac{|R|}{|S|} + 4.$$

Therefore we finally obtain

$$\left(\beta - \frac{\beta^- - 4}{2}\right)\cdot(\mathrm{rank}_{\mathcal{M}_1}(R_2)+\mathrm{rank}_{\mathcal{M}_2}(R_1)) \geq |R|.$$

Then, as $(\beta^- - 4)\cdot|S| \leq |R|$ and $\mathrm{rank}_{\mathcal{M}_1}(R_2)+\mathrm{rank}_{\mathcal{M}_2}(R_1) \leq \mathrm{rank}_{\mathcal{M}_1}(C_1)+\mathrm{rank}_{\mathcal{M}_2}(C_2) = \mu(V')$ (because $R_{3-l} \subseteq C_l$) we finally have $\left(\beta - \frac{\beta^- -4}{2}\right)\cdot\mu(V') \geq (\beta^- - 4)\cdot|S|$. Now using (1), we obtain:

$$\mu(V) = |O\backslash S| + |S| \leq \left(1 + \frac{\beta}{\beta^- - 4} - \frac{1}{2}\right)\cdot\mu(V') = \left(\frac{1}{2} + \frac{\beta}{\beta^- - 4}\right)\cdot\mu(V') \leq \left(\frac{3}{2}+\varepsilon\right)\cdot\mu(V'),$$

as $(\beta^- - 4)\cdot(1+\varepsilon) \geq \beta$. This concludes the proof. □

**Remark 26.** *We can observe that $\beta$ and $\beta^-$ can be of order $O(1/\varepsilon)$ to satisfy the constraints of Theorem 6. From now on we will suppose that $\beta, \beta^-$ are $O(1/\varepsilon)$.*

# 4 Application to One-Way Communication

Given two matroids $\mathcal{M}_1 = (V, \mathcal{I}_1)$ and $\mathcal{M}_2 = (V, \mathcal{I}_2)$, in the one-way communication model, Alice and Bob are given $V_A$ and $V_B = V\backslash V_A$ respectively, and the goal is for Alice to send a small message to Bob so that Bob can output a large intersection of matroids $\mathcal{M}_1$ and $\mathcal{M}_2$. Here we will show that if Alice communicates an appropriate Density-Constrained Subset of $V_A$, with parameters $\beta, \beta^-$ of order $O(1/\varepsilon)$, then Bob is able to get a $3/2+\varepsilon$ approximation of the optimal intersection.

**Theorem 8.** *There exists a one-way communication protocol that, given any $\varepsilon > 0$, computes a $3/2+\varepsilon$ approximation to maximum matroid intersection using a message of size $O(\mu(V)/\varepsilon)$ from Alice to Bob, where $\mu(V)$ denotes the size of the optimal solution of the matroid intersection problem.*

By Theorem 23 we know that a DCS in the two restricted matroids $\mathcal{M}_1|V_A$ and $\mathcal{M}_2|V_A$ always exists, and by Proposition 22 we know that the number of elements sent by Alice is at most $O(\mu(V)/\varepsilon)$. Hence we only need to prove the following lemma.

**Lemma 27.** *Let $\varepsilon > 0$, $\beta$ and $\beta^-$ be parameters such that $\beta \geq \beta^- + 7$ and $(\beta^- - 4) \cdot (1+\varepsilon) \geq \beta$, if $V'$ is a $(\beta, \beta^-)$-DCS of the two matroids $\mathcal{M}_1|V_A$ and $\mathcal{M}_2|V_A$, then $(3/2+\varepsilon) \cdot \mu(V' \cup V_B) \geq \mu(V)$.*

**Remark 28.** *In [AB19] the proof is done by showing that the combination of the subgraphs built in the first and second phase of the algorithm contains a subgraph which is an EDCS with respect to some subgraph containing the optimal solution. Here we are unable to imitate this approach because adding several elements to the matroids can change significantly the density decompositions. Therefore our approach is to directly adapt the proof of Theorem 4. This remark also applies to Lemma 31 in the next section.*

*Proof.* Let $O$ be an optimal solution in $V$. Let $O_A = O \cap V_A$ and $O_B = O \cap V_B$. Let $C_1$ and $C_2$ be sets such that $C_1 \cup C_2 = V' \cup O_B$, $C_1 \cap C_2 = \emptyset$ and they minimize the sum $\text{rank}_{\mathcal{M}_1}(C_1) + \text{rank}_{\mathcal{M}_2}(C_2)$. By Theorem 6 we know that $\text{rank}_{\mathcal{M}_1}(C_1) + \text{rank}_{\mathcal{M}_2}(C_2) = \mu(V' \cup O_B)$, the maximum size of a common independent set in $V' \cup O_B$.

As in the proof of Theorem 4, we will build an auxiliary set $S$, starting with $S = \emptyset$. If there exists an element $o_1 \in O$ such that $o_1 \notin \text{span}_{\mathcal{M}_1}(C_1) \cup \text{span}_{\mathcal{M}_2}(C_2)$, then we add $o_1$ into $S$ and we next consider the contracted matroids $\mathcal{M}_1/S$ and $\mathcal{M}_2/S$. We keep the same sets $C_1$ and $C_2$ and we try again to find an element $o_2 \in O \backslash S$ such that $o_2 \notin \text{span}_{\mathcal{M}_1/S}(C_1) \cup \text{span}_{\mathcal{M}_2/S}(C_2)$, and we add $o_2$ to $S$. We repeat that operation until it is no longer possible to add into $S$ another element of $O$ satisfying the aforementioned constraint. Note that here all the elements $o_i$ added to $S$ come necessarily from $O_A$.

As a result, as $O \backslash S$ is a common independent subset in $\mathcal{M}_1/S$ and $\mathcal{M}_2/S$, and because of Theorem 6, the size of $O \backslash S$ is upper-bounded by $\text{rank}_{\mathcal{M}_1/S}(C_1) + \text{rank}_{\mathcal{M}_2/S}(C_2) \leq \text{rank}_{\mathcal{M}_1}(C_1) + \text{rank}_{\mathcal{M}_2}(C_2) = \mu(V' \cup O_B)$, as in the proof of Theorem 4.

Now we need to upper-bound the value of $|S|$. We will use the same construction as that of the proof of Theorem 4, as there is no difference in the algorithms that construct the sets $R_{l,i}$ and $Q_{l,i}$ (those remain subsets of $V'$, we can just follow the same procedures described in Lemmas 24 and 25).

After similar computations, we get the inequality:

$$\beta \geq \frac{|R|}{2 \cdot |S|} + \frac{|R|}{\text{rank}_{\mathcal{M}_1}(R_2) + \text{rank}_{\mathcal{M}_2}(R_1)}.$$

As the elements of $S$ are from $O_A \subset V_A$, and because of Property (ii) of Definition 3, we know that for all $o_i \in S$,

$$\beta^- \leq \tilde{\rho}_{\mathcal{M}_1}(o_i) + \tilde{\rho}_{\mathcal{M}_2}(o_i) \leq |R_{1,i}| + |R_{2,i}| + 4,$$

so by averaging over all the elements of $S$ we get

$$\beta^- \leq \frac{|R|}{|S|} + 4.$$

Therefore we derive

$$\left(\beta - \frac{\beta^- - 4}{2}\right) \cdot (\text{rank}_{\mathcal{M}_1}(R_2) + \text{rank}_{\mathcal{M}_2}(R_1)) \geq |R|.$$

Then, as $(\beta^- - 4) \cdot |S| \leq |R|$ and $\text{rank}_{\mathcal{M}_1}(R_2) + \text{rank}_{\mathcal{M}_2}(R_1) \leq \text{rank}_{\mathcal{M}_1}(C_1) + \text{rank}_{\mathcal{M}_2}(C_2) = \mu(V' \cup O_B)$ we finally have $\left(\beta - \frac{\beta^- - 4}{2}\right) \cdot \mu(V' \cup O_B) \geq (\beta^- - 4) \cdot |S|$, and therefore:

$$\mu(V) = |O \backslash S| + |S| \leq \left(\frac{1}{2} + \frac{\beta}{\beta^- - 4}\right) \cdot \mu(V' \cup O_B) \leq \left(\frac{3}{2} + \varepsilon\right) \cdot \mu(V' \cup O_B) \leq \left(\frac{3}{2} + \varepsilon\right) \cdot \mu(V' \cup V_B),$$

as $(\beta^- - 4) \cdot (1+\varepsilon) \geq \beta$. □

# 5 Application to Random-Order Streams

Now we consider our problem in the random-order streaming model. As our algorithm builds on that of Bernstein [Ber20] for the unweighted simple matching, let us briefly summarize his approach. In the first phase of the streaming, he constructs a subgraph that satisfies only a weaker definition of EDCS in Definition 1 (only Property (i) holds). In the second phase of the streaming, he collects the "underfull" edges, which are those edges that violate Property (ii). He shows that in the end, the union of the subgraph built in the first phrase and the underfull edges collected in the second phase, with high probability, contains a $3/2+\varepsilon$ approximation and that the total memory used is in the order of $O(k \cdot poly(\log(k), 1/\varepsilon))$ (there $k$ refers to the number of vertices in the graph). As we will show below, this approach can be adapted to our context of matroid intersection.

