Worst Case Analysis of Batch Arrivals with the Increasing Convex Ordering *

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Abstract. We consider a finite buffer queue with one deterministic server fed by packets arriving in batches. We assume that we are not able to fully describe the batch distribution: only the maximal size and the average number of packets are supposed known. Indeed, these two quantities are simple to measure in a real system. We additionally allow the batch distribution to be state dependent. We analyze the worst case distribution of the queue length and the expectation of lost packets per slot. We show that the increasing convex ordering provides tight bounds for such a system.

1 Introduction

In the case when we do not have complete information but some qualitative and quantitative information, a quite natural approach in many fields of applied probability consists in finding an extremal distribution. For instance, in reliability modelling, one can compute the worst case Increasing Failure Rate distribution knowing the first moment (for the definitions and method see Barlow and Proschan [2, p. 113]).

In Performance Evaluation such a method has received less attention. The major exception are the $(max, +)$ linear equations which naturally arise when one models Stochastic Event Graphs, a subset of Petri Nets (see the book by Baccelli et al. [1] for a considerable survey on these topics). Most of the results obtained in this book can be generalized to models exhibiting stochastic linear recurrence equations in some semirings: for instance (min, max) semiring or $(min, +)$ semiring. These results are based on the properties of the semirings: when we consider more complex algebraic structures most results do not apply any more.

A completely different idea was recently proposed by P. Buchholz [5]. The main assumption is that the modellers do not know the real transition probabilities. Thus, one wants to model a system by a family of Markov chains where the transition probabilities belong to an interval. One has to derive the worst case (or the best case) for all the matrices in the set. The theoretical arguments

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rely on Courtois's polyhedral approach. The algorithms are very accurate as the bounds can be reached by a matrix in the set. Unfortunately the complexity is quite high. Very recently a similar problem was solved independently by Haddad and Moreaux [9]. Again one has to find the best and the worst matrices in a set. However, Haddad and Moreaux's approach is based on strong stochastic ordering (st-ordering). The algorithm is simpler but the bounds are generally less accurate. To the best of our knowledge, the two approaches have not been compared on some benchmarking problems.

Our approach combines some of these ideas. We analyze a finite buffer queue with a deterministic service fed by a batch process. The batch distribution can be state dependant. We assume that we know the maximal batch size and the average number of packets in a batch. Note that both quantities are simple to obtain from the specifications of a system or from simple measurements. The maximal size of the batch is the number of inputs in a slotted system and the mean batch size is easily related to the load. A natural question when we analyze such a system is to find the worst batch distribution when we compute the distribution of the queue size, its average, or the average packet loss. Even if the system exhibits a simple evolution equation, the analysis is quite difficult. Indeed, due to the buffer finiteness this equation is based on three operators: max, $+$, and min, and the theory developed in [1] does not apply.

The infinite buffer case has already been studied by several authors [10, 11]. In that case the model has an evolution equation on max and + operators and the analysis is in general much simpler. Unfortunately, the finite buffer case introduces min operator and the underlying monotonicity disappears on the boundary of the state space.

We consider here a different approach based on Markov chains rather than evolution equations. We design an upper bounding monotone chain for the considered system in the sense of the increasing convex order (icx-order). This order is known for a long time [13] but only recently an algorithmic derivation of icx-monotone chains has been proposed [4]. The main advantage of this order is that it is possible to obtain a bound with the same mean as the initial distribution. Such a property is very important here to obtain tight bounds. This property is not valid with the usual st-ordering. Indeed, if X is smaller than Y in the sense of the st-order and if the expectation of X is equal to the expectation of Y, then X equals Y.

The problem we consider is related to the dimensioning of finite buffer in systems with fixed size packets: for instance ATM [14] or optical packet networks like ROM [8]. Such systems are slotted, thus discrete time chains provide natural models. The time slot is the service time and arrivals occur in batches of packets. The maximum batch size is the number of wavelengths in the optical transmission part of the network. The real distribution of batches is unknown and the traffic can be state-dependent. Instead of trying to give more and more details on the traffic, we try to derive a more pessimistic traffic. This traffic will be used to dimension the buffer. Hence our approach is quite different from the traditional traffic engineering approach.

The remaining of the paper is as follows. In Sect. 2 we briefly introduce the icx-order and the useful results proved in [13] and [4]. We also describe the worst case batch distribution in the sence of the icx-order. In Sect. 3 we construct an upper bounding icx-monotone Markov chain for a Batch/D/1/N queue where only the maximal and the average batch size are known, and we show that this chain provides a worst case bound for the queue length. We additionally show how we can use this bound to derive bounds on the number of lost packets. Finally in Sect. 4 we present some numerical results.

2 Some Preliminaries on Stochastic Bounds and the Icx-Worst Case Batch Distribution

In this section we first give some basic definitions and theorems of the stochastic comparison. We refer to [13] for proofs and further details. Then we consider a batch distribution whose average is known and we recall the worst case (largest) distribution for the icx-ordering [15].

2.1 Stochastic Comparison under the Icx Order

Definition 1. *Let X and Y be two random variables taking values on a totally ordered space $\mathcal{E}$. Then we say that X is smaller than Y in the increasing convex sense (icx),*

$$X \preceq_{icx} Y \;\; \textit{if} \;\; E(f(X)) \leq E(f(Y)), \;\textit{for all increasing and convex functions } f,$$

whenever the expectations exist.

In the case of a finite state space $\mathcal{E} = \{0, \ldots, N\}$, we have the following characterization of icx-comparison of two random variables.

