Homework Assignment 2

October 27, 2023

Please write your homework exericse using Latex. Please send me your answers by the 9th of November.

You are welcome to discuss among yourselves. But everyone must write down his/her own answers.

Question 1

Given a connected (and not necessarily simple) graph $G = (V, E)$ where each vertex has degree at least k (and to avoid triviality, assume that $|V| \geq 3$), prove the following: there exist at least two pairs of vertices (u, v) so that there are k edge-disjoint paths linking u and v.

You may want to look at the definition of Gomory-Hu trees very carefully.

Question 2

In this exercise, we again show a very cool application of Gomory-Hu tree.

Edmonds showed that the following linear program describes the convex hull of all perfect matchings in a general graph $G = (V, E)$.

$$\begin{aligned} x_e &\geq 0, \forall e \in E \\ \sum_{e \in \delta(v)} x_e &= 1, \forall v \in V \\ \sum_{e \in \delta(U)} x_e &\geq 1, \forall U \subseteq V, |U| \text{ is odd}, |U| \geq 3 \end{aligned}$$

This fact implies that to find a maximum weight perfect matching, given $w : E \to \mathbb{R}_{\geq 0}$, we just need to optimize over $\sum_{e \in E} w(e) x_e$. And this can be done by Ellipsoid algorithm—but there is a trouble: the number of inequalities is exponential. So we have to design a separation oracle. Given a fractional x, we need to find out which inequality is violated. The first two sets of inequalities are trivial to check. But the third set is problematic.

Let us proceed as follows: define the capacity of an edge $e \in E$ as x_e. The question then boils down to how to find an odd set $U \in V$, $|U| \geq 3$, whose cut size $\sum_{e \in \delta(U)} x_e$ is minimized.

Let us do the following. First build a Gomory-Hu tree $H = (V, F)$ on all vertices. For each edge $f \in F$, define V_f as the set of vertices corresponding to either of the two components in H, after f is removed. Suppose that U is the odd set with the minimum cut-size. Let $f_1, f_2, \cdots, f_t \in F$ denote the edges connecting a vertex in U and a vertex in $V \backslash U$ in H.

- Prove that the symmetric differences of $V_{f_1}, V_{f_2}, \cdots, V_{f_t}$ is equivalent to either U or $V \backslash U$.
- Then use this fact to design an algorithm to check whether some odd set (of size ≥ 3) has its cut size less than 1.

Question 3

Given two matroids M_1 and M_2, we have explained in class how Edmonds' algorithm can find the maximum common independent set in polynomial time. Here is a much simpler heuristics: initially let A be an empty set. If we can increase the size of A while guaranteeing that A is a common independent set in M_1 and M_2 by

- either adding an element into A,
- or adding two elements into A and removing one element from A at the same time,

then do it. We continue this process until A cannot be increased any further.

Prove that the final A is a 2/3-approximation of the maximum common independent set. A hint is to look carefully at the exchange graph $G(A)$ that is used by Edmonds' algorithm.