Vendredi 28 Janvier 2011. Examen du cours ENS/DI "Structures et Algorithmes Aléatoires", Partie II: "Chaînes de Markov".

Exercise 1. Consider the following graph described by its set of vertices $E = \{0, 1, 2, \ldots, n, n+1, \ldots, 2n\}$ and its edges consisting of two sets

$(0, 1), (1, 2), \ldots, (n-1, n), (n, 0)$ (the "circle"), and

$(0, n+1), (n+1, n+2), \ldots, (2n-2, 2n-1), (2n-1, 2n)$ (the "tail")

Perform a random walk on this graph.

1. What is the average time between two consecutive passages in vertex 0?

2. What is the average time between two successive passages in the successive states $1, 0, n+1$ in this order? (When in 1 next step you are in 0 and second next step you are in $n+1$.)

Exercise 2. Usually, a laboratory mouse moves in a labyrinth of rooms (forming a finite set E) in the following way. When in room i, it chooses one of the d_i doors of room i (leading to different other rooms) at random. Our mouse in this experiment has another strategy. If it is in room i, it chooses one of the adjacent room at random (say, j), it inspects this room from where it is. If it finds that it has less doors ($d_j \leq d_i$) it enters room j. However, if it has more doors ($d_j > d_i$) it enters room j with the probability $\frac{d_i}{d_j}$, and otherwise stays in room i.

1. Under what condition is the transition matrix of the corresponding Markov chain irreducible?

2. Admitting that it is irreducible, under what condition is it aperiodic?

3. Give the transition matrix of the corresponding Markov chain.

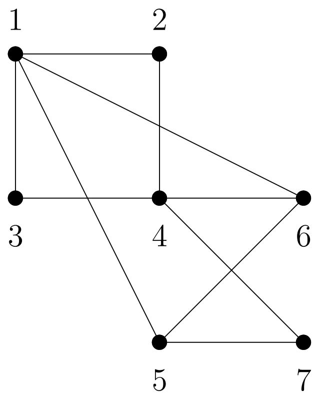
4. Under the condition of irreducibility, what is its stationary distribution?

5. Assuming irreducibility and aperiodicity, at some large given time N, the position of the mouse is recorded and one then records the time taken by the mouse to come back to where it was at time N. What is the expectation of this time?

Exercise 43 Consider the graph on $S = \{1, 2, 3, 4, 5, 6, 7\}$ in the figure below. Let the phase space be $\Lambda = \{-1, +1\}$. For a configuration $x \in \Lambda^S$, denote by $n(x)$ the number of positive *bonds*, that is, the number of edges of the graph for which the phases of the adjacent sites coincide. Define a probability distribution π on Λ^S by $\pi(x) = \frac{e^{-n(x)}}{Z}$.

1. Give the full Gibsian description of this random field (neighborhhoods, cliques, potential function).

2. Give its local characteristics.



Figure

Exercise 4. Let $\{S_n\}_{n\geq 1}$ be an IID sequence of random variables with values in $\mathbb{N}_+$ (the positive integers) and with the probability distribution $P(S_1 = k) = f_k$ assumed non-lattice. Define for $n \geq 0$, $R_{n+1} = R_n + S_{n+1}$, where $R_0 = 0$. Define for each $n \geq 0$ the backward recurrence time B_n and the forward recurrence time F_n by

$$B_n = n - L_n, \; F_n = N_n - n,$$

where $L_n = \sup\{R_k; \; k \geq 0, \; R_k \leq n\}$ and $N_n = \inf\{R_k; k > 0, \; R_k > n\}$. In particular, if $n = R_m$ for some m, then $B_n = 0$ and $F_n = R_{m+1} - R_m = S_{m+1}$. Observe that $F_n \geq 1$ for all $n \geq 0$. Also, if $n \in [R_m, R_{m+1})$, then $B_n + F_n = S_{m+1}$.

1. Give the asymptotic distribution of (B_n, F_n), that is, compute for all $i, j \in \mathbb{N}$ the quantity $\lim_{n\uparrow\infty} P(B_n = i, F_n = j)$ in terms of the distribution of S_1.

2. When the typical interrenewal time is geometric, compute $\lim_{n\uparrow\infty} P(F_n + B_n = k)$, $\lim_{n\uparrow\infty} P(F_n = k)$, and $\lim_{n\uparrow\infty} P(F_n + B_n = k)$.

3. Under what circumstances do we have $\lim_{n\uparrow\infty} P(F_n + B_n = k) = P(S_1 = k)$ for all $k \geq 1$?