**Definition 29.** *We say that a subset $V'$ has* bounded density $\beta$ *if for every element $v \in V'$, $\tilde{\rho}_{\mathcal{M}_1}(v) + \tilde{\rho}_{\mathcal{M}_2}(v) \leq \beta$.*

**Definition 30.** *Let $V'$ be a subset of $V$ with bounded density $\beta$. For any parameter $\beta^-$, we say that an element $v \in V \backslash V'$ is* $(V', \beta, \beta^-)$-underfull *if $\tilde{\rho}_{\mathcal{M}_1}(v) + \tilde{\rho}_{\mathcal{M}_2}(v) < \beta^-$.*

As in [Ber20], we can get a good approximation by combining a subset $V'$ of bounded density $\beta$ and the set of $(V', \beta, \beta^-)$-underfull elements in $V \backslash V'$. The proof of the following lemma is quite similar to that of Theorem 4, so we will only highlight the points where the proofs differ.

**Lemma 31.** *Let $\varepsilon > 0$, $\beta$ and $\beta^-$ be parameters such that $\beta \geq \beta^- + 7$ and $(\beta^- - 4) \cdot (1+\varepsilon) \geq \beta$. Given a subset $V' \subseteq V$ with bounded density $\beta$, if $X$ contains all elements in $V \backslash V'$ that are $(V', \beta, \beta^-)$-underfull, then $(3/2+\varepsilon) \cdot \mu(V' \cup X) \geq \mu(V)$.*

*Proof.* Let $O$ be an optimal solution in $V$. Let $X^{\text{opt}} = X \cap O$. Let $C_1$ and $C_2$ be sets such that $C_1 \cup C_2 = V' \cup X^{\text{opt}}$, $C_1 \cap C_2 = \emptyset$, and they minimize the sum $\text{rank}_{\mathcal{M}_1}(C_1) + \text{rank}_{\mathcal{M}_2}(C_2)$. By Theorem 6 we know that $\text{rank}_{\mathcal{M}_1}(C_1) + \text{rank}_{\mathcal{M}_2}(C_2) = \mu(V' \cup X^{\text{opt}})$, the maximum size of a common independent set in $V' \cup X^{\text{opt}}$.

As in the proof of Theorem 4, we will build an auxiliary set $S$, starting with $S = \emptyset$. If there exists an element $o_1 \in O$ such that $o_1 \notin \text{span}_{\mathcal{M}_1}(C_1) \cup \text{span}_{\mathcal{M}_2}(C_2)$, then we add $o_1$ into $S$ and we now consider the contracted matroids $\mathcal{M}_1/S$ and $\mathcal{M}_2/S$. We keep the same sets $C_1$ and $C_2$ and we try again to find an element $o_2 \in O \backslash S$ such that $o_2 \notin \text{span}_{\mathcal{M}_1/S}(C_1) \cup \text{span}_{\mathcal{M}_2/S}(C_2)$, and we add $o_2$ to $S$. We repeat that operation until it is no longer possible to add another element to $S$ satisfying the aforementioned constraints.

As a result, as $O \backslash S$ is a common independent subset in $\mathcal{M}_1/S$ and $\mathcal{M}_2/S$, and because of Theorem 6, the size of $O \backslash S$ is upper-bounded by $\text{rank}_{\mathcal{M}_1/S}(C_1) + \text{rank}_{\mathcal{M}_2/S}(C_2) \leq \text{rank}_{\mathcal{M}_1}(C_1) + \text{rank}_{\mathcal{M}_2}(C_2) = \mu(V' \cup X^{\text{opt}})$, as in the proof of Theorem 4.

Now we need to upper-bound the value of $|S|$. We will use the same construction as that of the proof of Theorem 4, as there is no difference in the algorithms construct the sets $R_{l,i}$ and $Q_{l,i}$ (those remain subsets of $V'$, we can just follow the same procedures described in Lemmas 24 and 25).

Then after similar computations, we get the inequality:

$$\beta \geq \frac{|R|}{2 \cdot |S|} + \frac{|R|}{\text{rank}_{\mathcal{M}_1}(R_2) + \text{rank}_{\mathcal{M}_2}(R_1)}.$$

As the elements of $S$ are not underfull (observe that here we use this fact, instead of using Property (ii) of Definition 3 as we have done in the proof of Theorem 4), we know that for all $o_i \in S$,

$$\beta^- \leq \tilde{\rho}_{\mathcal{M}_1}(o_i) + \tilde{\rho}_{\mathcal{M}_2}(o_i) \leq |R_{1,i}| + |R_{2,i}| + 4,$$

so by averaging over all the elements of $S$ we get

$$\beta^- \leq \frac{|R|}{|S|} + 4.$$

Therefore we obtain

$$\left(\beta - \frac{\beta^- - 4}{2}\right) \cdot (\text{rank}_{\mathcal{M}_1}(R_2) + \text{rank}_{\mathcal{M}_2}(R_1)) \geq |R|.$$

Then, as $(\beta^- - 4) \cdot |S| \leq |R|$ and $\text{rank}_{\mathcal{M}_1}(R_2) + \text{rank}_{\mathcal{M}_2}(R_1) \leq \text{rank}_{\mathcal{M}_1}(C_1) + \text{rank}_{\mathcal{M}_2}(C_2) = \mu(V' \cup X^{\text{opt}})$ we finally have $\left(\beta - \frac{\beta^- - 4}{2}\right) \cdot \mu(V' \cup X^{\text{opt}}) \geq (\beta^- - 4) \cdot |S|$, and therefore:

$$\mu(V) = |O \backslash S| + |S| \leq \left(\frac{1}{2} + \frac{\beta}{\beta^- - 4}\right) \cdot \mu(V' \cup X^{\text{opt}}) \leq \left(\frac{3}{2} + \varepsilon\right) \cdot \mu(V' \cup X^{\text{opt}}),$$

as $(\beta^- - 4) \cdot (1 + \varepsilon) \geq \beta$. □

Here we recall a classic probabilistic tool that we will use in the analysis of our algorithm.

**Proposition 32** (Hoeffding's inequality)**.** *Let $X_1, \ldots, X_t$ be $t$ negatively associated random variables that take values in $[0, 1]$. Let $X := \sum_{i=1}^{t} X_i$. Then, for all $\lambda > 0$ we have:*

$$\mathbb{P}(X - \mathbb{E}[X] \geq \lambda) \leq \exp\left(-\frac{2\lambda^2}{t}\right).$$

The following ideas for the streaming algorithm come from a recent paper originally intended for $b$-matchings [HS22]. For sake of completeness, we reproduce the details in the following, with some slight adaptations to our more general case of matroid intersection.

**Algorithm 2** Algorithm for computing an intersection of two matroids in a random-order stream

1: $V' \leftarrow \emptyset$
2: $\forall\, 0 \leq i \leq \log_2 k$, $\alpha_i \leftarrow \left\lfloor \frac{\varepsilon \cdot n}{\log_2(k) \cdot (2^{i+2}\beta^2 + 1)} \right\rfloor$
3: **for** $i = 0 \ldots \log_2 k$ **do**
4:     ProcessStopped $\leftarrow$ False
5:     **for** $2^{i+2}\beta^2 + 1$ iterations **do**
6:         FoundUnderfull $\leftarrow$ False
7:         **for** $\alpha_i$ iterations **do**
8:             let $v$ be the next element in the stream
9:             **if** $\tilde{\rho}_{\mathcal{M}_1}(v) + \tilde{\rho}_{\mathcal{M}_2}(v) < \beta^-$ **then**
10:                 add $v$ to $V'$
11:                 FoundUnderfull $\leftarrow$ True
12:                 **while** there exists $v' \in V' : \tilde{\rho}_{\mathcal{M}_1}(v') + \tilde{\rho}_{\mathcal{M}_2}(v') > \beta$ **do**
13:                     remove $v'$ from $V'$
14:         **if** FoundUnderfull = False **then**
15:             ProcessStopped $\leftarrow$ True
16:             **break** from the loop
17:     **if** ProcessStopped = True **then**
18:         **break** from the loop
19: $X \leftarrow \emptyset$
20: **for** each $v$ remaining element in the stream **do**
21:     **if** $\tilde{\rho}_{\mathcal{M}_1}(v) + \tilde{\rho}_{\mathcal{M}_1}(v) < \beta^-$ **then**
22:         add $v$ to $X$
23: **return** the maximum common independent set in $V' \cup X$

The algorithm, formally described in Algorithm 2, consists of two phases. The first phase, corresponding to Lines 3-18, constructs a subset $V'$ of bounded density $\beta$ using only an $\varepsilon$ fraction of the stream $V^{\text{early}}$. In the second phase, the algorithm collects the underfull elements in the remaining part of the stream $V^{\text{late}}$. As in [Ber20] we use the idea that if no underfull element was found in an interval of size $\alpha$ (see Lines 6-13), then with high probability the number of underfull elements remaining in the stream is bounded by some value $\gamma = 4\log(n)\frac{n}{\alpha}$. The issue is therefore how to choose the right size of interval $\alpha$, because we ignore the order of magnitude of $\mu(V)$ the optimal solution: if we do as in [Ber20] by choosing only one fixed size of intervals $\alpha$, then if $\alpha$ is too small, the value of $\gamma$ will be too big compared to $\mu(V)$, whereas if the value of $\alpha$ is too large we will be unable to terminate the first phase of the algorithm within the early fraction of size $\varepsilon m$. Therefore, the idea in the first phase of the algorithm is to "guess" the value of $\log_2 \mu(V)$ by trying successively larger and larger values of $i$ (see Line 3). In fact, by setting $i_0 = \lceil \log_2 \mu(V) \rceil$, we know that the number of insertion/deletion operations that can be performed on a $(\beta, \beta^-)$-DSC is bounded by $2^{i_0+2}\beta^2$ (see the proof of Theorem 23). As a result we know that the first phase should always stop at a time where $i$ is smaller than or equal to $i_0$, and therefore at a time when $\alpha_i \geq \alpha_{i_0}$. Then we can prove that with high probability the number of remaining underfull elements in the stream is at most $\gamma_i = 4\log(n)\frac{n}{\alpha_i}$.