Proposition 1. *Let X and Y be two random variables with probability vectors $p = (p_i)_{i=0}^N$ and $q = (q_i)_{i=0}^N$ ($p_i = P(X = i)$ and $q_i = P(Y = i)$, $\forall i$). Then,*

$$X \preceq_{icx} Y \Longleftrightarrow \sum_{k=i}^{N}(k-i+1)\; p_k \leq \sum_{k=i}^{N}(k-i+1)\; q_k,\; \forall i \in \{1, \ldots, N\}.$$

Recall that the usual strong stochastic order (st) is generated by the family of all increasing functions. Obviously, $X \preceq_{st} Y$ implies $X \preceq_{icx} Y$, as the family of all increasing functions is larger. Characterization of the st-comparison on a finite space $\mathcal{E} = \{0, \ldots, N\}$ is given by

$$X \preceq_{st} Y \Longleftrightarrow \sum_{k=i}^{N} p_k \leq \sum_{k=i}^{N} q_k,\; \forall i \in \{1, \ldots, N\}.$$

Example 1. Let us consider $\mathcal{E} = \{0, \ldots, 3\}$, and let

$$x = (0.5, 0.1, 0.1, 0.3),\ y = (0.3, 0.3, 0.1, 0.3),\ \text{and } z = (0.3, 0.2, 0.4, 0.1)$$

be probability vectors on $\mathcal{E}$. Then $x \preceq_{st} y$ and, therefore, $x \preceq_{icx} y$. The vectors x and z are not icx-comparable (and, consequently, not st-comparable), as $x_3 = 0.3 > 0.1 = z_3$, but $x_1 + 2x_2 + 3x_3 = 1.2 < 1.3 = z_1 + 2z_2 + 3z_3$. Finally, vectors y and z are not st-comparable, but $z \preceq_{icx} y$.

The stochastic comparison can be also defined on a process level.

Definition 2. *Let $\{X_k\}_{k\geq 0}$ and $\{Y_k\}_{k\geq 0}$ be two homogeneous Markov chains. Then,*

$$\{X_k\} \preceq_{icx} \{Y_k\},\ \textit{if } X_k \preceq_{icx} Y_k,\ \textit{for all } k \geq 0.$$

Let us now introduce the comparison and the monotonicity property for stochastic matrices. It is shown in Theorem 5.2.11. of [13, p.186] that comparison and monotonicity of the transition matrices of homogeneous discrete time Markov chains yield sufficient conditions to stochastically compare the underlying chains. Notice that Definitions 2, 3, 4, and Theorem 1 are also valid for the st-order.

Definition 3. *Let $\mathbf{P}$ and $\mathbf{Q}$ be two stochastic matrices. We say that $\mathbf{P} \preceq_{icx} \mathbf{Q}$ if*

$$P_{i,*} \preceq_{icx} Q_{i,*}, \quad \forall i \in \{0, \ldots, N\}$$

where $P_{i,*}$ denotes the i^{th} row of matrix $\mathbf{P}$.

Definition 4. *A stochastic matrix $\mathbf{P}$ is said to be icx-monotone if for any probability vectors p and q,*

$$p \preceq_{icx} q \implies p\mathbf{P} \preceq_{icx} q\mathbf{P}.$$

Theorem 1. *Two homogeneous Markov chains $\{X_k\}_{k\geq 0}$ and $\{Y_k\}_{k\geq 0}$ with the transition matrices $\mathbf{P}$ and $\mathbf{Q}$ satisfy $\{X_k\} \preceq_{icx} \{Y_k\}$, if*

- $X_0 \preceq_{icx} Y_0$,
- $\mathbf{P} \preceq_{icx} \mathbf{Q}$
- *at least one of matrices $\mathbf{P}$ or $\mathbf{Q}$ is icx-monotone.*

Definition 4 is not very useful in practical applications. We give here the algebraic characterization of icx-comparison for the finite space case. We refer to [3, 4] for the proof. Characterization for the icx-monotonicity for $\mathcal{E} = \mathbb{Z}$ can be found in [12].

Let $\mathbf{P}$ be a stochastic matrix taking values on $\mathcal{E} = \{0, \ldots, N\}$. Let us first introduce the following notations:

$$\begin{aligned} \phi_{i,j}(\mathbf{P}) &= \textstyle\sum_{k=j}^{N} (k - j + 1) P_{i,k}, && 0 \leq i \leq N,\ 0 \leq j \leq N, \\ \Delta_{i,j}(\mathbf{P}) &= P_{i,j} - P_{i-1,j}, && 1 \leq i \leq N,\ 0 \leq j \leq N. \end{aligned}$$

We will denote by $\phi(\mathbf{P})$ the matrix $\phi(\mathbf{P}) = (\phi_{i,j}(\mathbf{P}))_{i,j=0}^{N}$.

Proposition 2. *A stochastic matrix* $\mathbf{P}$ *taking values on* $\mathcal{E} = \{0, \ldots, N\}$ *is icx-monotone if and only if the vector*

$$\phi_{*,j}(\mathbf{P}) = (\phi_{i,j}(\mathbf{P}))_{i=0}^{N} \text{ is increasing and convex, for all } j \in \{1, \ldots, N\},$$

i.e.

$$\phi_{1,j}(\mathbf{P}) \geq \phi_{0,j}(\mathbf{P}) \text{ and } \phi_{i+1,j}(\mathbf{P}) + \phi_{i-1,j}(\mathbf{P}) \geq 2\phi_{i,j}(\mathbf{P}),$$

for all $i \in \{1, \ldots, N-1\}$, $j \in \{1, \ldots, N\}$. *Notice that the vector* $\phi_{*,j}(\mathbf{P})$ *is increasing and convex if and only if the vector* $\Delta_{*,j}(\phi(\mathbf{P}))$ *is non-negative and increasing.*

Example 2. Let us consider the two matrices

$$\mathbf{P} = \begin{pmatrix} 0.2 & 0.5 & 0.3 \\ 0.3 & 0.3 & 0.4 \\ 0.2 & 0.3 & 0.5 \end{pmatrix} \text{ and } \mathbf{Q} = \begin{pmatrix} 0.5 & 0.4 & 0.1 \\ 0.3 & 0.3 & 0.4 \\ 0.1 & 0.4 & 0.5 \end{pmatrix}.$$

Using Proposition 2 it can be easily shown that the matrix $\mathbf{P}$ is icx-monotone, while the matrix $\mathbf{Q}$ is not.

