Level Crossing Ordering of Markov Chains: Computing End to End Delays in an All Optical Network

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ABSTRACT

We advocate the use of level crossing ordering of Markov chains and we present two applications of this ordering to analyze the deflection routing in an all optical packet network. As optical storage of packets is not available, we assume that the routing protocol is based on deflection. This routing strategy does not allow packet loss. However it keeps the packets inside the network, increases the delay and reduces the bandwidth. Thus the transport delay distribution is the key performance issue for these networks. Here, we consider the deflection routing of a packet in a hypercube. First we assume that the deflection probability is known and we build an absorbing Markov chain to model the packet inside the network. Then we present a more abstract model of the topology and we show that under weak assumptions bounds on the deflection probability provide bounds on the end to end delay. This result is based on level-crossing comparison of Markov chains. Then we present an approximate model of the switch to obtain a fixed point system between two sub-models. The first subsystem describes the global network performance while the other one models the stochastic behavior of the packet. The fixed point system is solved by a numerical algorithm and the convergence of this algorithm is proved using again the theory of level crossing comparison of Markov chains. Proving convergence is a new application for the theory of Markov chain comparison and this example can be generalized to many algorithms based on Markov chains.

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1. INTRODUCTION

All optical packet networks have received considerable attention during the last years. However with actual technology, all-optical networks do not allow the buffering of packets inside the network. Fiber delay loops allow to keep some packets inside a switch for some time but they are not functionally equivalent to a Random Access Memory. Therefore packets have to be sent immediately to the next switch along the path. Old algorithms like Deflection Routing [2] have recently received attention to overcome this weakness [9, 11]. This routing strategy does not allow packet loss but it keeps the packets inside the network, increases the delay and reduces the bandwidth. In Shortest-Path Deflection Routing, switches attempt to forward packets along a shortest hop path to their destination. Each link can send a finite number of packets per time-slot: this is the link capacity. If the number of packets which require a link is larger than the capacity, only some of them will use the link they ask for and the other ones have to be misdirected or deflected and they will travel through longer paths. This is the major drawback of this routing technique. The tail of the transportation delay and the average usable bandwidth are therefore two major measures of interest. As acknowledgments in networking protocols must arrive before some timer expiration, heavily deflected packets will be considered as lost because they experience delays larger than the transport time-out. Packets are never physically lost due to physical errors or buffer congestion, but they can be logically lost because the transport delay is too large.

Deflection routing has been studied for a long time but to the best of our knowledge all the analysis published so far has studied the average throughput for some simple topology and simple switch architecture. For instance one can find in [3, 1] models for networks based on 2×2 switching blocks without the queueing of new packets. Recently, Fabrega and Muñoz [6] have modeled a network with deflection routing using an approximate model based on Markov chains. However, they have only considered 2×2 switches and a topology such that only one shortest path exists between the source and the destination. Yao et al. have presented in [14] an approximate model for a quite restrictive topology which do not contain any directed cycle. Clearly, all these methods only provide approximate mean delay or throughput while the important measure is the tail of the delay distribution. Therefore, new methods to obtain the distribution of the delay are still necessary.

As we assume that there is no memory in the switches, it is difficult to apply classical queueing theory to analyze the delay. Thus we advocate the use of Markov chain to build the model of a packet and comparison of chains to obtain upper bounds on the end to end delay in the network. Instead of the classical sample-path ordering associated to strong stochastic monotonicity we consider a weaker comparison technique: the level crossing comparison of Markov chains. This new concept has been introduced by Irle and Gani in [10] to compare stochastic processes, motivated by word detection in a sequence of independent drawings of letters over a finite alphabet.

The level crossing ordering of Markov chains is based on the intuitive idea that the smaller chain should take longer to cross any fixed level in the state space. Thus it is naturally associated to absorption time. Absorption time computation problem arises naturally in various domains. For instance the end to end delay of a packet in a network, lifetime of a component of a system or a task duration can be modeled as an absorbing Markov chain. We have often only some partial information on the parameters of the model, so we must find a bounding chain for a family of chains corresponding to all the potential values of unknown parameters. Of course strong stochastic ordering of Markov chains also allows to compare distributions of time to reach an absorbing state. But the stochastic monotonicity associated with this ordering usually implies less accurate bounds.

The remaining of the paper is as follows: in section 2 we present a brief introduction to the level crossing ordering of Markov chains. Section 3 is devoted to the description of the model of a packet routing inside a hypercube. In section 4 we prove the comparison theorems to bound the end to end delay. Then in section 5 we provide a general algorithm which builds a fixed point system on the deflection probability and we prove the convergence of this algorithm under some intuitive assumptions. Again this proof is based on the level crossing comparison of chains and the bounded convergence theorem. In section 6 we present a simple approximation to model the deflection probability as the result of independent packets in a fair competition and we check the assumptions of the general algorithm we have formerly proved. We advocate that this proof of convergence may be generalized to many algorithms based on fixed point systems on Markov chains which are generally left unproved. To the best of our knowledge this proof is a new application of the theory of Markov chain comparison.

2. A BRIEF INTRODUCTION TO LEVEL CROSSING ORDERING OF MARKOV CHAINS

Let us first define the strong stochastic ordering of random variables ("st-ordering" for short). This ordering is defined by means of the set of increasing functions. We consider here only discrete (finite or infinite) random variables. Indeed, we will consider the set $\{0, 1, 2, \dots, n\}$ for the state space of a Markov chain or $\mathbb{N} \cup \infty$ for the comparison of absorption times. For a far more general introduction to stochastic orders we refer to [12].

Definition 1. *For two random variables X and Y we say that X is smaller than Y in a strong stochastic sense, denoted by $X \leq_{st} Y$, if*

$$E[f(X)] \leq E[f(Y)],$$

for all increasing real functions f.

