1. INTRODUCTION

Network Calculus (NC) is a theory of deterministic queuing systems encountered in communications networks. It is based on \((\min, +)\) algebra and it can be seen as a \((\min, +)\) filtering theory by analogy with the \((+, \times)\) filtering theory used in traditional system theory. More than just a formalism, it enables to analyze complex systems and to prove deterministic bounds on delays, backlogs and other Quality-of-Service (QoS) parameters. The analysis usually focuses on worst-case performances. The information about the system is stored in functions such as arrival curves shaping the traffic or service curves quantifying the service guaranteed at the network nodes. These functions can be combined together thanks to special Network Calculus operations, in order to analyze the system and compute bounds on local performances (i.e. maximum buffer size at a node) or on end-to-end performances (i.e. maximum end-to-end delay). At the present time, the theory has developed and yielded accomplished results which are mainly recorded in two reference books: Chang (2000) and Le Boudec and Thiran (2001). A nice survey of NC including recent results can be found in Fidler (2010).

Nevertheless it remains difficult to draw the exact borders of Network Calculus at the time being. One of the main obstacles comes from the apparent variety of service curve definitions in the literature which might lead to different types of models. Readers are often warned to stick to the definition chosen in each paper in order to ensure the relevance of the model and the validity of its analysis. However it is rarely questioned whether another choice may lead to the same model or at least to the same performance evaluation.

Our general objective is to unveil the differences between models yielded by the different definitions of service curves. As a first step, we study their expressiveness. Such comparisons exist as folklore in the Network Calculus literature, but they are scattered, some proofs are missing and some comparisons are unset. Moreover the question of canonical service specifications has not been tackled. Given a system constrained by a family of service curves of a fixed type, is it possible to reduce or transform this family into a canonical one? Is it possible to translate it into a family of another type of service curves? We investigate those key issues in this paper. Those are necessary premises to present NC as an unified theory.

Section 2 introduces the NC framework and the service curve definitions we wish to classify: \textit{variable capacity node service, strict service, weakly strict service, simple service}. In Section 3, we compare those different definitions which fit into each other but with some large gaps. We fully characterize the cases when they are equivalent and for instance we show that the equivalence between variable capacity node service and strict service, often quoted as a folklore result, is often true but not always. We also state several results about the translation of service curve families into other families of the same type or of a different type. As soon as simple services are involved, these are mainly impossibility results. Section 4 concludes the paper with first assessments about those comparisons and discusses future works required to complete the big picture.

Note that the study also concerns alternative theories like Real-Time Calculus (RTC - Thiele et al. (2000); Wandeler (2006) or Sensor Calculus (SC - Schmitt and Roedig (2005)) that use extremely close formalisms (envelope-based models, \((\min, +)\) algebra). In particular, RTC models are described as a combination of RTC Greedy Processing Components which have exactly the same behavior as NC Variable Capacity Nodes, as shown in Bouillard et al. (2009).
2. DEFINITIONS AND NOTATION

2.1 NC functions and operations

Network Calculus’ primal objective is the performance analysis of communication networks. Flows and services in the network are modelled by non-decreasing functions \( t \mapsto f(t) \) where \( t \) is time and \( f(t) \) an amount of data. There are different models depending on whether \( t \) (resp. \( f(t) \)) takes discrete or continuous values, e.g. \( t \in \mathbb{N} \) or \( t \in \mathbb{R} \). In this paper, we will present the results in a fluid model where time and data quantities belong to \( \mathbb{R}_+ \), but it can be easily checked that they directly apply to models where time or data are discrete. Note also that we do not exactly use the term fluid as in Le Boudec and Thiran (2001) where the fluid model adds the condition that the manipulated functions are continuous.

In Network Calculus, one must distinguish two kinds of objects: the real movements of data and the constraints that these movements satisfy. The real movements of data are mainly modeled by cumulative functions: a cumulative function \( f(t) \) counts the total amount of data that has achieved some condition up to time \( t \) (e.g. the total amount of data which has gone through a given place in the network). In all the paper, we make the usual assumption that cumulative functions are left-continuous. This is not a huge restriction for the modeler. This assumption has nevertheless a technical importance in the Network Calculus edifice (e.g. when defining the start of backlogged periods). On the contrary, no assumption of (left- or right-)continuity is imposed to the constraint functions.

In the paper, Network Calculus functions will belong to \( \mathcal{F} \) the set of functions from \( \mathbb{R}_+ \) into \( \mathbb{R} = \mathbb{R} \cup \{ -\infty, +\infty \} \). Cumulative functions usually belong to \( \mathcal{F}_1 = \{ f \in \mathcal{F} \mid f \text{ non-decreasing, left-continuous, } f(0) = 0 \} \).