**Claim 33.** *With probability at least $1 - \exp(-2 \cdot \varepsilon^2 \cdot \mu(V))$ the late part of the stream $V^{\text{late}}$ contains at least a $(1 - 2\varepsilon)$ fraction of the optimal solution. Moreover, in expectation $V^{\text{late}}$ contains a $(1 - \varepsilon)$ fraction of the optimal solution.*

*Proof.* Consider an optimal solution $O = \{o_1, \ldots, o_{\mu(V)}\}$. We define the random variables $X_i = \mathbb{1}_{o_i \in V^{\text{early}}}$. Hence we have $\mathbb{E}[\sum X_i] = \varepsilon \cdot |O|$. Moreover, the random variables $X_i$ are negatively associated, so we can use Hoeffding's inequality (see Proposition 32) to get

$$\mathbb{P}\left[\sum_{i=1}^{\mu(V)} X_i \geq 2\varepsilon \cdot \mu(V)\right] \leq \exp\left(-\frac{2 \cdot \varepsilon^2 \cdot \mu(V)^2}{\mu(V)}\right) = \exp\left(-2 \cdot \varepsilon^2 \cdot \mu(V)\right).$$

□

Recall that we defined $i_0 = \lceil \log_2 \mu(V) \rceil$. Algorithm 2 works when $\mu(V)$ is not too big (otherwise we may use intervals of size $\alpha_{i_0} = \lfloor \frac{\varepsilon \cdot n}{\log_2(k) \cdot (2^{i_0+2}\beta^2+1)} \rfloor = 0$). Here we will first argue that this case can be handled anyway.

**Claim 34.** *We can assume that $\frac{\varepsilon \cdot n}{\log_2(k) \cdot (2^{i_0+2}\beta^2+1)} \geq 1$.*

*Proof.* If this is not the case, then we can just store all the elements of $V$ as the number of elements $n$ is bounded by $\frac{\log_2(k) \cdot (2^{i_0+2}\beta^2+1)}{\varepsilon} = O(\mu(V) \cdot \log(k) \cdot (1/\varepsilon)^3)$ (as $\beta$ is $O(1/\varepsilon)$, see Remark 26). As a result, if at some point of the first phase we have not stopped and we have $\alpha_i = 0$, then we store all the remaining elements of $V^{\text{late}}$ and we will be able to get a $(1 - \varepsilon)$ approximation in expectation and a $(1 - 2\varepsilon)$ approximation with high probability (more precisely, at least $1 - \exp(-2 \cdot c \cdot \varepsilon^5 \cdot n/\log(k))$, for some constant $c > 0$, see Claim 33), using $O(\mu(V) \cdot \log(k) \cdot (1/\varepsilon)^3)$ memory. □

From now on we will assume that $\frac{\varepsilon \cdot n}{\log_2(k) \cdot (2^{i_0+2}\beta^2+1)} \geq 1$. Then we can move on to our main algorithm. The following lemma is very similar to the one used in [Ber20].

**Lemma 35.** *The first phase of Algorithm 2 uses $O(\beta \cdot \mu(V))$ memory and constructs a subset $V' \subseteq V$, satisfying the following properties:*

1. *The first phase terminates within the first $\varepsilon \cdot n$ elements of the stream.*
2. *When the first phase terminates after processing some element, we have:*
   (a) *$V'$ has bounded density $\beta$, and contains at most $O(\beta \cdot \mu(V))$ elements.*
   (b) *With probability at least $1 - n^{-3}$, the total number of $(V', \beta, \beta^-)$-underfull elements in the remaining part of the stream is at most $\gamma = O(\mu(V) \cdot \log(n) \cdot \log(k) \cdot \beta^2 \cdot 1/\varepsilon)$.*

*Proof.* First, in each interval of size $\alpha_i$ processed until the first phase terminates (except the last interval), at least one insertion/deletion operation that is performed (as described in the proof of Theorem 23), and therefore the total number of such processed intervals is bounded by $2\beta^2 \cdot \mu(V) + 1$. As a result, the first phase ends with some $i \leq i_0 = \lceil \log_2 \mu(V) \rceil$, and the total number of elements processed in the first phase is therefore bounded by $\varepsilon \cdot n \cdot \frac{i_0}{\log_2(k)} \leq \varepsilon \cdot n$. For Property 2.a, as the subset $V'$ built always keeps a bounded density $\beta$, Proposition 22 implies that $V'$ uses $O(\beta \cdot \mu(V)) = O(\mu(V) \cdot 1/\varepsilon)$ memory.

Now we turn to the last property. As mentioned previously, the intuition is simple: the algorithm only exits the first phase if it fails to find a single underfull element in an entire interval (Line 14-16), and since the stream is random, such an event implies that there are most likely few underfull elements left in the stream.

To formalize this, we call the $j$-th time that Lines 7-13 are processed the *epoch* $j$. Let $\mathcal{A}_j$ be the event that FOUNDUNDERFULL is set to FALSE in epoch $j$. Let $\mathcal{B}_j$ be the event that the number of $(V', \beta, \beta^-)$-underfull elements in the remaining part of the stream is larger than some $\gamma$. Note that the last property fails to hold if and only if we have $\mathcal{A}_j \wedge \mathcal{B}_j$ for some $j$, so we want to upper bound $\mathbb{P}[\mathcal{A}_j \wedge \mathcal{B}_j]$. Let $V_j^r$ contains all elements in $V$ that have not yet appeared in the stream at the *beginning* of epoch $j$ (r for remaining). Let $V_j^e$ be the elements that appear in epoch $j$ (e for epoch), and note that $E_j^e$ is a subset of size $\alpha_i \geq \alpha_{i_0} = \alpha_{\lceil \log_2 \mu(V) \rceil} = \alpha$ chosen uniformly at random from $V_j^r$. Define $V_j'$ to be the subset $V'$ at the beginning of epoch $j$, and define $V_j^u \subseteq E_j^r$ to be the set of remaining underfull elements with respect to $V_j'$, $\beta$, and $\beta^-$. Observe that because of event $\mathcal{A}_j$, the subset $V'$ remains the same throughout epoch $j$, so an element that is underfull at any point during the epoch will be underfull at the end as well. Thus, $\mathcal{A}_j \wedge \mathcal{B}_j$ is equivalent to the event that $|V_j^u| > \gamma$ and $V_j^u \cap V_j^e = \emptyset$.

Let $\mathcal{A}_j^k$ be the event that the $k$-th element of epoch $j$ is not in $V_j^u$. We have that $\mathbb{P}[\mathcal{B}_j \wedge \mathcal{A}_j] \leq \mathbb{P}[\mathcal{A}_j \,|\, \mathcal{B}_j] \leq \mathbb{P}[\mathcal{A}_j^1 \,|\, \mathcal{B}_j] \prod_{k=2}^{\alpha} \mathbb{P}[\mathcal{A}_j^k \,|\, \mathcal{B}_j, \mathcal{A}_j^1, \ldots, \mathcal{A}_j^{k-1}]$, where the second inequality comes from that $V_j^e$ is of size larger or equal to $\alpha = \alpha_{\lceil \log_2 \mu(V) \rceil}$.

Now, observe that $\mathbb{P}[\mathcal{A}_j^1 \,|\, \mathcal{B}_j] < 1 - \frac{\gamma}{n}$ because the first element of the epoch is chosen uniformly at random from the set of $\leq n$ remaining elements, and the event fails if the chosen element is in $V_j^u$, where $|V_j^u| > \gamma$ by definition of $\mathcal{B}_j$. Similarly, for any $k$, $\mathbb{P}[\mathcal{A}_j^k \,|\, \mathcal{B}_j, \mathcal{A}_j^1, \ldots, \mathcal{A}_j^{k-1}] < 1 - \frac{\gamma}{m}$ because conditioning on the previous events $\mathcal{A}_j^t$ implies that no element from $V_j^u$ has yet appeared in this epoch, so there remain still at least $\gamma$ element from $V_j^u$ left in the stream.

We now set

$$\gamma = 4\log(n) \cdot \frac{n}{\alpha} = 4\log(n) \cdot n \cdot \left\lfloor \frac{\varepsilon \cdot n}{\log_2(k) \cdot (2^{i_0+2}\beta^2 + 1)} \right\rfloor^{-1},$$

and as we assumed that $\frac{\varepsilon \cdot n}{\log_2(n) \cdot (2^{i_0+2}\beta^2+1)} \geq 1$ (and as a factor of at most 2 separates $\lfloor x \rfloor$ and $x$ when $x \geq 1$) we have $\gamma = O(\mu(V) \cdot \log(n) \cdot \log(k) \cdot (1/\varepsilon)^3)$.

Combining the above equations yields that $\mathbb{P}[\mathcal{B}_j \wedge \mathcal{A}_j] \leq (1 - \frac{\gamma}{n})^\alpha = (1 - \frac{4\log(n)}{\alpha})^\alpha \leq n^{-4}$. There are clearly at most $n$ epochs, so union bounding over all of them shows that the last property fails with probability at most $n^{-3}$, as desired. □

Then we can combine the previous results to obtain the following theorem.