For discrete random variables, we use the following algebraic equivalent formulation which is far more convenient (see [12] for the equivalence) :

Definition 2. *If X and Y are discrete random variables having respectively p and q as probability distribution vectors, then X is said to be less than Y in the strong stochastic sense, that is $X \leq_{st} Y$, if*

$$\sum_{j \geq k} p_j \leq \sum_{j \geq k} q_j, \text{ for all } k.$$

Let us now illustrate definition 2 by an example:

Example 1. *Let $\alpha = (0.1, 0.3, 0.4, 0.2)$ and $\beta = (0.1, 0.1, 0.5, 0.3)$. It follows then that $\alpha \leq_{st} \beta$ since:*

$$\begin{bmatrix} 0.2 & \leq & 0.3 \\ 0.2 + 0.4 & \leq & 0.3 + 0.5 \\ 0.2 + 0.4 + 0.3 & \leq & 0.3 + 0.5 + 0.1 \end{bmatrix}$$

For two Markov chains the st-comparison is usually defined as the comparison at each time step:

Definition 3. *Let $\{X_k\}_{k\geq 0}$ and $\{Y_k\}_{k\geq 0}$ be two DTMC on the state space $\{0, \dots, n\}$. We say that the chain $\{X_k\}$ is st-smaller than $\{Y_k\}$, denoted by $\{X_k\} \leq_{st} \{Y_k\}$, if*

$$X_k \leq_{st} Y_k, \text{ for all } k \geq 0.$$

Note that bounds on a distribution imply bounds on performance measures that are increasing functions of the state indices (see definition 1).

We can also compare the transition matrices of Markov chains. We denote by $P_{i,*}$ row i of matrix P. The st-comparison of two transition matrices is defined as the st-comparison of the corresponding rows.

Definition 4 (st-comparison of trans. matrices). *Let P and R be two transition matrices. We say P is st-smaller than R, denoted by $P \leq_{st} R$, if $P_{i,*} \leq_{st} R_{i,*}$ for all i, that is if $\sum_{k=j}^{n} P_{i,k} \leq \sum_{k=j}^{n} R_{i,k}$ for all i and j between 0 and n.*

Definition 5 (st-monotonicity). *Let P be a transition matrix. P is st-monotone if $P_{i-1,*} \leq_{st} R_{i,*}$ for all $i > 1$, that is $\sum_{k=j}^{n} P_{i-1,k} \leq \sum_{k=j}^{n} P_{i,k}$, for all i between 1 and n and for all j between 0 and n.*

It is known for some time that monotonicity and comparability of transition probability matrices yield sufficient conditions for the st-comparison of Markov chains and their transient and steady-state distributions [12]:

THEOREM 1. *Let $\{X_k\}_{k\geq 0}$ and $\{Y_k\}_{k\geq 0}$ be two DTMC and P and R be their respective transition matrices. If*

- $X_0 \leq_{st} Y_0$,
- *at least one transition matrix P or R is st-monotone,*
- $P \leq_{st} R$,

then $X_k \leq_{st} Y_k$, for all $k \geq 0$. If X and Y have steady-state distributions π_X and π_Y, then $\pi_X \leq_{st} \pi_Y$.

Transient and steady-state distributions provide often enough informations on the performance of a studied system. However, in optical networks modeling the far more important measure is the end to end delay which can be modeled as the absorption time in a Markov chain representing the distance to destination (see section 3). Clearly, the state 0 will be absorbing. Moreover, this is the only absorbing state of the chain.

Consider now two absorbing DTMC $\{X_k\}$ and $\{Y_k\}$ with unique absorbing state 0. Denote by T^X and T^Y the respective absorption times into the state 0. Then, under the same conditions as in theorem 1, we have also the comparison of distributions of absorption time into 0:

PROPOSITION 1. *Let $\{X_k\}_{k\geq 0}$ and $\{Y_k\}_{k\geq 0}$ be two absorbing DTMC with unique absorbing state 0, and let P and R be their respective transition matrices. If $X_0 \leq_{st} Y_0$, at least one transition matrix P or R is st-monotone and $P \leq_{st} R$, then $T^X \leq_{st} T^Y$.*

PROOF. Theorem 1 implies $X_k \leq_{st} Y_k, \forall k \geq 0$. In particular, $P(X_k \geq 1) \leq P(Y_k \geq 1), \forall k \geq 0$. As the state 0 is absorbing, we have the following relation between the random variables T^X and X_k (resp. between T^Y and Y_k):

$$T^X \leq k \iff X_k = 0 \quad (\text{resp. } T^Y \leq k \iff Y_k = 0).$$

Therefore, $P(T^X \leq k) = P(X_k = 0) = 1 - P(X_k \geq 1) \geq 1 - P(Y_k \geq 1) = P(Y_k = 0) = P(T^Y \leq k)$, $\forall k \geq 0$. Since $P(T^X \geq k) = 1 - P(T^X \leq k-1)$ (resp. $P(T^Y \geq k) = 1 - P(T^Y \leq k-1)$), the former equation can be rewritten as:

$$P(T^Y \geq k) \leq P(T^X \geq k), \ \forall k \geq 1.$$

Finally, for $k = 0$ this is trivially verified as $P(T^X \geq 0) = P(T^Y \geq 0) = 1$. Thus, $T^Y \leq_{st} T^X$. □

Unfortunately, the Markov chain we use to model the network is not always st-monotone as we will see in the next section. For the non-monotone case we will use a similar result based on level-crossing ordering of Markov chains and a particular structure of the chain of our model.

Irle and Gani have introduced [10] the level crossing ordering of Markov chains based on the intuitive idea that the smaller chain should take longer to cross any fixed level.

DEFINITION 6 (LEVEL CROSSING ORDERING). *Let $X = \{X_k\}_{k\geq 0}$ and $Y = \{Y_k\}_{k\geq 0}$ be two DTMC on the state space $\{0,\dots,n\}$. Let $S^X_{i,l}$ (resp. $S^Y_{i,l}$) be the first passage time in the subset $\{l, l+1, \dots, n\}$ for the chain $\{X_k\}$ (resp. $\{Y_k\}$) when the initial state is i:*

$$S^X_{i,l} = \inf\{k \geq 0 \ : \ X_k \geq l\}, \quad S^Y_{i,l} = \inf\{k \geq 0 \ : \ Y_k \geq l\},$$

with notation $\inf \emptyset = \infty$. We say that $X \leq_{lc} Y$ if

$$S^Y_{i,l} \leq_{st} S^X_{i,l}, \ \forall i, \ l.$$

Note that we need to verify the above relation only for $i < l$, since for $i \geq l$ we have trivially $S^X_{i,l} = S^Y_{i,l} = 0$. It might seem not intuitive at the first sight that we say that X is lc-smaller than Y if $S^Y_{i,l} \leq_{st} S^X_{i,l}, \forall i, l$. However this means that X takes longer to cross any fixed level, i.e. that the chain X is slower.

Irle and Gani showed in [10] that under the same conditions of theorem 1 we have also lc-comparison of Markov chains. Furthermore, they showed that lc-ordering of two Markov chains can be also established for skip-free chains that are non-necessarily st-monotone.