Beyond usual operations like the minimum or the addition of functions, Network Calculus makes use of several classical operations which are the translations of \((+,:\times)\) filtering operations into the \((\min,\max)\) operations which are the translations of \((+,:\times)\) filtering operations into the \((\min,\max)\) operations. In all the paper, we will present the results in a fluid model where time and data quantities belong to \( \mathbb{R}_+ \), but it can be easily checked that they directly apply to models where time or data are discrete. Note also that we do not exactly use the term fluid as in Le Boudec and Thiran (2001) where the fluid model adds the condition that the manipulated functions are continuous.

Such operations have interesting algebraic properties (e.g. see Baccelli et al. (1992)). Network Calculus formulas use such operations to combine the curves constraining the traffic and the services in the network, in order to output worst-case performance bounds.

2.2 NC input/output systems

An NC model for a communication network usually consists in a partition of the network into subsystems which may have different scales (from elementary hardware like a processor to large sub-networks), a description of data flows, where each flow follows a path through a specified sequence of subsystems and where each flow is shaped by some arrival curve just before entering the network, a description of the behavior of each subsystem, that is service curves bounding the performances of each subsystem, as well as service policies in case of multiplexing (several flows entering the same subsystem and thus sharing its service).

Systems or sub-systems are described as input/output systems (where the number of inputs is the same as the number of outputs). An (acceptable) trajectory for a system crossed by \( p \) flows is a set of cumulative functions \((A_k)_1\leq k \leq p \) and \((B_k)_1\leq k \leq p \) in \( \mathcal{F}_1 \) (where \( A_k \) and \( B_k \) respectively correspond to the cumulative functions of flow \( k \) at the input and the output of the system). For now, a system \( \mathcal{S} \) over \( p \) flows will be simply defined as the set of all its acceptable trajectories, that is \( \mathcal{S} \subseteq \mathcal{F}_1^p \times \mathcal{F}_1^p \).

Such a black boxed view is usual in classical filtering theory and enables to deal with any scale of system. Note also that this definition allows to consider deterministic dynamics (one output for one input) and non-deterministic dynamics (several possible outputs for one input).

2.3 NC main performance measures: backlog & delay

Let \((A,B)\) be an input/output trajectory for a flow in a system. Then the global backlog of the flow at time \( t \) is \( b(t) = A(t) - B(t) \) and the delay (under the FIFO policy assumption) endured after \( z \) input bits is \( d(z) = B^{-1}(z) - A^{-1}(z) \) where for all \( f \in \mathcal{F} \), \( f^{-1}(z) = \inf \{ t \geq 0 \mid f(t) \geq z \} \) (pseudo-inverse). Now for a system \( \mathcal{S} \), the worst-case backlog over \( \mathcal{S} \) is \( \max_{(A,B)\in\mathcal{S}} \sup_{t\geq 0} A(t) - B(t) \) and the worst-case delay over \( \mathcal{S} \) is \( \max_{(A,B)\in\mathcal{S}} \sup_{z\geq 0} B^{-1}(z) - A^{-1}(z) \).

Given a trajectory \((A,B)\in\mathcal{F}_1^p \times \mathcal{F}_1^p \), a backlogged period is an interval \( I \subseteq \mathbb{R}_+ \) of time during which the backlog is non-null, i.e. \( \forall u \in I, A(u) > B(u) \). Let \( t \in \mathbb{R}_+ \), the start of the backlogged period of \( t \) is \( \text{start}(t) = \sup \{ u \leq t \mid A(u) = B(u) \} \). Since the cumulative functions \( A \) and \( B \) are assumed left-continuous, we also have \( A(\text{start}(t)) = B(\text{start}(t)) \). If \( A(t) = B(t) \), then \( \text{start}(t) = t \). For any \( t \in \mathbb{R}_+ \), \( \text{start}(t), \text{end}(t) \) is a backlogged period \((\text{start}(t), \text{end}(t)) \) if \( A(t) > B(t) \).

In the definition of backlogged period, the interval \( I \) can be closed, semi-closed or open. Such a flexible definition is convenient in some future definitions or proofs where the precise description of trajectories requires a particular type of intervals, e.g. semi-closed rather than open (see the consequences of such choices in Section 2.6). Note that in the literature, backlogged periods have been sometimes defined for open intervals only Bouillard et al. (2008) (page 885) or without worrying about this question Le Boudec and Thiran (2001) (Definition 1.3.2, page 21).
2.4 NC arrival curves: one definition

Given a data flow traversing a network, let $A \in \mathcal{F}_1$ be its cumulative function at some point in the network, i.e. $A(t)$ is the number of bits that have gone through this point until time $t$, with $A(0) = 0$. A function $\alpha \in \mathcal{F}$ is an arrival curve for $A$ if $\forall s, t \in \mathbb{R}_+$, $0 \leq s \leq t$, we have $A(t) - A(s) \leq \alpha(t - s)$.