2.2 Icx-Worst Case Batch Distribution

We now study the existence and description of the worst case distribution within the family of all distributions with the same mean. Formally, let $\mathcal{F}_\alpha$ be the family of all probability distributions on the space $\mathcal{E} = \{0, \ldots, N\}$ having the same mean α. This family admits a greatest distribution under the icx-order.

Proposition 3. *The distribution* $q = (1 - \frac{\alpha}{N}, 0, \ldots, 0, \frac{\alpha}{N})$ *satisfies*

$$q \in \mathcal{F}_\alpha \text{ and } p \preceq_{icx} q, \text{ for all } p \in \mathcal{F}_\alpha.$$

See Theorem 2.A.9 of [15] for a proof.

Note that the family $\mathcal{F}_\alpha$ does not admit a greatest element under the st-order. Indeed, if for two random variables X and Y, $Y \preceq_{st} X$ and $E(X) = E(Y)$, then X and Y have the same distribution (see Theorem 1.2.9. of [13, p.5]).

In the next section we consider a finite capacity single server queue with batch arrivals and we are interested in queue length worst case analysis. Distribution q from Proposition 3 will be used to model the unknown batch distribution of a given mean, thus we will refer to it in the following shortly as to "the worst case batch".

Finally, it is worthy to remark that this distribution q is also an icx-bound for batch distributions whose mean is smaller than α.

3 Worst Case Analysis of a Batch/D/1/N Queue

We consider a finite capacity queue with a single server. The queue capacity is N. The service is deterministic and equals to one time slot. The queue is fed by a batch arrival process. We do not assume that the batch arrivals are i.i.d., for instance they can be state dependent. We suppose that we know the maximal size K of the batch. More precisely, let $A_i = (a_0^{(i)}, \ldots, a_K^{(i)})$ denote the distribution of the batch arrivals at state i. The exact values of $a_k^{(i)}$ $(0 \leq k \leq K)$ are unknown. We only know the mean batch size $\alpha = E(A_i)$. In order to have the mean load less than 1, we assume that $\alpha < 1$. Note that the maximum batch size is generally determined from the underlying physical system. For instance, in the case of optical networks the batch size is upper bounded by the number of wavelengths. Both parameters α and K are quite simple to measure or obtain from specifications.

3.1 Upper Bound for the Queue Length

We suppose that $K << N$ and that $0 < \alpha < 1$. We are interested in upper bounding the queue length of an arbitrary Batch/D/1/N queue with the maximal batch size equal to K and the mean batch size equal to α.

First step consists in finding the transition matrix $\mathbf{P}$ such that

$$\mathbf{R} \preceq_{icx} \mathbf{P},$$

for each transition matrix $\mathbf{R}$ of a Batch/D/1/N queue with the maximal batch size equal to K and the mean batch size equal to (or smaller than) α. From the description of icx-worst case batch in the previous section, we easily get:

$$\mathbf{P} = \begin{cases} P_{0,0} = 1 - \frac{\alpha}{K} & P_{0,K} = \frac{\alpha}{K} \\ i = 1, \ldots, N-K+1 : & P_{i,i-1} = (1 - \frac{\alpha}{K}) \quad P_{i,i+K-1} = \frac{\alpha}{K} \\ i = N-K+2, \ldots, N-1 : & P_{i,i-1} = (1 - \frac{\alpha}{N-i+1}) \quad P_{i,N} = \frac{\alpha}{N-i+1} \\ P_{N,N-1} = (1-\alpha) & P_{N,N} = \alpha \end{cases}$$

Notice that the rows $i = N-K+2, \ldots, N$ are obtained by taking the worst case batch (Proposition 3) with the mean batch size equal to α and the maximal batch size equal to $N-i+1$ (and not K), since we need to assure the icx-comparison of the unknown matrix $\mathbf{R}$ and the matrix $\mathbf{P}$, i.e. $\mathbf{R}_{i,*} \preceq_{icx} \mathbf{P}_{i,*}$, for all i. We want to emphasize that the matrix $\mathbf{P}$ actually belongs to the family of queues we want to bound. However, this matrix is not icx-monotone so we cannot directly apply Theorem 1.

