Homework Assignment

November 20, 2018

Please type down your answer using Latex. Please hand in your answers before the last class of the course (Dec 6).

**Question 1**

Given a graph $G = (V, E)$, where $|V|$ is of even size, and a family $\mathcal{F} = \{F_1, F_2, \cdots \}$ of subsets of vertices, we say $\mathcal{F}$ is a 2-packing if for each edge $e \in E$, $|\{F_i| e \in \delta(F_i), F_i \in \mathcal{F}\}| \leq 2$. Prove that $G$ does not have a perfect matching if there is a 2-packing $\mathcal{F}$ so that (1) $|\mathcal{F}| > n$, and (2) each member $F_i \in \mathcal{F}$ is of odd size.

Tutte’s theorem might be relevant here.

**Question 2**

Given a bipartite graph $G = (A \cup B, E)$ and a weight function $w \rightarrow \{1, 2\}$, below is an algorithm that solves the maximum weight matching problem.

First consider the subgraph $G' = (A \cup B, E')$ consisting of only edges $E' = \{e| w(e) = 2\}$. Compute a maximum cardinality matching $M_1$ in $G'$. Find a minimum integral vertex cover $C_1$. (Here we write $C_1(v) = 1$ if $v \in C_1$, otherwise, $C_1(v) = 0$). Now create a second graph $G'' = (A \cup B, E'')$ consisting of edges $E'' = \{e = (a, b)| w(e) - C_1(a) - C_1(b) = 1\}$.

Now again try to find a maximum cardinality matching $M_2$ in $G''$ under the condition that $M_2$ is derived from $M_1$ by augmenting (so you should first prove that all edges in $M_1$ are still part of $E''$.)

Prove that $M_2$ will be the optimal solution.

**Question 3**

Given a graph $G = (V, E)$ and an integral weight function $w \rightarrow \mathbb{Z}_{\geq 0}$, we need to define an integral potential $\pi$ over the vertices so that (1) for every edge $e = (a, b)$, $\pi(a) + \pi(b) \geq w(e)$, (2) $\pi(v) \in \mathbb{Z}_{\geq 0}$, $\forall v \in A \cup B$, and (3) under the previous two conditions, $\sum_{v \in V} \pi(v)$ is minimised.
When talking about the Hungarian method, we have shown that the problem can be solved in polynomial time if $G$ is bipartite. But if $G$ is non-bipartite, this problem becomes NP-complete (why?) Nonetheless, we can design approximation algorithms for this problem.

**Question 3.1**

Design a 2-approximation algorithm using the following ideas.

Write a linear program for this problem. Suppose that $x^*$ is a fractional optimal solution. (Then we can use the objective value of $x^*$ to lower bound the optimal solution.) Show that we can transform $x^*$ into a half-integral solution $y^*$ while keeping the objective undiminished. Then “round” $y^*$ to get an integral solution.

**Question 3.2**

Suppose that all edges weights $w(e) \geq 2$ (and still integral). Design a $\frac{4}{3}$-approximation algorithm.

**Question 4**

Assume that an undirected network $G = (V, E)$ is given. Along every edge $e \in E$, we assume that we can send flow in both directions.

We can define a circulation $\vec{f}$ as a function mapping every edge $e = (u, v)$ to an orientation $\vec{f}_e \in \{u, v\}$ and a volume $|\vec{f}_e| \geq 0$, so that for every node $v \in V$, $\sum_{e \in \delta(v)} |\vec{f}_e| = \sum_{e \in \delta(v), \vec{f}_e \neq u} |\vec{f}_e|$. Notice that if we define a circulation $\vec{f}$ like this, a directed cycle of $\vec{f}$ can be easily understood without ambiguity.

Given two circulation $\vec{f}^1$ and $\vec{f}^2$, we can define their difference $\vec{f}^1 - \vec{f}^2$ as follows. For every edge $e = (u, v)$,

1. if $\vec{f}^1_e = \vec{f}^2_e = u$ and $|\vec{f}^1_e| \geq |\vec{f}^2_e|$, then $\vec{f}^1_e - \vec{f}^2_e = u$ and $|\vec{f}^1_e - \vec{f}^2_e| = |\vec{f}^1_e| - |\vec{f}^2_e|$.
2. if $\vec{f}^1_e = \vec{f}^2_e = u$ and $|\vec{f}^1_e| < |\vec{f}^2_e|$, then $\vec{f}^1_e - \vec{f}^2_e = v$ and $|\vec{f}^1_e - \vec{f}^2_e| = |\vec{f}^2_e| - |\vec{f}^1_e|$.
3. if $\vec{f}^1_e = u$ and $\vec{f}^2_e = v$, then $\vec{f}^1_e - \vec{f}^2_e = u$ and $|\vec{f}^1_e - \vec{f}^2_e| = |\vec{f}^1_e| + |\vec{f}^2_e|$.

It can be easily verified that $\vec{f}^1 - \vec{f}^2$ is also a circulation.

Prove or disprove the following statement.

There always exists an agreeing cycle $C$ defined as follows. It is a directed cycle in $\vec{f}^i$, $i = 1$ or 2, so that on every edge $e = (u, v) \in C$,

1. $|\vec{f}^1_e| > 0$,
2. $|\vec{f}^1_e - \vec{f}^2_e| = 0$ or $\vec{f}^1_e - \vec{f}^2_e = \vec{f}^i_e$. 


Question 5

Consider the min-cost flow problem. Recall one optimality condition that we have used is the following:

Let $\pi : V \rightarrow Z$. For every edge $e = (u, v)$ in the residual network $G(f)$, define $c^\pi_e = c(e) - \pi(u) + \pi(v)$. $f$ is a min-cost flow if $c^\pi_e \geq 0$ for all edges $e \in G(f)$.

The above optimality condition in fact has an interpretation based on linear programming duality. First observe that the min-cost flow problem can be written as follows.

$$\begin{align*}
\min & \sum_{e \in E} c(e)x_e \\
\text{s.t.} & \sum_{e \in \delta^+(v)} x_e - \sum_{e \in \delta^-(v)} x_e = b(v) \quad \forall v \in V \\
& x_e \leq u(e) \quad \forall e \in E \\
& x_e \geq 0 \quad \forall e \in E
\end{align*}$$

Figure 1: The min-cost flow program

Question 5.1

Write down the dual program. Then flesh out the complementary slackness conditions for a pair of primal and dual optimal solutions.

Question 5.2

Prove next that given a pair of primal and dual optimal solutions, we can guarantee that there exists $\pi : V \rightarrow Z$ so that $c^\pi_e \geq 0$ for all edges $e \in G(f)$, where $f$ is the flow based on the primal solution.