The set of all arrival curves for $A \in \mathcal{F}_1$ admits a minimum which remains an arrival curve; it is $\alpha = A \otimes A$ and it can be called the canonical arrival curve for $A$.

2.5 NC service curves: several definitions

In the literature, the definitions of service curves usually concern:

- minimum service curves which are lower bounds on the service provided in a system (useful for upper bounds on worst case performances).
- single flow systems $\mathcal{S}$, that is $S \subseteq \mathcal{F}_1 \times \mathcal{F}_1$.

Note that in NC models with multiplexing, the aggregation of all the flows entering the system is often considered as a single flow to which the minimum service is applied.

![Fig. 1. A single flow input/output system.](image)

For each type $T$ of service curve, we define for any $\beta \in \mathcal{F}$ and for any input/output trajectory $(A, B) \in \mathcal{F}_1 \times \mathcal{F}_1$ the conditions so that $\beta$ is a $T$-service curve for $(A, B)$ (we also say that $(A, B)$ admits $\beta$ as a $T$-service curve). We then define for all $\beta \in \mathcal{F}$, $S_T(\beta)$ the set of all trajectories admitting $\beta$ as a $T$-service curve. We say that a system $\mathcal{S}$ admits $\beta$ as a $T$-service curve if it is true for all its trajectories, i.e. $S \subseteq S_T(\beta)$.

- **Simple service curve**: $S_{\text{simple}}(\beta) = \{(A, B) \in \mathcal{F}_1 \times \mathcal{F}_1 \mid A \geq B \geq A \oplus B\}$.
- **Strict service curve (weak sense)**: $S_{\text{wstrict}}(\beta) = \{(A, B) \in \mathcal{F}_1 \times \mathcal{F}_1 \mid A \geq B, \text{ and } \forall t \geq 0, B(t) \geq B(\text{start}(t)) + \beta(t - \text{start}(t))\}$.
- **Strict service curve**: $S_{\text{strict}}(\beta) = \{(A, B) \in \mathcal{F}_1 \times \mathcal{F}_1 \mid A \geq B, \text{ and } a \text{ backlogged period } [s, t], B(t) \geq B(s) + \beta(t - s)\}$.

Two classical functions used as service curves are:

- Pure delay $T \in \mathbb{R}_+ \cup \{+\infty\}$: $\delta_T(t) = 0$ if $t \leq T$, $= +\infty$ otherwise.
- Constant rate $r \in \mathbb{R}_+ \cup \{+\infty\}$: $\lambda_r(t) = rt$ (if $r = +\infty$, we set $\lambda_r = \delta_0$).

Nothing prevents us from using some service curves which are not in $\mathcal{F}_1$, e.g. with negative values, decreasing parts or left-discontinuities. Nevertheless note that it is usually required that at least $\beta(0) \leq 0$, otherwise $\beta(0) > 0$ implies that $B(0) > A(0)$ and thus $S_{\text{simple}}(\beta) = S_{\text{wstrict}}(\beta) = S_{\text{strict}}(\beta) = S_{\text{vcn}}(\beta) = \emptyset$.

The definition of strict service curves presented here is the one used by Schmitt and Zdarsky (2006); Schmitt et al. (2006). Some papers do not choose exactly the same definition for strict service curves Bouillard et al. (2007, 2008); they replace the backlogged interval $[s, t]$ in the definition by $[s, t]$ (both definitions allow $B(s) = A(s)$, but this variant also allows $B(t) = A(t)$). For $\beta \in \mathcal{F}$, let us denote $S_{\text{strict}}^{(\beta)}$ the set of trajectories satisfying this variant of our definition. How do those slightly different definitions compare? It is clear that $\forall \beta \in \mathcal{F}$, $S_{\text{strict}}(\beta) \subseteq S_{\text{strict}}^{(\beta)}$ and the equality holds if $\beta$ is left-continuous. If $\beta$ is not left-continuous we may have a strict inclusion as illustrated by Figure 2 where $A(t) = 1/2$ if $t > 0$ and $= 0$ if $t = 0$, $B(t) = \min(t/2, 1/2)$, $\beta(t) = [t]$, and $(A, B) \in S_{\text{strict}}(\beta)$ but $(A, B) \notin S_{\text{strict}}^{(\beta)}$ (see Bouillard et al. (2009) for details).

![Fig. 2. Beware of the definition of strict service curves.](image)

2.6 Remark on strict service curves

**Lemma 1.** Let $A, C \in \mathcal{F}_1$ and $B \in \mathcal{F}$ such that $\forall t \geq 0$, $B(t) = \inf_{0 \leq s \leq t} [A(s) + C(t) - C(s)]$. Then $B \in \mathcal{F}_1$ and $\forall t \geq 0, B(t) = A(\text{start}(t)) + C(t) - C(\text{start}(t))$.