Now we apply to $\mathbf{P}$ a linear transform which does not modify the steady-state distribution,

$$\mathbf{Q} = \delta\mathbf{P} + (1-\delta)\mathbf{Id},$$

where δ is a real constant, $0 < \delta < 1$. This transform was shown to improve the accuracy for st-bounds [6]. Here it has a crucial role as it allows to move some probability mass to the diagonal elements (see (3) and the proof of Theorem 2).

$$\mathbf{Q} = \begin{cases} Q_{0,0} = 1 - \delta\frac{\alpha}{K} & Q_{0,K} = \delta\frac{\alpha}{K} \\ i = 1, \ldots, N-K+1 : & \\ Q_{i,i-1} = \delta(1 - \frac{\alpha}{K}) & Q_{i,i} = 1-\delta \qquad Q_{i,i+K-1} = \delta\frac{\alpha}{K} \\ i = N-K+2, \ldots, N-1 : & \\ Q_{i,i-1} = \delta(1 - \frac{\alpha}{N-i+1}) & Q_{i,i} = 1-\delta \qquad Q_{i,N} = \delta\frac{\alpha}{N-i+1} \\ Q_{N,N-1} = \delta(1-\alpha) & Q_{N,N} = 1-\delta+\delta\alpha \end{cases} \tag{1}$$

Finally, we define the matrix $\mathbf{B}$ as follows:

$$\mathbf{B} = \begin{cases} B_{0,0} = 1 - \delta\frac{\alpha}{K} & B_{0,K} = \delta\frac{\alpha}{K} \\ i = 1, \ldots, N-K+1: & \\ B_{i,i-1} = \delta(1 - \frac{\alpha}{K}) & B_{i,i} = 1 - \delta \qquad B_{i,i+K-1} = \delta\frac{\alpha}{K} \\ i = N-K+2, \ldots, N-1: & \\ B_{i,i-1} = f_i & B_{i,i} = e_i \qquad B_{i,N} = \delta\frac{\alpha}{K}(i - N + K) \\ B_{N,N-1} = \delta(1-\alpha) & B_{N,N} = 1 - \delta + \delta\alpha \end{cases} \tag{2}$$

where $e_i = 1 - \delta + \delta\alpha - (N - i + 1)B_{i,N}$ and $f_i = 1 - e_i - B_{i,N}$.

This matrix $\mathbf{B}$ will be used to derive the worst case bounds for the underlying system. The proof of the following theorem is given in Appendix.

Theorem 2. *Suppose that*

$$\delta \leq \frac{1}{1+\alpha U}, \tag{3}$$

where $U = \max_{r=2\ldots K-1} \frac{r(K-r+1)}{K}$. *Then,*

1. $\mathbf{B}$ *is a stochastic matrix.*
2. $\mathbf{B}$ *is irreducible.*
3. $\mathbf{Q} \preceq_{icx} \mathbf{B}$.
4. $\mathbf{B}$ *is icx-monotone.*

Now $\mathbf{Q} \preceq_{icx} \mathbf{B}$ gives $\delta\mathbf{R} + (1-\delta)\mathbf{Id} \preceq_{icx} \mathbf{B}$, for each transition matrix $\mathbf{R}$ of a Batch/D/1/N queue with the mean batch size smaller or equal to α. Note that $\mathbf{B}$ is also icx-monotone. Therefore, it follows from Theorem 1 that

$$\pi_{\delta\mathbf{R}+(1-\delta)\mathbf{Id}} \preceq_{icx} \pi_{\mathbf{B}},$$

where $\pi_{\mathbf{A}}$ denotes the steady-state distribution, provided that it exists, of a Markov chain with the transition matrix $\mathbf{A}$. Since $\delta\mathbf{R} + (1-\delta)\mathbf{Id}$ and $\mathbf{R}$ have the same steady-state distribution, the matrix $\mathbf{B}$ provides an upper bound for the steady-state queue length distribution of a queue given by matrix $\mathbf{R}$, i.e.

$$\pi_{\mathbf{R}} \preceq_{icx} \pi_{\mathbf{B}}.$$

3.2 Deriving Bounds on Lost Packets

The bounds on the queue length we obtained in Sect. 3.1 can be also used to compute the bounds on the average number of lost packets per slot. As we consider the icx-order, we must prove that the rewards describing the mean number of lost packets are increasing and convex. Unfortunately they are not in general, thus we upper bound the rewards by an increasing and convex function. Recall that we do not know the real batch distribution.

Let us remind that we consider the state dependant batches, where $A_i = (a_0^{(i)}, \ldots, a_K^{(i)}) \in \mathcal{F}_\alpha$ denotes the distribution of batch arrivals in state i. Let us define a reward g, with $g(i)$ equal to the mean number of lost packets in state i,

$$g(i) = \begin{cases} 0, & 0 \leq i \leq N-K+1 \\ \sum_{k=0}^{K} P(A_i = k)(i - 1 + k - N)^+, & N-K+2 \leq i \leq N. \end{cases}$$

Proposition 4. *The reward g is upper bounded by the increasing and convex function h,*

$$h(i) = \begin{cases} 0, & 0 \le i \le N - K + 1 \\ r\frac{\alpha}{K}, & i = N - K + 1 + r,\ 1 \le r \le K - 1. \end{cases}$$

Proof. From $A_i \in \mathcal{F}_\alpha$ and Proposition 3 it follows that

$$A_i \preceq_{icx} q = (1 - \frac{\alpha}{K}, 0, \ldots, 0, \frac{\alpha}{K}), \quad (4)$$

for all $i \in \{0, \ldots, N\}$. On the other hand, for $i = N - K + 1 + r$,

$$g(i) = \sum_{k=0}^{K} a_k^{(i)} (i - 1 + k - N)^+ = \sum_{k=K-r+1}^{K} a_k^{(i)} (k - K + r),$$

for all $r \in \{1, \ldots, K - 1\}$. Now from (4) and Proposition 1 it follows that

$$g(i) = \sum_{k=K-r+1}^{K} a_k^{(i)} (k - K + r) \le \sum_{k=K-r+1}^{K} q_k (k - K + r),$$

for all $i = N - K + 1 + r,\ r \in \{1, \ldots, K - 1\}$. Notice that only the last term of the right side in the above equation is strictly positive, thus

$$g(i) \le r\frac{\alpha}{K} = h(i),$$

for all $i = N - K + 1 + r,\ r \in \{1, \ldots, K - 1\}$, and, therefore, $g \le h$. □

Finally we can bound the average number of lost packets per slot by the expectation of the reward h on the steady state distribution of matrix $\mathbf{B}$.