DEFINITION 7 (SKIP FREE). *Let P be a transition matrix of a Markov chain. The transition matrix (and the chain) is skip free to the right if $P_{i,j} = 0$ for all $j > i+1$. The transition matrix (and the chain) is skip free to the left if $P_{i,j} = 0$ for all $j < i-1$.*

THEOREM 2 (THEOREM 4.1, P. 73 [10]). *If two DTMC X and Y with transitions matrices P and R satisfy the following conditions:*

1. $P \leq_{st} R$
2. *X and Y are skip free to the right.*

then $X \leq_{lc} Y$.

REMARK 1. *Ferreira and Pacheco proved in [7] that only the slower chain (chain X in theorem 2) needs to be skip-free to the right.*

We are interested here in comparison of absorption times to 0. Thus we rewrite the above result for the first passage times into the sets $\{0,\dots,l-1,l\}$. We first define the dual lc-ordering of Markov chains in which the smaller chain takes longer to cross any fixed level to the left.

DEFINITION 8. *Let $X = \{X_k\}_{k\geq 0}$ and $Y = \{Y_k\}_{k\geq 0}$ be two DTMC on the state space $\{0,\dots,n\}$. Let $Z^X_{i,l}$ (resp. $Z^Y_{i,l}$) be the first passage time in the subset $\{0,\dots,l-1,l\}$ of the chain $\{X_k\}$ (resp. $\{Y_k\}$) when the initial state is i:*

$$Z^X_{i,l} = \inf\{k \geq 0 \ : \ X_k \leq l\}, \quad Z^Y_{i,l} = \inf\{k \geq 0 \ : \ Y_k \leq l\},$$

We say that $X \leq^*_{lc} Y$ if

$$Z^Y_{i,l} \leq_{st} Z^X_{i,l}, \ \forall i, \ l.$$

By taking into account Ferreira and Pacheco's less restrictive assumptions (remark 1) in theorem 2, we can obtain the following sufficient conditions for $\leq^*_{lc}$-comparison:

COROLLARY 1. *If two finite DTMC X and Y with transitions matrices P and R satisfy the following conditions:*

1. $R \leq_{st} P$
2. *X is skip free to the left.*

then $X \leq^*_{lc} Y$.

PROOF. Let us consider two new chains $\widetilde{X}$ and $\widetilde{Y}$ on the state space $\{0,\ldots,n\}$ obtained from X and Y by inversing the order of the states ($i \leftrightarrow n-i$, $\forall i$). Denote by $\widetilde{P}$ and $\widetilde{R}$ the transition matrices of $\widetilde{X}$ and $\widetilde{Y}$. We have the following relations between chains X and $\widetilde{X}$ (the equivalent relations are valid for chains Y and $\widetilde{Y}$):

- Transition matrices P and $\widetilde{P}$ satisfy:
$$\widetilde{P}_{i,j} = P_{n-i,n-j}. \tag{1}$$
- X is skip free to the left if and only if $\widetilde{X}$ is skip free to the right. This follows directly from (1).
- The first passage time to the set $\{0,\ldots,l-1,l\}$ for the chain X starting from i is equivalent to the first passage time to the set $\{n-l,n-l+1,\ldots,n\}$ for the chain $\widetilde{X}$ starting from $n-i$:
$$Z^X_{i,l} = S^{\widetilde{X}}_{n-i,n-l},\ \forall i,l. \tag{2}$$

Using equation (1) and definition 2 it is easy to show that:
$$R \leq_{st} P \iff \widetilde{P} \leq_{st} \widetilde{R}.$$
Since $R \leq_{st} P$, and P is skip-free to the left, we have $\widetilde{P} \leq_{st} \widetilde{R}$ and $\widetilde{P}$ is skip-free to the right. Now theorem 2 and remark 1 imply that $\widetilde{X} \leq_{lc} \widetilde{Y}$. Finally, equation (2) implies that $\widetilde{X} \leq_{lc} \widetilde{Y} \iff X \leq^*_{lc} Y$. $\square$

Let us return now to our problem of comparison of absorption times to 0. If X and Y are two DTMC with absorbing state 0, then $Z^X_{i,0}$ and $Z^Y_{i,0}$ are absorption times to 0 when the initial state is i. Corollary 1 provides thus the sufficient conditions for comparison of absorption times for chains knowing that they start from the same state. We show that this remains valid if we consider that the chains X and Y have arbitrary initial distributions ν^X and ν^Y that are st-comparable. We use the following lemma:

LEMMA 1. *For a DTMC Y with a transition matrix R that is skip-free to the left the first crossing times of a fixed level to the left satisfy:*
$$Z^Y_{i,l} \leq_{st} Z^Y_{j,l},\ \textit{for all}\ i \leq j,\ \textit{for all}\ l.$$

PROOF. Follows directly from the skip-free to the left structure of the chain. For $l \geq i$ the above relation is trivial as $Z^Y_{i,l} = 0$. Consider now $l < i < j$. A chain starting from a state $j > i$ needs to reach first the state i in order to cross the level l since the only way to reach the states $\{0,\ldots,i-1\}$ is to use the transition $(i,i+1)$. Thus clearly $Z^Y_{i,l} \leq_{st} Z^Y_{j,l}$. $\square$

PROPOSITION 2. *Let X and Y be two DTMC with transitions matrices P and R, and initial distributions ν^X and ν^Y. If the chains X and Y satisfy:*

1. $\nu^X \leq_{st} \nu^Y$,
2. $P \leq_{st} R$,
3. *Y is skip free to the left,*

then $T^X \leq_{st} T^Y$.

PROOF. We need to show that:
$$P(T^X \geq k) \leq P(T^Y \geq k),\ \forall k \geq 0.$$
Consider $k \geq 0$ arbitrary and fixed. We have:
$$P(T^Y \geq k) = \sum_{i=0}^{n} P(Z^Y_{i,0} \geq k)\nu^Y_i.$$
We will show the following relation by induction on m:
$$\begin{aligned} P(T^Y \geq k) \ &\geq\ \sum_{i=0}^{m} P(Z^Y_{i,0} \geq k)\nu^Y_i + \sum_{i=m+1}^{n} P(Z^Y_{i,0} \geq k)\nu^X_i \\ &+\ P(Z^Y_{m+1,0} \geq k)\Big(\sum_{i=m+1}^{n} \nu^Y_i - \sum_{i=m+1}^{n} \nu^X_i\Big), \\ &\text{for } 0 \leq m < n. \end{aligned} \tag{3}$$