**Theorem 9.** *Let $1/4 > \varepsilon > 0$. One can extract from a randomly-ordered stream of elements a common independent subset in two matroids with an approximation ratio of $3/2 + \varepsilon$ in expectation, using $O(\mu(V) \cdot \log(n) \cdot \log(k) \cdot (1/\varepsilon)^3)$ memory, where $\mu(V)$ denotes the size of the optimal solution, and $k$ is the smaller rank of the two given matroids. Moreover the approximation ratio is worse than $3/2 + \varepsilon$ only with probability at most $\exp(-1/32 \cdot \varepsilon^2 \cdot \mu(V)) + n^{-3}$.*

*Proof.* Using Lemma 31 on the graph $V' \cup V^{\text{late}}$ we get $(3/2 + \varepsilon) \cdot \mu(V' \cup X) \geq \mu(V' \cup V^{\text{late}})$. Applying Claim 33, we know that in expectation $(1 - \varepsilon)^{-1} \cdot \mu(V' \cup V^{\text{late}}) \geq \mu(V)$. Hence in expectation we also have

$$(3/2 + \varepsilon) \cdot (1 - \varepsilon)^{-1} \cdot \mu(V' \cup X) \geq \mu(V).$$

Moreover, by Lemma 35, the memory consumption is bounded by $O(\mu(V) \cdot \log(n) \cdot \log(k) \cdot (1/\varepsilon)^3)$ with probability at least $1 - n^{-3}$. Hence we can decide that, if during the execution of the algorithm at some point the memory consumption reaches the bound defined in Lemma 35 (recall that this bound can be computed as it depends only on the epoch when the first phase stopped), then we discard the remaining elements. As this event happens only with probability $1 - n^{-3}$, this is not harmful for the expectation of the approximation ratio.

Moreover, using Claim 33, we know that a $(1-2\varepsilon)^{-1} \cdot (3/2+\varepsilon)$ approximation of the optimal common independent subset is contained in $\mu(V \cup X)$ with probability at least $1 - \exp(-2 \cdot \varepsilon^2 \cdot \mu(V))$. As the memory consumption of $O(\mu(V)\cdot\log(n)\cdot\log(k)\cdot(1/\varepsilon)^3)$ is guaranteed with probability at least $1-n^{-3}$ (see Lemma 35), then with probability at least $1-(\exp(-2\cdot\varepsilon^2\cdot\mu(V))+n^{-3})$ (by union bound), we can obtain a $(1-2\varepsilon)^{-1}\cdot(3/2+\varepsilon)$ approximation using $O(\mu(V)\cdot\log(n)\cdot\log(k)\cdot(1/\varepsilon)^3)$ memory. As $\varepsilon<1/4$, we have $(1-2\varepsilon)^{-1}\cdot(3/2+\varepsilon)\le(3/2+8\varepsilon)$, and therefore to get a $3/2+\varepsilon$, we have to use an $\varepsilon'=\varepsilon/8$ so that the probability to have an approximation ratio worse than $3/2+\varepsilon$ approximation is at most $\exp(-2\cdot(\varepsilon/8)^2\cdot\mu(V))+n^{-3}$. □

# A Deferred Proofs

*Proof of Lemma 20.* In the following, we will use the notation $P_{a,b}=U_a^{\text{old}}\cap U_b^{\text{new}}$ for $a,b\in[\![1,k]\!]$.

We prove (i) by strong induction. We start with the case $j=1$. Let $i_0$ be the largest $i$ set such as $P_{1,i}\neq\emptyset$. Then we know that for all $v\in U_1^{\text{old}}$,

$$\begin{aligned}
\tilde\rho^{\text{new}}_{\mathcal M}(v) &\ge \rho_{\mathcal M'^{\text{new}}/(\bigcup_{i=1}^{i_0-1}U_i^{\text{new}})}(U_{i_0}^{\text{new}}) && \text{by Proposition 17}\\
&\ge \rho_{\mathcal M'^{\text{new}}/(\bigcup_{i=1}^{i_0-1}U_i^{\text{new}})}(P_{1,i_0}) && \text{by maximality of } U_{i_0}^{\text{new}}\\
&\ge \rho_{\mathcal M'^{\text{new}}/(\bigcup_{i=1}^{i_0-1}P_{1,i})}(P_{1,i_0}) && \text{as } \bigcup_{i=1}^{i_0-1}P_{1,i}\subseteq\bigcup_{i=1}^{i_0-1}U_i^{\text{new}} \text{ (Proposition 14)}\\
&= \rho_{\mathcal M'^{\text{old}}/(\bigcup_{i=1}^{i_0-1}P_{1,i})}(P_{1,i_0}) && \text{as } u^{\text{new}}\notin U_1^{\text{old}}\\
&\ge \rho_{\mathcal M'^{\text{old}}}(U_1^{\text{old}}). && \text{by Proposition 15}
\end{aligned}$$

For the induction step, let $2\le j\le k$. Suppose that the property is true for all $i\in[\![1,j-1]\!]$. We want to prove that the property is also true for $j$. Let $i_0$ be the largest $i$ such that $P_{j,i}\neq\emptyset$. Then we know that for all $v\in U_j^{\text{old}}$,

$$\tilde\rho^{\text{new}}_{\mathcal M}(v)\ge\rho_{\mathcal M'^{\text{new}}/(\bigcup_{i=1}^{i_0-1}U_i^{\text{new}})}(U_{i_0}^{\text{new}})\ge\rho_{\mathcal M'^{\text{new}}/(\bigcup_{i=1}^{i_0-1}U_i^{\text{new}})}(P_{j,i_0}).$$

Then we have two cases:

- We have $\bigcup_{i=1}^{j-1}U_i^{\text{old}}\subseteq\bigcup_{i=1}^{i_0-1}U_i^{\text{new}}$. In that case we can write, similarly as in the previous case,

$$\begin{aligned}
\tilde\rho^{\text{new}}_{\mathcal M}(v) &\ge \rho_{\mathcal M'^{\text{new}}/(\bigcup_{i=1}^{i_0-1}U_i^{\text{new}})}(P_{j,i_0})\\
&\ge \rho_{\mathcal M'^{\text{new}}/(\bigcup_{i=1}^{j-1}U_i^{\text{old}}\cup\bigcup_{i=1}^{i_0-1}P_{j,i})}(P_{j,i_0}) && \text{as } \bigcup_{i=1}^{j-1}U_i^{\text{old}}\cup\bigcup_{i=1}^{i_0-1}P_{j,i}\subseteq\bigcup_{i=1}^{i_0-1}U_i^{\text{new}}\\
&= \rho_{\mathcal M'^{\text{old}}/(\bigcup_{i=1}^{j-1}U_i^{\text{old}}\cup\bigcup_{i=1}^{i_0-1}P_{j,i})}(P_{j,i_0}) && \text{as } u^{\text{new}}\notin\bigcup_{i=1}^{j-1}U_i^{\text{old}}\cup\bigcup_{i=1}^{i_0}P_{j,i}\\
&\ge \rho_{\mathcal M'^{\text{ old}}}(U_j^{\text{old}}). && \text{by Proposition 15}
\end{aligned}$$

- Otherwise, there exists $u'\in U_{i_1}^{\text{old}}$ such that $i_1<j$ and $u'\notin\bigcup_{i=1}^{i_0-1}U_i^{\text{new}}$. It means that $u'\in U_{i_2}^{\text{new}}$ for some $i_2\ge i_0$, and hence $\tilde\rho^{\text{new}}_{\mathcal M}(u')\le\tilde\rho^{\text{new}}_{\mathcal M}(v)$. Then we have

$$\tilde\rho^{\text{new}}_{\mathcal M}(v)\ge\tilde\rho^{\text{new}}_{\mathcal M}(u')\ge\rho_{\mathcal M'^{\text{old}}/(\bigcup_{i=1}^{i_1-1}U_i^{\text{old}})}(U_{i_1}^{\text{old}})\ge\rho_{\mathcal M'^{\text{old}}/(\bigcup_{i=1}^{j-1}U_i^{\text{old}})}(U_j^{\text{old}}),$$

  where the second inequality uses the strong induction hypothesis and the third uses Proposition 17.

This concludes the proof of (i).

Now we move to (ii). Observe that (i) implies the result for $v\in V'^{\text{old}}$. If $v\notin\text{span}_{\mathcal M}(V'^{\text{old}})$, then it is clear that the density associated to that element can only increase (recall that by Definition 19, $\tilde\rho^{\text{old}}_{\mathcal M}(v)=0$). From now on we suppose that $v\in\text{span}_{\mathcal M}(V'^{\text{old}})$ and we denote $j=\min\{j\in$

$[\![1,k]\!] : v \in \mathrm{span}_{\mathcal{M}}(\bigcup_{i=1}^{j} U_i^{\mathrm{old}})\}$. By (i) we know that for all $v' \in \bigcup_{i=1}^{j} U_i^{\mathrm{old}}$, we have $\tilde{\rho}_{\mathcal{M}}^{\mathrm{new}}(v') \geq \rho_{\mathcal{M}'^{\mathrm{old}}/(\bigcup_{i=1}^{j-1} U_i^{\mathrm{old}})}(U_j^{\mathrm{old}}) = \tilde{\rho}_{\mathcal{M}}^{\mathrm{old}}(v') \geq \tilde{\rho}_{\mathcal{M}}^{\mathrm{old}}(v)$, hence we also have $\tilde{\rho}_{\mathcal{M}}^{\mathrm{new}}(v) \geq \min_{v' \in \bigcup_{i=1}^{j} U_i^{\mathrm{old}}} \tilde{\rho}_{\mathcal{M}}^{\mathrm{old}}(v') \geq \tilde{\rho}_{\mathcal{M}}^{\mathrm{old}}(v)$, as the associated density of $v$ will be at least equal to the smallest density of the elements in $\bigcup_{i=1}^{j} U_i^{\mathrm{old}}$, as once all those elements are in the decomposition we are sure that $v$ is spanned.