4 Numerical Results

As the matrices considered here are very small (up to one thousand states) we use GTH [7], a direct elimination algorithm which is known to be very accurate. First we show that the monotonicity constraints we impose on matrix $\mathbf{B}$ does not have a very important effect on the accuracy of the bound. Recall that the matrix $\mathbf{B}$ was constructed in three steps. First we found the matrix $\mathbf{P}$, the largest transition matrix in the sense of icx-order. There exists a state dependent batch which allows to reach this largest batch matrix. Then we compute matrix $\mathbf{Q}$ which has the same steady state distribution as $\mathbf{P}$. Finally, matrix $\mathbf{B}$ is built from $\mathbf{Q}$ to prove the monotone icx-bound at the steady state. Only the last step of the method can add some perturbation. Tables 1 and 2 illustrate the quality of the bound.

In Table 1 we report the average queue length. Clearly the relative errors are not very large when the load is light or moderate. At heavy load ($\alpha > 0.95$) they are still smaller than 0.5%.

	K=10			K=100		
α	**P**	**B**	rel. error	**P**	**B**	rel. error
0.5	5.000e+00	5.000e+00	<1.0e-15	5.000e+01	5.000e+01	5.292e-06
0.8	1.880e+01	1.880e+01	<1.0e-15	1.962e+02	1.965e+02	1.708e-03
0.9	4.140e+01	4.140e+01	1.602e-12	3.909e+02	3.924e+02	3.895e-03
0.95	8.645e+01	8.645e+01	4.452e-08	6.038e+02	6.060e+02	3.585e-03
0.99	3.984e+02	3.984e+02	1.670e-05	8.990e+02	8.999e+02	1.085e-03

Table 1. Comparison of the mean queue length at the steady-state between the "largest-batch" queue (**P**) and the monotone upper bound (**B**) for $N = 1000$

Let us know consider the probability that the queue is full (Table 2). The bounds are now less accurate, especially when the load is light. Even though the relative errors are significant, the probabilities are very small and the absolute errors are not so important. So we advocate that the bounds are tight. The analysis provides a bound which is very close to one matrix in the feasible set. We give in Figure 1 the evolution of the average queue length for the bound when we change the load or the maximum batch size.

	K=10			K=100		
α	**P**	**B**	rel. error	**P**	**B**	rel. error
0.5	1.375e-60	2.667e-60	9.404e-01	4.169e-07	3.299e-06	6.913e+00
0.8	1.646e-21	2.265e-21	3.759e-01	5.589e-03	1.341e-02	1.400e+00
0.9	9.240e-11	1.094e-10	1.838e-01	8.069e-02	1.261e-01	5.624e-01
0.95	1.154e-05	1.258e-05	9.056e-02	2.889e-01	3.619e-01	2.527e-01
0.99	1.057e-01	1.076e-01	1.788e-02	7.820e-01	8.184e-01	4.648e-02

Table 2. Comparison of $\pi(N)$ at the steady-state between the "largest-batch" queue (**P**) and the monotone upper bound (**B**) for $N = 1000$

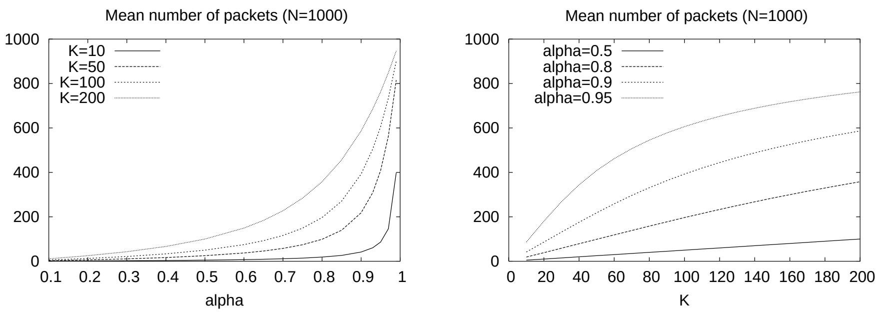

Fig. 1. Upper bounds for the mean number of packets for $N = 1000$

Now we consider a state dependent batch. We assume that the queue has some kind of back-pressure mechanism. When the queue size is large, a signal is sent to the sources of traffic to avoid congestion. We assume that this mechanism changes the variability of the traffic. The traffic still has the same average but the variability of the batch is now smaller. Typically a traffic shaper can have this effect. More formally we assume that the back-pressure signal is sent when the queue size is larger than 80% of the buffer size. We also assume that the signal instantaneously acts upon the source and that the effect ends when the queue size becomes smaller than the threshold. The batch distribution is the worst batch introduced in Sect. 2 when the queue size is small. When the queue becomes larger than the threshold we assume that the maximal batch size is now 2. Remember that the average batch size is still the same. We present in Table 3 the numerical results for the average number of packets in the queue.

	K=10			K=100		
α	**S**	**B**	rel. error	**S**	**B**	rel. error
0.5	5.000e+00	5.000e+00	<1.0e-15	5.000e+01	5.000e+01	2.755e-05
0.8	1.880e+01	1.880e+01	<1.0e-15	1.935e+02	1.965e+02	1.526e-02
0.9	4.140e+01	4.140e+01	8.916e-09	3.690e+02	3.924e+02	6.346e-02
0.95	8.644e+01	8.645e+01	9.122e-05	5.453e+02	6.060e+02	1.113e-01
0.99	3.780e+02	3.984e+02	5.396e-02	7.946e+02	8.999e+02	1.325e-01

Table 3. Comparison of the mean queue length at the steady-state between the state dependant "back-pressure mechanism batch" (**S**) and the monotone upper bound (**B**) for $N = 1000$

We compute the exact solution and the bound to check the accuracy of the approach for a large buffer ($N = 1000$). As expected, the bound is very accurate at light load for small and large values of K. At heavy load the relative errors are larger but it is still a good estimate.