- Base: $m = n-1$. We have trivially $P(T^Y \geq k) = \sum_{i=0}^{n-1} P(Z^Y_{i,0} \geq k)\nu^Y_i + P(Z^Y_{n,0} \geq k)\nu^X_n + P(Z^Y_{n,0} \geq k)(\nu^Y_n - \nu^X_n)$.
- Suppose that the relation is valid for an m such that $0 < m < n$. Then it is also valid for $m-1$. Indeed, using lemma 1 we have $Z^Y_{m,0} \leq_{st} Z^Y_{m+1,0}$, thus:
$$P(Z^Y_{m+1,0} \geq k) \geq P(Z^Y_{m,0} \geq k). \tag{4}$$
Note that $\sum_{i=m+1}^{n} \nu^Y_i - \sum_{i=m+1}^{n} \nu^X_i \geq 0$, since $\nu^X \leq_{st} \nu^Y$. By using the induction hypothesis and relation (4) we obtain:
$$\begin{aligned} P(T^Y \geq k) &\geq \sum_{i=0}^{m-1} P(Z^Y_{i,0} \geq k)\nu^Y_i + \sum_{i=m+1}^{n} P(Z^Y_{i,0} \geq k)\nu^X_i \\ &\quad + P(Z^Y_{m,0} \geq k)\Big(\nu^Y_m + \Big(\sum_{i=m+1}^{n} \nu^Y_i - \sum_{i=m+1}^{n} \nu^X_i\Big)\Big) \\ &= \sum_{i=0}^{m-1} P(Z^Y_{i,0} \geq k)\nu^Y_i + \sum_{i=m}^{n} P(Z^Y_{i,0} \geq k)\nu^X_i \\ &\quad + P(Z^Y_{m,0} \geq k)\Big(\sum_{i=m}^{n} \nu^Y_i - \sum_{i=m}^{n} \nu^X_i\Big). \end{aligned}$$

Thus relation (3) is valid for all m such that $0 \leq m < n$. In particular, for $m = 0$ we obtain:
$$\begin{aligned} P(T^Y \geq k) \ &\geq\ P(Z^Y_{0,0} \geq k)\nu^Y_0 + \sum_{i=1}^{n} P(Z^Y_{i,0} \geq k)\nu^X_i \\ &+\ P(Z^Y_{1,0} \geq k)\Big(\sum_{i=1}^{n} \nu^Y_i - \sum_{i=1}^{n} \nu^X_i\Big). \end{aligned}$$
From lemma 1 it follows that $P(Z^Y_{1,0} \geq k) \geq P(Z^Y_{0,0} \geq k)$, and from $\nu^X \leq_{st} \nu^Y$ that $\sum_{i=1}^{n} \nu^Y_i - \sum_{i=1}^{n} \nu^X_i \geq 0$. Thus:
$$\begin{aligned} P(T^Y \geq k) \ &\geq\ \sum_{i=1}^{n} P(Z^Y_{i,0} \geq k)\nu^X_i + P(Z^Y_{0,0} \geq k)\Big(1 - \sum_{i=1}^{n} \nu^X_i\Big) \\ &=\ \sum_{i=0}^{n} P(Z^Y_{i,0} \geq k)\nu^X_i. \end{aligned}$$

Corollary 1 implies that $Y \leq^*_{lc} X$. Thus $Z^X_{i,0} \leq_{st} Z^Y_{i,0}$, $\forall i$, i.e.
$$P(Z^Y_{i,0} \geq k) \geq P(Z^Y_{i,0} \geq k),\ \forall k.$$
Therefore $P(T^Y \geq k) \geq P(T^X \geq k)$, $\forall k$, i.e. $T^X \leq_{st} T^Y$. $\square$

In the following we give two applications of the level crossing ordering. First we provide a bound of the end to end delay based on a bound of the deflection probability. This is a rather usual application of Markov chain ordering. We also consider a quite different approach: we prove that an approximate analysis has a fixed point solution. The convergence of approximate analysis based on mean interactions between Markovian submodels is usually not proved and we show how level crossing ordering can provide such a framework for a proof. Before proceeding with the general case we introduce first the deflection routing on an hypercube which is used to illustrate both approaches.

3. DEFLECTION ROUTING ON A HYPERCUBE

We model a hypercube of dimension n (see figure 1 for a hypercube of dimension 4). A hypercube is a simple generalization of a cube with an arbitrary size. The nodes are the vectors with n components taking values in $\{0,1\}$. We consider the directed and symmetrical version of the graph (see definition 9). Nodes which differ only by one component are connected by two directed edges. Thus a hypercube of dimension n has 2^n nodes and $n2^n$ directed edges and the switches have an indegree and outdegree equal to n. The shortest path distance in the hypercube is equal to the Hamming distance among binary vectors. The diameter of a hypercube with dimension n is thus n.

Let x and y be two nodes. All the directions such as $x_i \neq y_i$ are good directions for the routing algorithm to send a packet from x to y using shortest paths. All the others are bad directions. Therefore a packet at distance k of its destination has k good directions for the next step of routing. In figure 1 we give an example of a packet with current position 1011 and destination 0101. Thus the distance to destination of the packet is 3 and it has 3 good directions (depicted in bold): 0011, 1111 and 1001.

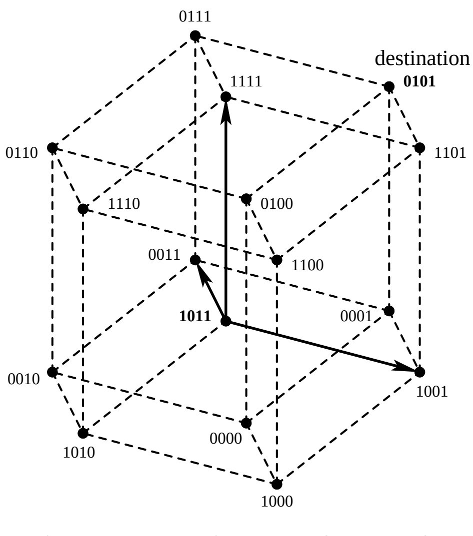

Figure 1: Routing on a hypercube

As previous authors, we assume that the couples source-destination of traffic follow an uniform distribution. We also assume that packets are independent.

Routing algorithm.

The packet will select at random with uniform distribution one direction among the good ones. If this direction is not given to the packet by the routing algorithm, the packet is deflected. We say that this direction is not available. We consider a two phases algorithm instead of a greedy choice. During the first phase the packets which are not deflected are routed and the deflected packets are kept. Then during the second phase the deflected packets are sent among the directions which are still available after the first phase. A deflected packet uses a direction at random with uniform distribution among all available directions. As all the packets are equivalent from the probabilistic point of view, we consider an arbitrary packet in an arbitrary switch. Note that due to the topology of a hypercube and traffic assumptions all the switches are statistically equivalent.