For (iii), we only have to prove the upper bound (the lower bound comes from (ii)). Suppose that in the new decomposition, $u^{\mathrm{new}}$ appears in $U_{i_0}^{\mathrm{new}}$. Then it means that for all $i < i_0$, we have $U_i^{\mathrm{new}} = U_i^{\mathrm{old}}$, as the previous sets have been made using the very same elements.

If $u^{\mathrm{new}} \notin \mathrm{span}_{\mathcal{M}/(\bigcup_{i=1}^{i_0-1} U_i^{\mathrm{old}})}(U_{i_0}^{\mathrm{new}} \backslash \{u^{\mathrm{new}}\})$, then the only possibility is $\rho_{\mathcal{M}'^{\mathrm{new}}/(\bigcup_{i=1}^{i_0-1} U_i^{\mathrm{old}})}(U_{i_0}^{\mathrm{new}}) = 1$ and we have our upper bound. From now on we assume that $u^{\mathrm{new}} \in \mathrm{span}_{\mathcal{M}/(\bigcup_{i=1}^{i_0-1} U_i^{\mathrm{old}})}(U_{i_0}^{\mathrm{new}} \backslash \{u^{\mathrm{new}}\})$.

Let $j = \min\{j \in [\![1,k]\!] : u^{\mathrm{new}} \in \mathrm{span}_{\mathcal{M}}(\bigcup_{i=1}^{j} U_i^{\mathrm{old}})\}$ ($j$ is well-defined by our assumption immediately before). Let $P_{i,i_0} = U_i^{\mathrm{old}} \cap U_{i_0}^{\mathrm{new}}$. Let $i_1$ be the largest $i$ such that $P_{i,i_0} \neq \emptyset$. As $u^{\mathrm{new}} \in \mathrm{span}_{\mathcal{M}}(\bigcup_{i=1}^{i_0-1} U_i^{\mathrm{old}} \cup \bigcup_{i=1}^{i_1} P_{i,i_0}) \subseteq \mathrm{span}_{\mathcal{M}}(\bigcup_{i=1}^{i_1} U_i^{\mathrm{old}})$ (by observing that $i_1 \geq i_0$ as $U_i^{\mathrm{new}} = U_i^{\mathrm{old}}$ for $i < i_0$), it means that $i_1 \geq j$. As a result we have

$$\begin{aligned}
\tilde{\rho}_{\mathcal{M}}^{\mathrm{new}}(u^{\mathrm{new}}) &= \rho_{\mathcal{M}'^{\mathrm{new}}/(\bigcup_{i=1}^{i_0-1} U_i^{\mathrm{old}})}(U_{i_0}^{\mathrm{new}}) \\
&\leq \rho_{\mathcal{M}'^{\mathrm{new}}/(\bigcup_{i=1}^{i_0-1} U_i^{\mathrm{old}} \cup \bigcup_{i=1}^{i_1-1} P_{i,i_0})}(P_{i_1,i_0} \cup \{u^{\mathrm{new}}\}) && \text{by Proposition 15} \\
&\leq \rho_{\mathcal{M}'^{\mathrm{new}}/(\bigcup_{i=1}^{i_0-1} U_i^{\mathrm{old}} \cup \bigcup_{i=1}^{i_1-1} P_{i,i_0})}(P_{i_1,i_0}) + 1 \\
&= \rho_{\mathcal{M}'^{\mathrm{old}}/(\bigcup_{i=1}^{i_0-1} U_i^{\mathrm{old}} \cup \bigcup_{i=1}^{i_1-1} P_{i,i_0})}(P_{i_1,i_0}) + 1 && \text{as } u^{\mathrm{new}} \notin \bigcup_{i=1}^{i_0-1} U_i^{\mathrm{old}} \cup \bigcup_{i=1}^{i_1} P_{i,i_0} \\
&\leq \rho_{\mathcal{M}'^{\mathrm{old}}/(\bigcup_{i=1}^{i_1-1} U_i^{\mathrm{old}})}(P_{i_1,i_0}) + 1 && \text{by Proposition 14} \\
&\leq \rho_{\mathcal{M}'^{\mathrm{old}}/(\bigcup_{i=1}^{i_1-1} U_i^{\mathrm{old}})}(U_{i_1}^{\mathrm{old}}) + 1 && \text{by maximality of } U_{i_1}^{\mathrm{old}} \\
&\leq \rho_{\mathcal{M}'^{\mathrm{old}}/(\bigcup_{i=1}^{j-1} U_i^{\mathrm{old}})}(U_j^{\mathrm{old}}) + 1 && \text{by Proposition 17 and } i_1 \geq j \\
&= \tilde{\rho}_{\mathcal{M}}^{\mathrm{old}}(u^{\mathrm{new}}) + 1.
\end{aligned}$$

For property (iv), we can first observe that the elements having densities larger than $\tilde{\rho}_{\mathcal{M}}^{\mathrm{new}}(u^{\mathrm{new}})$, which is upper bounded by $\tilde{\rho}_{\mathcal{M}}^{\mathrm{old}}(u^{\mathrm{new}})+1$, will remain with the same densities (actually they will even remain in the same sets $U_i$ as it was observed in the proof of (iii)). So we just focus on the case $\tilde{\rho}_{\mathcal{M}}^{\mathrm{old}}(v) < \tilde{\rho}_{\mathcal{M}}^{\mathrm{old}}(u^{\mathrm{new}})$. As $\tilde{\rho}_{\mathcal{M}}^{\mathrm{old}}(u^{\mathrm{new}}) > 0$, we can set $j = \min\{j \in [\![1,k]\!] : u^{\mathrm{new}} \in \mathrm{span}_{\mathcal{M}}(\bigcup_{i=1}^{j} U_i^{\mathrm{old}})\}$.

Let $U_{\mathrm{small}} = \bigcup_{i=j+1}^{k} U_i^{\mathrm{old}}$ and $U_{\mathrm{big}} = \bigcup_{i=1}^{j} U_i^{\mathrm{old}}$. First, suppose that there exists an element $v \in U_{\mathrm{small}}$ such that $\tilde{\rho}_{\mathcal{M}}^{\mathrm{new}}(v) \geq \tilde{\rho}_{\mathcal{M}}^{\mathrm{old}}(u^{\mathrm{new}})$. Let $P = \{v \in U_{\mathrm{small}} : \tilde{\rho}_{\mathcal{M}}^{\mathrm{new}}(v) \geq \tilde{\rho}_{\mathcal{M}}^{\mathrm{old}}(u^{\mathrm{new}})\}$. Let $P_i = P \cap U_i^{\mathrm{new}}$ for all $i$, and let $i_0$ be the smallest index $i$ such that $P_i \neq \emptyset$. Then

$$\begin{aligned}
\tilde{\rho}_{\mathcal{M}}^{\mathrm{new}}(v) &\leq \rho_{\mathcal{M}'^{\mathrm{new}}/(\bigcup_{i=1}^{i_0-1} U_i^{\mathrm{new}})}(U_{i_0}^{\mathrm{new}}) && \text{by Proposition 17} \\
&\leq \rho_{\mathcal{M}'^{\mathrm{new}}/(\bigcup_{i=1}^{i_0} U_i^{\mathrm{new}} \backslash P_{i_0})}(P_{i_0}) && \text{by Proposition 15} \\
&\leq \rho_{\mathcal{M}'^{\mathrm{new}}/(U_{\mathrm{big}} \cup \{u^{\mathrm{new}}\})}(P_{i_0}) && \text{as } \bigcup_{i=1}^{i_0} U_i^{\mathrm{new}} \backslash P_{i_0} \subseteq \bigcup_{i=1}^{j} U_i^{\mathrm{old}} \cup \{u^{\mathrm{new}}\} \\
&= \rho_{\mathcal{M}'^{\mathrm{old}}/(U_{\mathrm{big}})}(P_{i_0}) && \text{as } u^{\mathrm{new}} \in \mathrm{span}_{\mathcal{M}}(U_{\mathrm{big}}) \\
&\leq \rho_{\mathcal{M}'^{\mathrm{old}}/(U_{\mathrm{big}})}(U_{j+1}^{\mathrm{old}}) && \text{by maximality of } U_{j+1}^{\mathrm{old}} \\
&< \rho_{\mathcal{M}'^{\mathrm{old}}/(U_{\mathrm{big}} \backslash U_j^{\mathrm{old}})}(U_j^{\mathrm{old}}) && \text{by Proposition 17} \\
&= \tilde{\rho}_{\mathcal{M}}^{\mathrm{old}}(u^{\mathrm{new}}),
\end{aligned}$$

a contradiction to the assumption that $i_0$ exists, implying that $P = \emptyset$.