Let us now consider the average number of lost packets per slot. We give in Figure 2 the evolution of the mean number of lost packets for the bound as a function of the load. In Figure 3 we compare this bound to the exact mean number of lost packets for the queue with the i.i.d. batch $(1 - \frac{\alpha}{K}, 0, \ldots, 0, \frac{\alpha}{K})$ ("0-K" batch). The approach is acceptable when the load is relatively light. At extremely heavy load (i.e. larger than 0.9) the bounds on the lost packets are not accurate.

5 Conclusion

In this paper, we have shown how we can provide a worst case analysis of a finite buffer queue with deterministic service and batch arrivals when the detailed description of the arrival process is not available. The approach is based on the derivation of a worst case matrix which is larger than the matrix in the set and

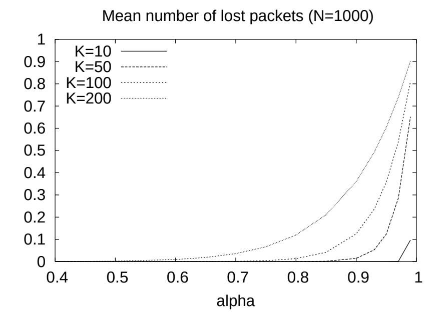

Fig. 2. Upper bounds for the mean number of lost packets.

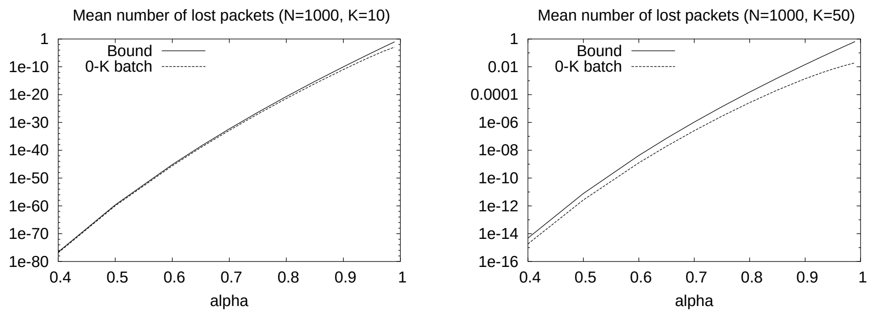

Fig. 3. Exact values and for the mean number of lost packets for "0-K" batch compared to the bound

which is also icx-monotone. Note that to the best of our knowledge it is not easy to apply the coupling method here because we use the icx-ordering rather than the st-ordering. We expect that such a method will help to dimension networking components because it is more and more difficult to really model the traffic characteristics and the worst case analysis is certainly a useful tool in the context of traffic engineering.

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Appendix

In this appendix we give the proof of Theorem 2. Let us first show some properties of diagonal and lower triangular entries of matrices $\mathbf{Q}$ and $\mathbf{B}$.

Lemma 1. *The diagonal entries of matrix $\phi(\mathbf{Q})$ have a constant value for all $i > 0$. Moreover, the diagonal of matrix $\phi(\mathbf{B})$ is equal to the diagonal of matrix $\phi(\mathbf{Q})$,*

$$\phi_{i,i}(\mathbf{B}) = \phi_{i,i}(\mathbf{Q}) = \begin{cases} 1 + \delta\alpha, & i = 0, \\ 1 - \delta + \delta\alpha, & \text{for all } i > 0. \end{cases}$$

Proof. Follows directly from the definitions of matrices $\mathbf{Q}$ and $\mathbf{B}$ (equations (1) and (2)). □

Lemma 2. *The lower triangle entries of matrices $\phi(\mathbf{Q})$ and $\phi(\mathbf{B})$ have the same values,*

$$\phi_{i,j}(\mathbf{Q}) = \phi_{i,j}(\mathbf{B}) = 1 - \delta + \delta\alpha + (i - j), \ j < i.$$

Proof. Notice that, for $0 \le i \le N, \ 0 \le j \le N - 1$,

$$\phi_{i,j}(\mathbf{P}) = \phi_{i,j+1}(\mathbf{P}) + \sum_{k=j}^{N} P_{i,k}. \tag{5}$$

The statement of the corollary follows directly from Lemma 1, (5), and the fact that $\sum_{k=j}^{N} Q_{i,k} = \sum_{k=j}^{N} P_{i,k} = 1$, for all i, j such that $j < i$. $\square$

Proof of Theorem 2.

1) $\mathbf{B}$ *is a stochastic matrix.* Notice that rows $0, \ldots, N-K+1$ and row N are the same for matrices $\mathbf{Q}$ and $\mathbf{B}$. For a row $i = N-K+r$, where $2 \leq r \leq K-1$, we have $0 < \frac{r}{K} < 1$ and $B_{i,N} = \delta\alpha\frac{r}{K}$, thus

$$0 < B_{i,N} < 1, \ i = N-K+2, \ldots, N-1. \tag{6}$$

It remains us to show that

$$B_{i,i} = e_i \geq 0 \text{ and } e_i + B_{i,N} \leq 1, \ i = N-K+2, \ldots, N-1.$$

Then $B_{i,i-1} = f_i \geq 0$ and $\sum_{j=0}^{N} B_{i,j} = 1, \ i = N-K+2, \ldots, N-1$. For a row $i = N-K+r$, where $2 \leq r \leq K-1$, we have

$$e_i = 1 - \delta + \delta\alpha - \delta\alpha\frac{r(K-r+1)}{K} \geq 1 - \delta + \delta\alpha - \delta\alpha U. \tag{7}$$

Now from (3) and $\delta\alpha > 0$ it follows that $B_{i,i} = e_i > 0, \ i = N-K+2, \ldots, N-1$.