Following the method developed in [8], we represent a tagged packet by its distance to destination. Thus the states of a Markov chain modelling the evolution of a tagged packet are between 0 and n and the state 0 is an absorbing state.

We give first the initial distribution for this chain. The network has 2^n nodes. In an uniform destination model, all the nodes (other than the source node) have probability $\frac{1}{2^n-1}$ to be addressed. The number of nodes at distance k in a hypercube with dimension n is equal to:

$$C(n,k) = \frac{n!}{k!(n-k)!}.$$

The initial distribution of the probability vector is therefore:

$$\pi_0 = \frac{1}{2^n-1}(0, C(n,1), \ldots, C(n,k), \ldots, C(n,n)).$$

In order to describe the transitions, we will denote by p the deflection probability in an arbitrary switch. Note that this probability is constant due to the routing hypothesis and the symmetry of a hypercube.

- Assume that the packet at distance k is not deflected, the distance decreases from k to $k-1$. This event has probability $(1-p)$.
- If the packet at distance k is deflected, it remains $(k-1)$ good directions among $(n-1)$ available directions. If the packet uses a good direction, its distance is now $(k-1)$; otherwise it is $(k+1)$. Thus we have the following transitions:
  - k to $k-1$ with probability $p\frac{k-1}{n-1}$
  - k to $k+1$ with probability $p\frac{n-k}{n-1}$

  When $k=1$, the deflected packet always uses a bad direction. When $k=n$, whether the packet is deflected or not, it uses a good direction.

We will denote by R the transition matrix of this chain. Matrix R is clearly tridiagonal.

Example 2. *For a hypercube of dimension $n=7$, the Markov chain has 8 states and the matrix R is:*

$$\begin{pmatrix}
1 & 0 & & & & & & \\
1-p & 0 & p & & & & & \\
 & 1-\frac{5p}{6} & 0 & \frac{5p}{6} & & & & \\
 & & 1-\frac{4p}{6} & 0 & \frac{4p}{6} & & & \\
 & & & 1-\frac{3p}{6} & 0 & \frac{3p}{6} & & \\
 & & & & 1-\frac{2p}{6} & 0 & \frac{2p}{6} & \\
 & & & & & 1-\frac{p}{6} & 0 & \frac{p}{6} \\
 & & & & & & 1 & 0
\end{pmatrix}$$

We assume that all optical technology will be used in the core network, the size of which is typically under 30 nodes for a national network and under 100 for a European one. The number of nodes in the graph is quite small and due to assumptions on the states of the Markov chain, the state space is also small. Furthermore, the matrix of the DTMC has a tridiagonal structure thus most of the numerical computations are easier. Unfortunately, it is very difficult to obtain the real value of the deflection probability which may depend on the topology but also of the load inside the network. Furthermore the number of deflections (and therefore the deflection probability) for a set of requested links depends on the algorithm used to select the packets [4]. Minimizing the number of deflections requires to compute the maximum matching in a bipartite graph. This problem can be solved in a polynomial time but due to severe timing constraints in an optical switch it is not possible to use this method. One can find in [11] a very efficient algorithm to solve this routing problem and find the packets to deflect for some particular topology. Due to these constraints, we assume that we can find upper and lower bounds of the deflection probability rather than exact results and we show that these bounds provide bounds for the end to end delays.

4. BOUNDING THE END TO END DELAY

We consider now a more general model of the network topology to prove the comparisons of the chains which describe the end to end delay. This topology is not completely characterized as a graph. We just give some necessary assumptions to obtain a tridiagonal DTMC to model the routing. First we assume that the directed graph is symmetrical, a quite natural assumption for Wide Area Network modeling.

DEFINITION 9. *A directed graph (digraph in the following) $G = (V, E)$ is symmetrical if and only if :*

$$(x, y) \in E \implies (y, x) \in E, \ \forall x, y \in V.$$

Because of this property, the evolution of the distance to destination is now much simpler.

PROPERTY 1. *Let $G = (V, E)$ be a symmetrical digraph. We assume that the routing inside G is based on the shortest path algorithm with deflection. If a packet is deflected (i.e. it asks a shortest path and its demand is rejected) then the distance to destination can:*

- *increase by one,*
- *decrease by one,*
- *stay identical.*

PROOF. Let x be the node with the packet, y the node finally obtained after the routing algorithm and z the node required by the packet as the input of the routing algorithm (figure 2). Let l be the length of the shortest path from x to the final destination.

- The distance cannot increase by more than 1. Indeed, assume that the packet is now in y. As the graph is symmetrical the directed link (y, x) exists and the packet can go back to x and follow its initial shortest path. Thus we have a path of length $l + 1$ and the shortest path must be smaller or equal to this value.

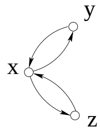

Figure 2: Deflection in a symmetrical digraph

- It may happen that the directed link (x, y) is also the first link of a shortest path to the destination. Remember the case of the hypercube in the former section. Thus the distance decreases by one because we move one hop along the shortest path.

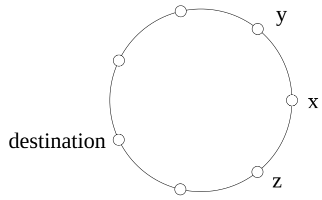

Figure 3: Deflection in an odd ring

- Finally, for some digraphs, it may be possible that after a deflection the packet reaches a node where the distance is still l. For instance, in an odd ring with size $2l + 1$, if the packet is in a node at distance l to its destination before the routing, it reaches another state at distance l if it is deflected. We illustrate this in figure 3. A packet in node x is at distance 3 to its destination. His only shortest path direction is node z. If the packet is deflected, it goes to node y which is also at distance 3 to destination.

  Note that this case only happens for some particular digraphs (odd rings but not even rings or hypercubes) and some particular nodes. □

We now present the main assumption about the topology, the traffic and the exact routing algorithm.

ASSUMPTION 1. *We assume that the traffic is uniform and that conflicts among packets are ruled by a fair competition where all the packets have the same probability to win. We also assume the the the choices of links are random with a uniform distribution among the possible directions. These are typical assumptions in network modelling. Finally we assume that the network topology has many symmetries such that it is possible to model the movements of the tagged packet inside the network by a DTMC whose states are the distance to destination. Rings and hypercubes are examples of such networks.*

Thus we can build the DTMC and due to property 1 its matrix is tridiagonal. Let m be the diameter of the digraph (i.e. the longest shortest path between any couple of states). Then the state space is $\{0, \ldots, m\}$, and the state 0 is absorbing. For an arbitrary state $i > 0$ the transitions are:

- at distance i, the packet request is rejected with probability p_i;

- If the request is rejected, distribution $(q_{i,-1}, q_{i,0}, q_{i,1})$ gives the probabilities that the link finally obtained respectively decreases the distance by one, keeps it constant, or increases the distance by one. Obviously, $q_{m,1} = 0$ as m is the maximal distance in the digraph.