As a result, the set of elements in $V'$ of density smaller than $\tilde{\rho}_{\mathcal{M}}^{\mathrm{old}}(u^{\mathrm{new}})$ remains the same (note that this set cannot become bigger because of property (i)). Thereby, the execution of Algorithm 1 can be decomposed into two phases, the early phase when the elements of $U_{\mathrm{big}} \cup \{u^{\mathrm{new}}\}$ are processed, and the late phase when the elements of $U_{\mathrm{small}}$ are processed. As $\mathrm{span}_{\mathcal{M}}(U_{\mathrm{big}} \cup \{u^{\mathrm{new}}\}) = \mathrm{span}_{\mathcal{M}}(U_{\mathrm{big}})$, the construction of the sets in the late phase is the same no matter $u^{\mathrm{new}}$ is in $V'$ or not. Hence the sets are the same and so are the associated densities. This concludes the proof of (iv). □

*Proof of Lemma 21.* Consider the behavior when $u^{\mathrm{old}}$ is added to $V' \backslash \{u^{\mathrm{old}}\}$: it is clear that Lemma 20 applies. As a result the points (i) and (ii) come easily from Lemma 20 (ii). For (iii), observe that from Lemma 20 (iii) we get that $\tilde{\rho}^{\mathrm{new}}_{\mathcal{M}}(u^{\mathrm{old}}) \leq \tilde{\rho}^{\mathrm{old}}_{\mathcal{M}}(u^{\mathrm{old}}) \leq \tilde{\rho}^{\mathrm{new}}_{\mathcal{M}}(u^{\mathrm{old}}) + 1$ and hence we obtain also (iii) here. For (iv) the bounds are a bit different from what we could get from Lemma 20 (iv) but using ideas similar to that from the previous proof one can easily show the desired result. □

*Proof of Theorem 23.* Start with an empty subset $V'$. Then apply the following local improvement steps repeatedly on $V'$, until it is no longer possible. If an element in $V'$ violates Property (i) of Definition 3, then remove it from $V'$; similarly, if an element in $V \backslash V'$ violates Property (ii), insert it into $V'$. Note that among the two local improvement steps, the priority is given to the deletion operations.

Observe that when no element violates Property (i), all the elements have densities bounded by $\beta$ in both matroids. To prove that this algorithm terminates in finite time and to show the existence of a DCS, we introduce a potential function:

$$\Phi(V') = (2\beta - 7) \cdot |V'| - \sum_{l \in \{1,2\}} \left[ \sum_{j=1}^{k} \left( \mathrm{rank}_{\mathcal{M}'_l/(\bigcup_{i=1}^{j-1} U_{l,i})}(U_{l,j}) \cdot (\rho_{\mathcal{M}'_l/(\bigcup_{i=1}^{j-1} U_{l,i})}(U_{l,j}))^2 \right) \right]$$

where $U_{l,1}, \ldots, U_{l,k}$ denotes the density-based decomposition of $V'$ in $\mathcal{M}_l$ for $l \in \{1, 2\}$. We can rewrite this function in a more convenient form:

$$\Phi(V') = (2\beta - 7) \cdot |V'| - \sum_{l \in \{1,2\}} \left[ \sum_{j=1}^{k} \rho^2_{l,j} \right]$$

where for $l \in \{1, 2\}$, the vector $\rho_l = (\rho_{l,1}, \ldots, \rho_{l,k})$ is the list of the densities of each set of the decomposition $U_{l,1}, \ldots, U_{l,k}$ counted with multiplicity equal to their rank (so that, for instance, $\rho_{\mathcal{M}'_l/(\bigcup_{i=1}^{j-1} U_i)}(U_j)$ appears $\mathrm{rank}_{\mathcal{M}'_l/(\bigcup_{i=1}^{j-1} U_i)}(U_j)$ times in that vector; we potentially add some zeros in the end so that the vector has exactly $k$ components).

The execution of the algorithm can be seen as a series of batches of operations, consisting of one insertion operation followed by some number of deletion operations. Each batch has a finite size because we can make only a finite number of deletions when no insertion is performed. At the end of each batch of operations, all the densities are bounded by $\beta$, hence using Proposition 22 (as for this result to hold it is only required that the densities are bounded by $\beta$) we have that $\Phi$ is bounded by $(2\beta - 7) \cdot \beta \cdot \mu(V)$. Then we have to show that $\Phi$ increases at each local improvement step by at least some constant amount and we will be done.

When Property (i) of Definition 3 is not satisfied by some element in $u^{\mathrm{old}} \in V'^{\mathrm{old}}$, then it is removed to get a new set $V'^{\mathrm{new}} = V'^{\mathrm{old}} \backslash \{u^{\mathrm{old}}\}$. Hence from the vectors $\rho^{\mathrm{old}}_l = (\rho_{l,1}, \ldots, \rho_{l,k})$ we get new vectors $\rho^{\mathrm{new}}_l = (\rho_{l,1} - \lambda_{l,1}, \ldots, \rho_{l,k} - \lambda_{l,k})$, with the following properties:

- $\lambda_{l,i} \geq 0$ (by Lemma 21 (ii));
- $\sum_{j=1}^{k} \lambda_{l,j} = 1$ for $l \in \{1, 2\}$ (as we always have $\sum_{j=1}^{k} \rho_{l,j} = |V'|$, see Proposition 18);
- $\lambda_{l,i} > 0 \Rightarrow \tilde{\rho}^{\mathrm{old}}_{\mathcal{M}_l}(u^{\mathrm{old}}) - 1 \leq \rho_{l,i} \leq \tilde{\rho}^{\mathrm{old}}_{\mathcal{M}_l}(u^{\mathrm{old}})$ for $l \in \{1, 2\}$ (by Lemma 21 (iv)).

As a result we get:

$$\Phi(V'^{\mathrm{new}}) - \Phi(V'^{\mathrm{old}}) = -(2\beta - 7) + \sum_{l \in \{1,2\}} \left[ \sum_{j=1}^{k} \rho^2_{l,j} - (\rho_{l,j} - \lambda_{l,j})^2 \right]$$

$$
\begin{aligned}
&= -2\beta + 7 + \sum_{l\in\{1,2\}} \left[\sum_{j=1}^{k} 2\rho_{l,j}\lambda_{l,j} - \sum_{j=1}^{k} \lambda_{l,j}^2\right] \\
&\geq -2\beta + 5 + \sum_{l\in\{1,2\}} \left[\sum_{j=1}^{k} 2\rho_{l,j}\lambda_{l,j}\right] \\
&= -2\beta + 5 + \sum_{l\in\{1,2\}} \left[\sum_{\tilde{\rho}^{\text{old}}_{\mathcal{M}_l}(u^{\text{old}})-1\leq\rho_{l,j}\leq\tilde{\rho}^{\text{old}}_{\mathcal{M}_l}(u^{\text{old}})} 2\rho_{l,j}\lambda_{l,j}\right] \\
&\geq -2\beta + 5 + \sum_{l\in\{1,2\}} \left[2\cdot(\tilde{\rho}^{\text{old}}_{\mathcal{M}_l}(u^{\text{old}})-1) \sum_{\tilde{\rho}^{\text{old}}_{\mathcal{M}_l}(u^{\text{old}})-1\leq\rho_{l,j}\leq\tilde{\rho}^{\text{old}}_{\mathcal{M}_l}(u^{\text{old}})} \lambda_{l,j}\right] \\
&= -2\beta + 5 + 2\cdot(\tilde{\rho}^{\text{old}}_{\mathcal{M}_1}(u^{\text{old}}) + \tilde{\rho}^{\text{old}}_{\mathcal{M}_2}(u^{\text{old}}) - 2) \\
&> -2\beta + 1 + 2\beta = 1.
\end{aligned}
$$

The first inequality comes from $\lambda_{l,i} \geq 0$ and $\sum_{j=1}^{k} \lambda_{l,j} = 1$, implying that $\sum_{j=1}^{k} \lambda_{l,j}^2 \leq 1$. The last inequality comes from $\tilde{\rho}^{\text{old}}_{\mathcal{M}_1}(u^{\text{old}}) + \tilde{\rho}^{\text{old}}_{\mathcal{M}_2}(u^{\text{old}}) > \beta$. Hence we get an increase of $\Phi$ of at least 1.

Similarly, when Property (ii) of Definition 3 is not satisfied by some element in $u^{\text{new}} \in V\backslash V'^{\text{old}}$, then it is added to get a new set $V'^{\text{new}} = V'^{\text{old}} \cup \{u^{\text{new}}\}$. Hence from the vectors $\rho_l^{\text{old}} = (\rho_{l,1}, \ldots, \rho_{l,k})$ we get new vectors $\rho_l^{\text{new}} = (\rho_{l,1} + \lambda_{l,1}, \ldots, \rho_{l,k} + \lambda_{l,k})$, with the following properties:

- $\lambda_{l,i} \geq 0$ (by Lemma 20 (ii));
- $\sum_{j=1}^{k} \lambda_{l,j} = 1$ for $l \in \{1,2\}$ (as we always have $\sum_{j=1}^{k} \rho_{l,j} = |V'|$, see Proposition 18);
- $\lambda_{l,i} > 0 \Rightarrow \tilde{\rho}^{\text{old}}_{\mathcal{M}_l}(u^{\text{new}}) \leq \rho_{l,i} \leq \tilde{\rho}^{\text{old}}_{\mathcal{M}_l}(u^{\text{new}}) + 1$ for $l \in \{1,2\}$ (by Lemma 20 (iv)).