For a row $i = N-K+r$, where $2 \leq r \leq K-1$,

$$e_i + B_{i,N} = 1 - \delta(1-\alpha) - \delta\alpha\frac{r(K-r)}{K} < 1, \tag{8}$$

since $0 < \alpha < 1$. Thus,

$$B_{i,i-1} = f_i = 1 - e_i - B_{i,N} > 0, \ i = N-K+2, \ldots, N-1, \tag{9}$$

and $\mathbf{B}$ is a stochastic matrix.

2) $\mathbf{B}$ *is irreducible.* Follows easily from (2) and the fact that $0 < \alpha, \delta < 1$.

3) $\mathbf{Q} \preceq_{icx} \mathbf{B}$*, i.e.* $\phi_{i,j}(\mathbf{Q}) \leq \phi_{i,j}(\mathbf{B}), \ i = 0, \ldots, N, \ j = 1, \ldots, N$. We need to consider only the rows $i = N-K+2, \ldots, N-1$, as the remaining ones are the same for both matrices. Furthermore, from Lemmas 1 and 2 it follows that

$$\phi_{i,j}(\mathbf{Q}) = \phi_{i,j}(\mathbf{B}), \ j \leq i.$$

On the other hand, from the definition of matrices $\mathbf{Q}$ and $\mathbf{B}$ we have $\phi_{i,j}(\mathbf{Q}) = (N-j+1)Q_{i,N}$ and $\phi_{i,j}(\mathbf{B}) = (N-j+1)B_{i,N}, \ N-K+2 \leq i < j < N$. Therefore, we need only to verify that

$$\phi_{i,N}(\mathbf{Q}) \leq \phi_{i,N}(\mathbf{B}), \ N-K+2 \leq i \leq N-1.$$

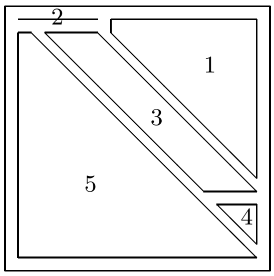
For a row $i = N-K+r, \ 2 \leq r \leq K-1$, we have

$$\begin{aligned} \phi_{i,N}(\mathbf{Q}) \leq \phi_{i,N}(\mathbf{B}) &\Leftrightarrow Q_{i,N} \leq B_{i,N} \\ &\Leftrightarrow \frac{1}{K-r+1} \leq \frac{1}{K}r \\ &\Leftrightarrow r^2 - (K+1)r + K \leq 0 \end{aligned}$$

The above second order equation has two real roots: 1 and K. Thus, for $r = 2, \ldots, K-1$, $r^2 - (K+1)r + K < 0$. Therefore, $\mathbf{Q} \preceq_{icx} \mathbf{B}$.

4) $\mathbf{B}$ *is icx-monotone.* After Proposition 2 this is equivalent to show that $\phi_{*,j}(\mathbf{B})$ is an increasing and convex vector, i.e. that $\Delta_{*,j}(\phi(\mathbf{B}))$ is a non-negative and increasing vector for all $j = 1, \ldots, N$.

We will consider the partition of matrix $\phi(\mathbf{B})$ into the following zones:



1. $i = 0$, $K+1 \le j \le N$ and $1 \le i \le N-K$, $i+K \le j \le N$
2. $i = 0$, $0 \le j \le K$
3. $1 \le i \le N-K+1$, $i+1 \le j \le i+K-1$
4. $N-K+2 \le i \le N-1$, $i+1 \le j \le N$
5. $1 \le i \le N$, $0 \le j \le i$

Matrix $\phi(\mathbf{B})$ can be then written as follows:

$$\text{Zone 1}: \phi_{i,j}(\mathbf{B}) = 0,$$

$$\text{Zone 2}: \phi_{0,0}(\mathbf{B}) = \delta(1+\alpha),\ \phi_{0,j}(\mathbf{B}) = (K-j+1)\delta\frac{\alpha}{K},\ 1 \le j \le K,$$

$$\text{Zone 3}: \phi_{i,j}(\mathbf{B}) = (i+K-j)\delta\frac{\alpha}{K}, \tag{10}$$

$$\text{Zone 4}: \phi_{i,j}(\mathbf{B}) = (i-N+K)(N-j+1)\delta\frac{\alpha}{K}, \tag{11}$$

$$\text{Zone 5}: \text{(Lemmas 1 and 2)}$$

$$\phi_{i,j}(\mathbf{B}) = 1 - \delta + \delta\alpha + (i-j). \tag{12}$$

Zone 1 is trivial as $\phi_{*,j}(\mathbf{B})$ has a constant value 0 within this zone for all j. Notice that for an arbitrary column $\phi_{*,j}(\mathbf{B})$, inside of zones 3, 4, and 5 we have a linear increase:

$$\Delta_{i,j}(\phi(\mathbf{B})) = \delta\frac{\alpha}{K},\ \text{for all } (i-1,j), (i,j) \text{ in zone 3}, \tag{13}$$

$$\Delta_{i,j}(\phi(\mathbf{B})) = (N-j+1)\delta\frac{\alpha}{K},\ \text{for all } (i-1,j), (i,j) \text{ in zone 4}, \tag{14}$$

$$\Delta_{i,j}(\phi(\mathbf{B})) = 1,\ \text{for all } (i-1,j), (i,j) \text{ in zone 5}. \tag{15}$$

Notice that $\delta\frac{\alpha}{K} \le (N-j+1)\delta\frac{\alpha}{K}$, since $j \le N$. Furthermore, inside of zone 4 we have $j \ge i+1 \ge N-K+3$. Thus, $(N-j+1)\delta\frac{\alpha}{K} \le \delta\alpha\frac{K-3}{K} < 1$, as $\delta\alpha < 1$.