The transition probabilities for a state $i > 0$ are therefore:

$$\begin{cases} i \longrightarrow i-1 & : \quad 1 - p_i + p_i q_{i,-1}, \\ i \longrightarrow i & : \quad p_i q_{i,0}, \\ i \longrightarrow i+1 & : \quad p_i q_{i,1}, \text{ for } i < m. \end{cases}$$

Consider now chains $\{X_k\}_{k\geq 0}$ and $\{Y_k\}_{k\geq 0}$ based on probabilities p_i^X, $(q_{i,-1}^X, q_{i,0}^X, q_{i,1}^X)$ and p_i^Y, $(q_{i,-1}^Y, q_{i,0}^Y, q_{i,1}^Y)$. Let us denote by A^X (resp. A^Y) the transition matrices of these chains and by ν^X (resp. ν^Y) their initial distribution. Finally, let T^X (resp. T^Y) be the absorption time for chain $\{X_k\}$ (resp. $\{Y_k\}$).

Proposition 3. *Transition matrices A^X and A^Y satisfy $A^X \leq_{st} A^Y$ if :*

$$\begin{array}{lcll} p_i^X & \leq & p_i^Y, & i > 0, \\ q_{i,-1}^X & \geq & q_{i,-1}^Y, & i > 0, \\ q_{i,1}^X & \leq & q_{i,1}^Y, & 0 < i < m. \end{array}$$

Proof. Indeed, we have:

$$\sum_{k\geq j} A_{i,k}^X = \begin{cases} 0, & j > i+1 \\ p_i^X q_{i,1}^X \leq p_i^Y q_{i,1}^Y, & j = i+1 \\ p_i^X (1 - q_{i,-1}^X) \leq p_i^Y (1 - q_{i,-1}^Y), & j = i \\ 1, & j < i \end{cases}$$

Thus $\sum_{k\geq j} A_{i,k}^X \leq \sum_{k\geq j} A_{i,k}^Y$, $\forall j$, i.e. $A^X \leq_{st} A^Y$ and the proof is completed. □

If A^X or A^Y is st-monotone, then we can use the *st*-comparison of Markov chains and proposition 1 to establish the comparison of absorption times. This is restated in the following theorem for the sake of clarity but it is a simple corollary of the theory presented in section 2.

Theorem 3. *If the following three conditions are satisfied:*

1. $\nu^X \leq_{st} \nu^Y$
   Note that they are usually equal when X and Y model the same network with the same uniform traffic assumption.
2. $A^X \leq_{st} A^Y$,
3. A^X *or* A^Y *is st-monotone*

then $T^X \leq_{st} T^Y$.

This is typically what has been used in [5] in order to prove that the deflection routing using a more complex algorithm on an odd 2D-torus is st-monotone for a partial ordering of the state space.

However, it may arrive that the st-monotonicity of the chain is not consistent with the topology of the network. For instance it is clear that matrix R in the previous section is not monotone. Due to the tridiagonal structure of the matrix we can use the level crossing ordering of Markov chains instead of the strong stochastic ordering and the st-monotonicity.

Theorem 4. *If $\nu^X \leq_{st} \nu^Y$ and if $A^X \leq_{st} A^Y$, then*

$$T^X \leq_{st} T^Y.$$

Proof. As matrix A^Y is tridiagonal it is clearly skip-free to the left. Thus, by proposition 2, $T^X \leq_{st} T^Y$. □

Due to theorem 4 and proposition 3 we get:

Corollary 2. *If $\nu^X \leq_{st} \nu^Y$ and if :*

$$\begin{array}{lcll} p_i^X & \leq & p_i^Y, & i > 0, \\ q_{i,-1}^X & \geq & q_{i,-1}^Y, & i > 0, \\ q_{i,1}^X & \leq & q_{i,1}^Y, & 0 < i < m, \end{array}$$

then $T^X \leq_{st} T^Y$.

Thus we can obtain upper and lower strong stochastic bounds on the end to end delay from upper and lower bounds of p_i^X, $q_{i,-1}^X$ and $q_{i,1}^X$. More precisely:

- An st-upper bound of T^X is obtained from upper bounds of p_i^X and $q_{i,1}^X$, and lower bounds of $q_{i,-1}^X$.
- We get an st-lower bound of T^X with lower bounds of p_i^X and $q_{i,1}^X$, and upper bounds of $q_{i,-1}^X$.

Note that for a hypercube the probabilities $q_{i,0}^X$ are all equal to 0. The model is therefore simpler to analyze. We do not present here a method to obtain bounds of deflection probabilities.

5. A FIXED POINT SYSTEM, AN ALGORITHM AND ITS PROOF

Let us now turn back again to the hypercube topology to improve another approach often considered in the literature (see for instance [14, 6, 8]) to compute an approximation of the average throughput.

Let the link capacity be f. Assume that the deflection probability is known. The former model of the end to end delay gives a relation between deflection probability and the distribution of the end to end delay. But matrix R can also be used to get the average delay before absorption (in this context it is the average end to end delay) and finally the average load due to Little's law. Now suppose that we are able to explain how the load of the links influences the deflection probability. We get a fixed point system that we can solve numerically. This is a typical mean interaction approximation.

However, we can do better that this traditional approach in performance modeling: due to some qualitative properties of the level crossing comparison of chains we can prove the existence of a solution of the fixed point system and provide a proved algorithm to find a solution, This is an improvement as the convergence of approximate methods based on a fixed point system is usually not provided. Let us now detail these results. In this section we give the general technique and we present in the next section the case of the unitary link capacity to illustrate the approach.

5.1 Average end to end delay

Let us first establish a new relation between the link utilization u and the deflection probability p.

Let $E(X)$ the expected number of customers in the network, λ the input rate in the global set of nodes from the electronic buffers and $E(T)$ the average end to end delay. $E(T)$ is the average number of hops (i.e. the average sojourn time in the optical part of the network). Following Little's law we get: $E(X) = \lambda E(T)$.