As a result we get:

$$
\begin{aligned}
\Phi(V'^{\text{new}}) - \Phi(V'^{\text{old}}) &= (2\beta - 7) - \sum_{l\in\{1,2\}} \left[\sum_{j=1}^{k} (\rho_{l,j}^2 + \lambda_{l,j})^2 - \rho_{l,j}^2\right] \\
&= 2\beta - 7 - \sum_{l\in\{1,2\}} \left[\sum_{j=1}^{k} 2\rho_{l,j}\lambda_{l,j} + \sum_{j=1}^{k} \lambda_{l,j}^2\right] \\
&\geq 2\beta - 9 - \sum_{l\in\{1,2\}} \left[\sum_{j=1}^{k} 2\rho_{l,j}\lambda_{l,j}\right] \\
&= 2\beta - 9 - \sum_{l\in\{1,2\}} \left[\sum_{\tilde{\rho}^{\text{old}}_{\mathcal{M}_l}(u^{\text{new}})\leq\rho_{l,j}\leq\tilde{\rho}^{\text{old}}_{\mathcal{M}_l}(u^{\text{new}})+1} 2\rho_{l,j}\lambda_{l,j}\right] \\
&\geq 2\beta - 9 - \sum_{l\in\{1,2\}} \left[2\cdot(\tilde{\rho}^{\text{old}}_{\mathcal{M}_l}(u^{\text{new}})+1) \sum_{\tilde{\rho}^{\text{old}}_{\mathcal{M}_l}(u^{\text{new}})\leq\rho_{l,j}\leq\tilde{\rho}^{\text{old}}_{\mathcal{M}_l}(u^{\text{new}})+1} \lambda_{l,j}\right] \\
&= 2\beta - 9 - 2\cdot(\tilde{\rho}^{\text{old}}_{\mathcal{M}_1}(u^{\text{new}}) + \tilde{\rho}^{\text{old}}_{\mathcal{M}_2}(u^{\text{new}}) + 2) \\
&> 2\cdot(\beta - \beta^-) - 13 \geq 1.
\end{aligned}
$$

The first inequality comes from $\lambda_{l,i} \geq 0$ and $\sum_{j=1}^{k} \lambda_{l,j} = 1$, implying that $\sum_{j=1}^{k} \lambda_{l,j}^2 \leq 1$. To move to the last line we use that $\tilde{\rho}^{\text{old}}_{\mathcal{M}_1}(u^{\text{new}}) + \tilde{\rho}^{\text{old}}_{\mathcal{M}_2}(u^{\text{new}}) < \beta^-$, and then that $\beta \geq \beta^- + 7$. Hence we also get an increase of $\Phi$ of at least 1.

As a result, a $(\beta, \beta^{-})$-DCS can be found in at most $2 \cdot \beta^2 \cdot \mu(V)$ such local improvement steps. $\square$

# References


[AB19] Sepehr Assadi and Aaron Bernstein. Towards a unified theory of sparsification for matching problems. In Jeremy T. Fineman and Michael Mitzenmacher, editors, *2nd Symposium on Simplicity in Algorithms, SOSA 2019, January 8-9, 2019, San Diego, CA, USA*, volume 69 of *OASICS*, pages 11:1–11:20. Schloss Dagstuhl - Leibniz-Zentrum für Informatik, 2019.

[AB21] Sepehr Assadi and Soheil Behnezhad. Beating two-thirds for random-order streaming matching. In Nikhil Bansal, Emanuela Merelli, and James Worrell, editors, *48th International Colloquium on Automata, Languages, and Programming, ICALP 2021, July 12-16, 2021, Glasgow, Scotland (Virtual Conference)*, volume 198 of *LIPIcs*, pages 19:1–19:13. Schloss Dagstuhl - Leibniz-Zentrum für Informatik, 2021.

[AB23] Amir Azarmehr and Soheil Behnezhad. Robust communication complexity of matching: EDCS achieves 5/6 approximation. In Kousha Etessami, Uriel Feige, and Gabriele Puppis, editors, *50th International Colloquium on Automata, Languages, and Programming, ICALP 2023, July 10-14, 2023, Paderborn, Germany*, volume 261 of *LIPIcs*, pages 14:1–14:15. Schloss Dagstuhl - Leibniz-Zentrum für Informatik, 2023.

[Ber20] Aaron Bernstein. Improved bounds for matching in random-order streams. In Artur Czumaj, Anuj Dawar, and Emanuela Merelli, editors, *47th International Colloquium on Automata, Languages, and Programming, ICALP 2020, July 8-11, 2020, Saarbrücken, Germany (Virtual Conference)*, volume 168 of *LIPIcs*, pages 12:1–12:13. Schloss Dagstuhl - Leibniz-Zentrum für Informatik, 2020.

[BHI18] Sayan Bhattacharya, Monika Henzinger, and Giuseppe F. Italiano. Dynamic algorithms via the primal-dual method. *Inf. Comput.*, 261:219–239, 2018.

[BK22] Soheil Behnezhad and Sanjeev Khanna. New trade-offs for fully dynamic matching via hierarchical EDCS. In Joseph (Seffi) Naor and Niv Buchbinder, editors, *Proceedings of the 2022 ACM-SIAM Symposium on Discrete Algorithms, SODA 2022, Virtual Conference / Alexandria, VA, USA, January 9 - 12, 2022*, pages 3529–3566. SIAM, 2022.

[BKS23] Sayan Bhattacharya, Peter Kiss, and Thatchaphol Saranurak. Sublinear algorithms for $(1.5+\epsilon)$-approximate matching. In Barna Saha and Rocco A. Servedio, editors, *Proceedings of the 55th Annual ACM Symposium on Theory of Computing, STOC 2023, Orlando, FL, USA, June 20-23, 2023*, pages 254–266. ACM, 2023.

[Bli21] Joakim Blikstad. Breaking $O(nr)$ for matroid intersection. In Nikhil Bansal, Emanuela Merelli, and James Worrell, editors, *48th International Colloquium on Automata, Languages, and Programming, ICALP 2021, July 12-16, 2021, Glasgow, Scotland (Virtual Conference)*, volume 198 of *LIPIcs*, pages 31:1–31:17. Schloss Dagstuhl - Leibniz-Zentrum für Informatik, 2021.

[BMNT23] Joakim Blikstad, Sagnik Mukhopadhyay, Danupon Nanongkai, and Ta-Wei Tu. Fast algorithms via dynamic-oracle matroids. *CoRR*, abs/2302.09796, 2023.

[BRR23] Soheil Behnezhad, Mohammad Roghani, and Aviad Rubinstein. Sublinear time algorithms and complexity of approximate maximum matching. In Barna Saha and Rocco A. Servedio, editors, *Proceedings of the 55th Annual ACM Symposium on Theory of Computing, STOC 2023, Orlando, FL, USA, June 20-23, 2023*, pages 267–280. ACM, 2023.

[BS15] Aaron Bernstein and Cliff Stein. Fully dynamic matching in bipartite graphs. In Magnús M. Halldórsson, Kazuo Iwama, Naoki Kobayashi, and Bettina Speckmann, editors, *Automata, Languages, and Programming - 42nd International Colloquium, ICALP 2015, Kyoto, Japan, July 6-10, 2015, Proceedings, Part I*, volume 9134 of *Lecture Notes in Computer Science*, pages 167–179. Springer, 2015.

[BS16] Aaron Bernstein and Cliff Stein. Faster fully dynamic matchings with small approximation ratios. In Robert Krauthgamer, editor, *Proceedings of the Twenty-Seventh Annual ACM-SIAM Symposium on Discrete Algorithms, SODA 2016, Arlington, VA, USA, January 10-12, 2016*, pages 692–711. SIAM, 2016.

[CCPV07] Gruia Călinescu, Chandra Chekuri, Martin Pál, and Jan Vondrák. Maximizing a submodular set function subject to a matroid constraint (extended abstract). In Matteo Fischetti and David P. Williamson, editors, *Integer Programming and Combinatorial Optimization, 12th International IPCO Conference, Ithaca, NY, USA, June 25-27, 2007, Proceedings*, volume 4513 of *Lecture Notes in Computer Science*, pages 182–196. Springer, 2007.

[CGQ15] Chandra Chekuri, Shalmoli Gupta, and Kent Quanrud. Streaming algorithms for submodular function maximization. In Magnús M. Halldórsson, Kazuo Iwama, Naoki Kobayashi, and Bettina Speckmann, editors, *Automata, Languages, and Programming - 42nd International Colloquium, ICALP 2015, Kyoto, Japan, July 6-10, 2015, Proceedings, Part I*, volume 9134 of *Lecture Notes in Computer Science*, pages 318–330. Springer, 2015.

[CKLV19] Flavio Chierichetti, Ravi Kumar, Silvio Lattanzi, and Sergei Vassilvitskii. Matroids, matchings, and fairness. In Kamalika Chaudhuri and Masashi Sugiyama, editors, *The 22nd International Conference on Artificial Intelligence and Statistics, AISTATS 2019, 16-18 April 2019, Naha, Okinawa, Japan*, volume 89 of *Proceedings of Machine Learning Research*, pages 2212–2220. PMLR, 2019.

[CLS+19] Deeparnab Chakrabarty, Yin Tat Lee, Aaron Sidford, Sahil Singla, and Sam Chiu-wai Wong. Faster matroid intersection. In David Zuckerman, editor, *60th IEEE Annual Symposium on Foundations of Computer Science, FOCS 2019, Baltimore, Maryland, USA, November 9-12, 2019*, pages 1146–1168. IEEE Computer Society, 2019.

[DFZ11] Randall Dougherty, Christopher F. Freiling, and Kenneth Zeger. Network coding and matroid theory. *Proc. IEEE*, 99(3):388–405, 2011.

[Edm70] Jack Edmonds. Submodular functions, matroids, and certain polyhedra. *Combinatorial Structures and Their Applications*, 1970.