For all j, $1 \le j \le N$, column $\Delta_{*,j}(\phi(\mathbf{B}))$ has thus a constant, non-negative value within each of the zones 1, 3, 4, and 5. Additionally, those constants are increasing with the respect of the number of the zone. Notice that the zones are ordered in such a way that each column j crosses the zones in increasing order with respect to the row index.

Some special care has to be done at the boundaries between different zones. We illustrate the procedure on the example of boundaries $3-4$ and $4-5$. The proof for other boundaries is simpler and it is omitted due to the lack of space.

Boundary $3-4$. We have to show that

$$\Delta_{N-K+1,j}(\phi(\mathbf{B})) \leq \Delta_{N-K+2,j}(\phi(\mathbf{B})) \leq \Delta_{N-K+3,j}(\phi(\mathbf{B})), \tag{16}$$

for all $N-K+3 \leq j \leq N$. From (13) for $j < N$, $\phi_{N-K+1,N}(\mathbf{B}) = \delta\frac{\alpha}{K}$, and $\phi_{N-K,N}(\mathbf{B}) = 0$ it follows that $\Delta_{N-K+1,j}(\phi(\mathbf{B})) = \delta\frac{\alpha}{K}$, $N-K+3 \leq j \leq N$. Equations (11) for $(N-K+2, j)$ and (10) for $(N-K+1, j)$ imply

$$\Delta_{N-K+2,j}(\phi(\mathbf{B})) = (N-j+1)\delta\frac{\alpha}{K},\ N-K+3 \leq j \leq N.$$

Thus, the left inequality in (16) holds.

Equation (14) implies $\Delta_{N-K+3,j}(\phi(\mathbf{B})) = (N-j+1)\delta\frac{\alpha}{K}$, $N-K+4 \leq j \leq N$. Thus, the right inequality in (16) holds for $N-K+4 \leq j \leq N$. It remains us to show $\Delta_{N-K+2,N-K+3}(\phi(\mathbf{B})) \leq \Delta_{N-K+3,N-K+3}(\phi(\mathbf{B}))$. Equations (12) for $(N-K+3, N-K+3)$ and (11) for $(N-K+2, N-K+3)$ imply $\Delta_{N-K+3,N-K+3}(\phi(\mathbf{B})) = 1-\delta+\delta\alpha-2(K-2)\delta\frac{\alpha}{K}$. Therefore,

$$\begin{aligned} &\Delta_{N-K+2,N-K+3}(\phi(\mathbf{B})) \leq \Delta_{N-K+3,N-K+3}(\phi(\mathbf{B})) \\ \Leftrightarrow\ & 1-\delta+\delta\alpha-\delta\alpha\frac{3(K-2)}{K} \geq 0. \end{aligned}$$

Proposition hypothesis (3) implies $\delta(1+\alpha\frac{3(K-2)}{K}) \leq 1$. Thus, the right inequality in (16) holds also for $j = N-K+3$.

Boundary $4-5$. We have to show that

$$\Delta_{i-1,i}(\phi(\mathbf{B})) \leq \Delta_{i,i}(\phi(\mathbf{B})),\ N-K+3 \leq i \leq N, \text{ and} \tag{17}$$

$$\Delta_{i,i}(\phi(\mathbf{B})) \leq \Delta_{i+1,i}(\phi(\mathbf{B})),\ N-K+3 \leq i \leq N-1. \tag{18}$$

From (11) for $(N-K+2, N-K+3)$, (10) for $(N-K+1, N-K+3)$, and (14) for $i > N-K+3$, it follows that

$$\Delta_{i-1,i}(\phi(\mathbf{B})) = (N-i+1)\delta\frac{\alpha}{K},\ N-K+3 \leq i \leq N.$$

Equations (12) for (i,i), (11) for $(i-1,i)$, and (15) give

$$\begin{aligned} \Delta_{i,i}(\phi(\mathbf{B})) &= 1-\delta+\delta\alpha-(i-1-N+K)(N-i+1)\delta\frac{\alpha}{K}, \\ &\qquad N-K+3 \leq i \leq N, \\ \Delta_{i+1,i}(\phi(\mathbf{B})) &= 1,\ N-K+3 \leq i \leq N-1. \end{aligned}$$

Since $\alpha < 1$ and $(i-N+K)(N-i) > 0$, $N-K+3 \leq i \leq N-1$, (18) holds.

In order to show (17), we have to show that

$$1-\delta+\delta\alpha-(i-N+K)(N-i+1)\delta\frac{\alpha}{K} \geq 0,\ N-K+3 \leq i \leq N. \tag{19}$$

Notice that, for $N-K+3 \leq i \leq N-1$, the left side of the above equation is equal to $B_{i,i} = e_i$ (see (7)), and we have already proved that, under the hypothesis of the proposition, $e_i \geq 0, N-K+2 \leq i \leq N-1$. It remains us to show (19) for $i = N$. We have

$$1-\delta+\delta\alpha-K\delta\frac{\alpha}{K} = 1-\delta \geq 0.$$

Thus, (18) holds for all i, $N-K+3 \leq i \leq N$. □