The computation of the average time before absorption in a DTMC is related to the number of visits to every state before being absorbed. Let $T_{i,j}$ be the average number of visits in j before being absorbed, knowing that the initial state is i. Clearly, $E(T) = \sum_i \sum_j \pi_0(i) T_{i,j}$. It is proved in Trivedi [13] that $T_{i,j}$ is the element with row index i and column index j in the matrix $(I - \tilde{R})^{-1}$, where $\tilde{R}$ known as the fundamental matrix of the absorbing chain is obtained from R by deletion of the rows and columns associated to absorbing states. As the remaining chain is transient, matrix $(I - \tilde{R})$ is not singular.

$$E(T) = \sum_i \sum_j \pi_0(i)(I - \tilde{R})^{-1}_{i,j} \tag{5}$$

As all the links are equivalent due to the hypercube topology and the traffic assumptions, we have 2^n nodes and each node has n input links. Thus: $u = \frac{E(X)}{n2^n}$. Finally, if we add that the link utilization is smaller than the link capacity f, we get:

$$u = min(f, \frac{\lambda E(T)}{n2^n}). \tag{6}$$

5.2 A general algorithm and its assumption

Now we need a relation which explains the influence of the load on deflection probability. To obtain such a relation we must have a detailed model of the link, how packets can change their wavelengths and how the deflection algorithm selects the packets to be deflected. With these two models we have a fixed point system: the relation proved in the former section shows how u changes when p moves and the model of the link explains the evolution of p due to u. We do not provide such a model in general but we show that if a very intuitive property is satisfied then we get a proved algorithm to find a solution of the fixed point system. So we suppose that such a model exists and that we have proved that $p = g(u)$.

ASSUMPTION 2. *We assume that:*

- *function g is increasing,*
- $g(0) \geq 0$,
- $g(f) < 1$.

We have a fixed point system and we must provide a proof of the existence of a solution and a numerical algorithm. In order to show the existence of a solution we will define the following sequence of values for deflection probability p:

1. Initialization: $p_0 = 0$
2. Let $p_i \in [0, 1]$. Then p_{i+1} is obtained as follows:
   (a) Compute $\tilde{R}$ with $p = p_i$
   (b) Inverse $(I - \tilde{R})$
   (c) Compute $E(T)$ using equation (5)
   (d) Compute u using equation (6)
   (e) Compute p_{i+1} from u using $p_{i+1} = g(u)$

The complexity of one iteration of the loop is $O(n^3)$ since we have to inverse a matrix of size n during the first step of the loop, while the other steps have at most a quadratic complexity. The number of iterations before convergence and the computation of function g are obviously unknown.

5.3 Proof

We do not use the Brouwer's theorem on the existence of a solution of a fixed point system. We build an increasing sequence which is upper bounded in a compact subset. The dominated convergence theorem proves that the solution exists and is the limit of the sequence.

LEMMA 2. *When u varies in $[0, f]$, p is in $[0, \Delta]$ with $\Delta = g(f) < 1$. And $[0, \Delta]$ is compact.*

LEMMA 3. *$E(T)$ is a non decreasing function of p.*

PROOF. It is based on the theory of stochastic comparison of finite discrete time Markov chains. Denote by $R(p)$ the transition matrix when deflection probability is equal to p. Clearly, we have the following properties.

PROPERTY 2. *If $p_1 \leq p_2$ then $R(p_1) \leq_{st} R(p_2)$.*

PROPERTY 3. *As matrix $R(p)$ is tridiagonal, it is obviously skip free to the left.*

Now from properties 2, 3, and proposition 2 from section 2 we obtain that T is stochastically increasing with p. Thus, $E(T)$ is increasing with p as an increasing function of T and the proof of lemma 3 is completed. □

LEMMA 4. *u is a non decreasing function of $E(T)$.*

PROOF. Trivial from equation (6). □

THEOREM 5. *The sequence converges to a fixed point.*

PROOF. Because of assumption 2 and lemmas 3 and 4, the sequence of the computed values of p is increasing. Lemma 2 proves that p is upper bounded by Δ. As an increasing and upper bounded sequence has a limit, there exists a solution (the limit) which is a fixed point of the system. □

The above theorem shows that there is an infinite sequence that leads to a fixed point of the system. Unfortunately, it not so easy to give a simple stopping criterion for this iterative computation since the distance between two consecutive iterations does not give any information on the distance from a fixed point. Therefore we propose to use a dichotomic search of a fixed point based on the following observations:

1. Let h denote the function that for a given deflection probability p computes the new deflection probability value after one iteration. h is increasing as a composition of increasing functions.
2. For each p_0, the sequence $p_i = h(p_{i-1}),\ i > 0$ is monotone. The sequence is increasing if $p_0 < p_1$ and decreasing if $p_0 > p_1$.

3. For each p_0, the sequence $p_i = h(p_{i-1}),\ i > 0$ converges to a fixed point in $[0,1]$ and all the points in the sequence are in $[0,1]$.

4. Let us suppose that for a given interval $[a,b]$, with $0 \le a < b \le 1$, for each $p_0 \in [a,b]$ the sequence $p_i = h(p_{i-1}),\ i > 0$ converges to a fixed point in $[a,b]$ and all points in the sequence are in $[a,b]$. Note that this is true for $a = 0$ and $b = 1$. Define $c = \frac{a+b}{2}$. Then $h(c) \in [a,b]$. Let us now compare c and $h(c)$.

   (a) If $h(c) > c$, then for all $p_0 \in [h(c), b]$ the sequence $p_i = h(p_{i-1}),\ i > 0$ converges to a fixed point in $[h(c), b]$. Indeed, this monotone sequence converges to a fixed point p in $[a,b]$; thus $p \le b$. On the other hand, $p_0 \ge h(c) > c$ and the fact that h is an increasing function imply that $p_i \ge h(c) > c, \forall i$, by induction on i. Therefore, $p \ge h(c)$. We can thus take $h(c)$ as the new value for a.

   (b) If $h(c) < c$, then for all $p_0 \in [a, h(c)]$, the sequence $p_i = h(p_{i-1}),\ i > 0$ converges to a fixed point in $[a, h(c)]$. The proof is similar to the previous case. We can take $b = h(c)$.

   (c) Finally, if $h(c) = c$, then c is a fixed point.

5. Stopping criterion : $b - a < \epsilon$. This algorithm computes clearly an ϵ-approximation of a fixed point.

It is worthy to remark that level crossing comparison provides two qualitative results here. First a reward is an increasing function of a parameter used to define the chain. And based on this property we can prove an approximation algorithm.

For our model of a hypercube, it is now sufficient to find a function g which describes how we get the deflection probability from the link utilization. Once we get this function, we must check if assumption 2 is satisfied. We now show how to proceed when the link capacity is one.