[Edm71] Jack Edmonds. Matroids and the greedy algorithm. *Mathematical Programming*, 1(1):127–136, 1971.

[Edm79] Jack Edmonds. Matroid intersection. In P L Hammer, E L Johnson, and B H Korte, editors, *Discrete Optimization I: Proceedings of the Advanced Research Institute on Discrete Optimization and Systems Applications of the Systems Science Panel of NATO and of the Discrete Optimization Symposium*, volume 4, pages 39–49. Elsevier, 1979.

[FHM+20] Alireza Farhadi, Mohammad Taghi Hajiaghayi, Tung Mai, Anup Rao, and Ryan A. Rossi. Approximate maximum matching in random streams. In Shuchi Chawla, editor, *Proceedings of the 2020 ACM-SIAM Symposium on Discrete Algorithms, SODA 2020, Salt Lake City, UT, USA, January 5-8, 2020*, pages 1773–1785. SIAM, 2020.

[FKK18] Moran Feldman, Amin Karbasi, and Ehsan Kazemi. Do less, get more: Streaming submodular maximization with subsampling. In Samy Bengio, Hanna M. Wallach, Hugo Larochelle, Kristen Grauman, Nicolò Cesa-Bianchi, and Roman Garnett, editors, *Advances in Neural Information Processing Systems 31: Annual Conference on Neural Information Processing Systems 2018, NeurIPS 2018, December 3-8, 2018, Montréal, Canada*, pages 730–740, 2018.

[FKM+05] Joan Feigenbaum, Sampath Kannan, Andrew McGregor, Siddharth Suri, and Jian Zhang. On graph problems in a semi-streaming model. *Theoretical Computer Science*, 348:207–216, 12 2005.

[FNSZ20] Moran Feldman, Ashkan Norouzi-Fard, Ola Svensson, and Rico Zenklusen. The one-way communication complexity of submodular maximization with applications to streaming and robustness. In Konstantin Makarychev, Yury Makarychev, Madhur Tulsiani, Gautam Kamath, and Julia Chuzhoy, editors, *Proccedings of the 52nd Annual ACM SIGACT Symposium on Theory of Computing, STOC 2020, Chicago, IL, USA, June 22-26, 2020*, pages 1363–1374. ACM, 2020.

[Fuj08] Satoru Fujishige. Theory of principal partitions revisited. In William J. Cook, László Lovász, and Jens Vygen, editors, *Research Trends in Combinatorial Optimization, Bonn Workshop on Combinatorial Optimization, November 3-7, 2008, Bonn, Germany*, pages 127–162. Springer, 2008.

[GJS21] Paritosh Garg, Linus Jordan, and Ola Svensson. Semi-streaming algorithms for submodular matroid intersection. In Mohit Singh and David P. Williamson, editors, *Integer Programming and Combinatorial Optimization - 22nd International Conference, IPCO 2021, Atlanta, GA, USA, May 19-21, 2021, Proceedings*, volume 12707 of *Lecture Notes in Computer Science*, pages 208–222. Springer, 2021.

[GKK12] Ashish Goel, Michael Kapralov, and Sanjeev Khanna. On the communication and streaming complexity of maximum bipartite matching. In Yuval Rabani, editor, *Proceedings of the Twenty-Third Annual ACM-SIAM Symposium on Discrete Algorithms, SODA 2012, Kyoto, Japan, January 17-19, 2012*, pages 468–485. SIAM, 2012.

[GKMS19] Buddhima Gamlath, Sagar Kale, Slobodan Mitrovic, and Ola Svensson. Weighted matchings via unweighted augmentations. In Peter Robinson and Faith Ellen, editors, *Proceedings of the 2019 ACM Symposium on Principles of Distributed Computing, PODC 2019, Toronto, ON, Canada, July 29 - August 2, 2019*, pages 491–500. ACM, 2019.

[GP13] Manoj Gupta and Richard Peng. Fully dynamic $(1+\varepsilon)$-approximate matchings. In *54th Annual IEEE Symposium on Foundations of Computer Science, FOCS 2013, 26-29 October, 2013, Berkeley, CA, USA*, pages 548–557. IEEE Computer Society, 2013.

[GS17] Guru Prashanth Guruganesh and Sahil Singla. Online matroid intersection: Beating half for random arrival. In Friedrich Eisenbrand and Jochen Könemann, editors, *Integer Programming and Combinatorial Optimization - 19th International Conference, IPCO 2017, Waterloo, ON, Canada, June 26-28, 2017, Proceedings*, volume 10328 of *Lecture Notes in Computer Science*, pages 241–253. Springer, 2017.

[GSSU22] Fabrizio Grandoni, Chris Schwiegelshohn, Shay Solomon, and Amitai Uzrad. Maintaining an EDCS in general graphs: Simpler, density-sensitive and with worst-case time bounds. In Karl Bringmann and Timothy Chan, editors, *5th Symposium on Simplicity in Algorithms, SOSA@SODA 2022, Virtual Conference, January 10-11, 2022*, pages 12–23. SIAM, 2022.

[GT84] Harold N. Gabow and Robert Endre Tarjan. Efficient algorithms for a family of matroid intersection problems. *J. Algorithms*, 5(1):80–131, 1984.

[HS22] Chien-Chung Huang and François Sellier. Maximum weight b-matchings in random-order streams. In Shiri Chechik, Gonzalo Navarro, Eva Rotenberg, and Grzegorz Herman, editors, *30th Annual European Symposium on Algorithms, ESA 2022, September 5-9, 2022, Berlin/Potsdam, Germany*, volume 244 of *LIPIcs*, pages 68:1–68:14. Schloss Dagstuhl - Leibniz-Zentrum für Informatik, 2022.

[Kap13] Michael Kapralov. Better bounds for matchings in the streaming model. In Sanjeev Khanna, editor, *Proceedings of the Twenty-Fourth Annual ACM-SIAM Symposium on Discrete Algorithms, SODA 2013, New Orleans, Louisiana, USA, January 6-8, 2013*, pages 1679–1697. SIAM, 2013.

[Kap21] Michael Kapralov. Space lower bounds for approximating maximum matching in the edge arrival model. In Dániel Marx, editor, *Proceedings of the 2021 ACM-SIAM Symposium on Discrete Algorithms, SODA 2021, Virtual Conference, January 10 - 13, 2021*, pages 1874–1893. SIAM, 2021.

[Kis22] Peter Kiss. Deterministic dynamic matching in worst-case update time. In Mark Braverman, editor, *13th Innovations in Theoretical Computer Science Conference, ITCS 2022, January 31 - February 3, 2022, Berkeley, CA, USA*, volume 215 of *LIPIcs*, pages 94:1–94:21. Schloss Dagstuhl - Leibniz-Zentrum für Informatik, 2022.

[KMM12] Christian Konrad, Frédéric Magniez, and Claire Mathieu. Maximum matching in semi-streaming with few passes. In Anupam Gupta, Klaus Jansen, José D. P. Rolim, and Rocco A. Servedio, editors, *Approximation, Randomization, and Combinatorial Optimization. Algorithms and Techniques - 15th International Workshop, APPROX 2012, and 16th International Workshop, RANDOM 2012, Cambridge, MA, USA, August 15-17, 2012. Proceedings*, volume 7408 of *Lecture Notes in Computer Science*, pages 231–242. Springer, 2012.

[Kon18] Christian Konrad. A simple augmentation method for matchings with applications to streaming algorithms. In Igor Potapov, Paul G. Spirakis, and James Worrell, editors, *43rd International Symposium on Mathematical Foundations of Computer Science, MFCS 2018, August 27-31, 2018, Liverpool, UK*, volume 117 of *LIPIcs*, pages 74:1–74:16. Schloss Dagstuhl - Leibniz-Zentrum für Informatik, 2018.

[Kus97] Eyal Kushilevitz. Communication complexity. In *Advances in Computers*, volume 44, pages 331–360. Elsevier, 1997.

[Mes06] Julián Mestre. Greedy in approximation algorithms. In Yossi Azar and Thomas Erlebach, editors, *Algorithms - ESA 2006, 14th Annual European Symposium, Zurich, Switzerland, September 11-13, 2006, Proceedings*, volume 4168 of *Lecture Notes in Computer Science*, pages 528–539. Springer, 2006.

[Mur99] Kazuo Murota. *Matrices and matroids for systems analysis*, volume 20. Springer Science & Business Media, 1999.

[Rec89] András Recski. *Matroid Theory and its Applications in Electric Network Theory and in Statics*. Springer, 1989.

[RSW22] Mohammad Roghani, Amin Saberi, and David Wajc. Beating the folklore algorithm for dynamic matching. In Mark Braverman, editor, *13th Innovations in Theoretical Computer Science Conference, ITCS 2022, January 31 - February 3, 2022, Berkeley, CA, USA*, volume 215 of *LIPIcs*, pages 111:1–111:23. Schloss Dagstuhl - Leibniz-Zentrum für Informatik, 2022.

[Sch03] Alexander Schrijver. *Combinatorial optimization: polyhedra and efficiency*. Springer, 2003.

[XG94] Ying Xu and Harold N. Gabow. Fast algorithms for transversal matroid intersection problems. In Ding-Zhu Du and Xiang-Sun Zhang, editors, *Algorithms and Computation, 5th International Symposium, ISAAC '94, Beijing, P. R. China, August 25-27, 1994, Proceedings*, volume 834 of *Lecture Notes in Computer Science*, pages 625–633. Springer, 1994.