6. AN APPROXIMATE MODEL FOR THE UNITARY LINK CAPACITY

We develop a new model to obtain a deflection probability based on independence of packets inside the switches when the link capacity is one. This assumption only states that there is no wavelength conversion inside the switches and we can analyze the network and the switches with one wavelength. This model provides a relation between the load and the deflection probability. It is based on the analysis of packets in conflict with the tagged packet for a given output link.

Let o_1 denote the output link requested by the tagged packet. The probability of deflection for the tagged packet is computed by conditioning on all configurations of arrivals. Let Z be the number of packets requesting o_1 other than the tagged packet. Note that the upper bound for Z is $n-1$ because the tagged customer uses one input link of the switch.

$$p = \sum_{i=0}^{n-1} Pr(\text{deflection of the tagged packet} \mid Z = i) \times Pr(Z = i) \qquad (7)$$

The arrival probabilities are obtained by an independence assumption. Let us denote by u the utilization of an arbitrary link. We have $n-1$ links. Each of them is used by a packet with probability u. When a packet enters a switch, it requires each output link with probability $1/n$.

$$Pr(Z = i) = C(n-1, i)(u/n)^i (1 - u/n)^{n-1-i} \qquad (8)$$

The probabilities of deflection are quite simple. The output capacity is one. We have $i+1$ packets in a fair competition. Thus the tagged packet wins with probability $1/(i+1)$ and is deflected with probability $i/(i+1)$.

$$Pr(\text{deflection of the tagged packet} \mid Z = i) = \frac{i}{i+1} \qquad (9)$$

Combining the two former equations we have:

$$p = \sum_{i=0}^{n-1} \frac{(n-1)!}{i!(n-i-1)!} \left(\frac{u}{n}\right)^i \left(1 - \frac{u}{n}\right)^{n-i-1} \frac{i}{i+1}.$$

After some algebraic manipulations, we finally get:

$$p = 1 - \frac{1}{u}\left(1 - \left(1 - \frac{u}{n}\right)^n\right). \qquad (10)$$

This relation has a simple physical interpretation. Note that it can be written as: $(1-p)\ n\ u = n\left(1 - \left(1 - \frac{u}{n}\right)^n\right)$. The left part is the average number of packets which are successfully sent by the router during one slot. On the right side $\left(1 - \frac{u}{n}\right)^n$ is the probability that an output link is not required by any packet. Thus $n\left(1 - \left(1 - \frac{u}{n}\right)^n\right)$ is the expected number of links required by the packets. The relation states that the expected number of packets going in a good direction is the expected number of links required by the packets. Indeed, each packet requires only one link and there is always one packet accepted when several are competing for the output.

To prove the algorithm we need to check assumption 2.

LEMMA 5. *p is an increasing function of u.*

PROOF. Consider relation (10) and develop $\left(1 - \left(1 - \frac{u}{n}\right)^n\right)$. The first two terms are canceled due to the other terms in the relation. Finally:

$$p = \sum_{i=2}^{n} \frac{n!}{i!(n-i)!} \left(\frac{-1}{n}\right)^i u^{i-1}. \qquad (11)$$

We group two consecutive terms of the summation. Typically we get:

$$(\frac{1}{n})^{2k} u^{2k-1} \frac{n!}{(2k)!(n-2k)!} - (\frac{1}{n})^{2k+1} \frac{n!\ u^{2k}}{(2k+1)!(n-2k-1)!}$$

After factorization:

$$(\frac{1}{n})^{2k} \frac{n!}{(2k)!(n-2k-1)!} u^{2k-1} \left(\frac{1}{n-2k} - \frac{u}{n(2k+1)}\right)$$

As $u < 1$ we can easily check that $\left(\frac{1}{n-2k} - \frac{u}{n(2k+1)}\right) > 0$. Thus p is increasing with u. $\square$

LEMMA 6. *Clearly $p = g(u)$ is positive and its maximal value is $(1-1/n)^n$. Obviously $\Delta = (1-1/n)^n < 1$ as $n > 1$.*

Thus we have derived the algorithm and its proof for the hypercube with unitary links.

Table 1: Number of iterations for the hypercube topology with unitary links: $n = 7$, $\lambda = 2^n/n$

ϵ	10^{-5}	10^{-10}	10^{-15}	10^{-20}
iterations	6	12	18	24

Table 2: Fixed point for the hypercube topology with unitary links: $n = 7$, $\lambda = 2^n/n$

p	u	$E[T]$
3.18×10^{-2}	7.56×10^{-2}	3.70

In table 1 we report the number of iterations needed for fixed point computation for the hypercube topology with unitary links when we change the accuracy of the algorithm. The dimension of hypercube is fixed to $n = 7$ (128 switches) and the input rate to $\lambda = 2^n/n = 18.29$. In table 2 we give deflection probability p, utilization u, and the expectation of the end to end delay $E[T]$ for $n = 7$ and $\lambda = 2^n/n = 18.29$.

Note that the algorithm is very fast and that the core topology is small. We can also solve the problem for a larger network (see table 3), even if the physical problem does not really make sense for a core network. For the size $n = 30$ (2^{30} nodes) and $\lambda = 2^n/n$ we obtain: $p = 8.11 \times 10^{-3}$, $u = 1.69 \times 10^{-2}$ and $E[T] = 15.18$. As we are only concerned here with the proof of the approximate analysis we had not checked the accuracy of the results versus a simulation, which is a clear extension of this paper.

7. CONCLUDING REMARKS

We have proved the algorithm for the topology of a hypercube with an arbitrary link capacity f. This algorithm may be generalized to any topology which satisfies the assumptions made in section 4. The proof of the algorithm is based on theorem 4 whose assumptions are more general than the ones we use for the end to end delay in an hypercube. For these topologies which satisfy the assumption of the theorem we just have to change the matrix and the number of links in the graph in equation (6) to correctly define the link utilization.

But the approach is even more general. In many problems in performance or reliability evaluation, we have to bound absorption times and we also have to prove algorithms which compute approximate solutions. To the best of our knowledge this is the first application of the comparison of Markov chains to prove the convergence of a numerical algorithm. We hope that such a result can foster new research activities on the proofs of algorithms based on Markov chains.

8. ACKNOWLEDGMENTS

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Table 3: Number of iterations for the hypercube topology with unitary links: $n = 30$, $\lambda = 2^n/n$

ϵ	10^{-5}	10^{-10}	10^{-15}	10^{-20}
iterations	6	10	14	18

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