The proof of that result is a bit technical and can be found in Bouillard et al. (2009).

3. COMPARISON OF NC SERVICE CURVES

3.1 Monotony

All the definitions from the literature share the same natural monotonic behavior about trajectories.

**Proposition 1.** (Monotony). For any type $T$ of service curve in the literature, for all $\beta, \beta' \in \mathcal{F}$ (not necessarily in $\mathcal{F}_1$), if $\beta \leq \beta'$ then $S_T(\beta) \supseteq S_T(\beta')$.

Moreover, for variable capacity node, strict and weakly strict service curves, one can replace service curves by some of their closures.

**Proposition 2.** Let $\beta \in \mathcal{F}$, then $S_{\text{wstrict}}(\beta) = S_{\text{wstrict}}(\beta_1), S_{\text{strict}}(\beta) = S_{\text{strict}}(\beta_1), S_{\text{strict}}(\beta) = S_{\text{strict}}(\beta_1), S_{\text{vcn}}(\beta) = S_{\text{vcn}}(\beta_1)$ and $S_{\text{vcn}}(\beta) = S_{\text{vcn}}(\beta_1)$.

The previous result is considered as folklore and the proof is rather simple. The next theorem is new.

**Theorem 1.** (Monotony refined). Let $\beta, \beta' \in \mathcal{F}$,

1. $S_{\text{simple}}(\beta) \supseteq S_{\text{simple}}(\beta') \iff \beta \leq \beta'$.
2. $S_{\text{simple}}(\beta) \supseteq S_{\text{simple}}(\beta') \Rightarrow \beta \leq \beta'$.
3. $S_{\text{simple}}(\beta) \supseteq S_{\text{simple}}(\beta') \Rightarrow \beta_1 \leq \beta'$.
4. $S_{\text{simple}}(\beta) \supseteq S_{\text{simple}}(\beta') \Rightarrow \beta_1 \leq \beta'$. 

(5) $S_{\text{simple}}(\beta_1) \supseteq S_{\text{simple}}(\beta'_1)$ if and only if $\beta_1 \leq \beta'_1$.
(6) $S_{\text{wstrict}}(\beta) \supseteq S_{\text{wstrict}}(\beta')$ if and only if $\beta \leq \beta'$.
(7) $S_{\text{strict}}(\beta) \supseteq S_{\text{strict}}(\beta')$ if and only if $(\beta_1)^{+} \leq (\beta'_1)^{+}$.
(8) $S_{\text{ecn}}(\beta) \supseteq S_{\text{ecn}}(\beta')$ if and only if $(\beta_1)^{+} \leq (\beta'_1)^{+}$.

Proof. (1) Proposition 1.

(2) Take $\beta'(t) = 0$ if $t = 0$ or $t \in [1, 2]$ and $\beta'(t) = +\infty$ otherwise and $\beta(t) = 0$ if $t \in [0, 1]$ or $t \in [2, +\infty]$ and $\beta(t) = +\infty$ otherwise. Then, $\beta \not\subseteq \beta'$ but, $\forall A \in \mathcal{F}_1$, $(A \ast \beta')(t) = A(t)$ if $t \in [0, 1]$, $\beta'(t)$ if $t \in [1, 2]$ and $\beta(t) = A(0)$ if $t \in [1, 2]$ and $A = 1$ otherwise. Then, $(A \ast \beta') \supseteq (A \ast \beta)$ and $S_{\text{simple}}(\beta) \supseteq S_{\text{simple}}(\beta')$. Figure 3 illustrates this construction.

Fig. 3. $S_{\text{simple}}(\beta) \supseteq S_{\text{simple}}(\beta') \not\Rightarrow \beta \leq \beta'$.

(3) $(\delta_0, \beta'_1) \in S_{\text{simple}}(\beta')$, so $(\delta_0, \beta'_1) \in S_{\text{simple}}(\beta)$ and then $\beta'_1 \geq (\delta_0 \ast \beta) = \beta_1$.

(4) Take $\beta'(t) = 0$ if $t = 0$ or $t \in [1, +\infty]$ and $\beta'(t) = +\infty$ otherwise, and $\beta = \beta' = \delta_0$. We have $\beta_1 \leq \beta'_1$ and $S_{\text{simple}}(\beta) = \{(A, A) \mid A \in \mathcal{F}_1\}$, but $\forall A \in \mathcal{F}_1$, $(A, A(\min(., 1))) \in S_{\text{simple}}(\beta')$. Figure 4 illustrates this construction.

Fig. 4. $S_{\text{simple}}(\beta) \supseteq S_{\text{simple}}(\beta') \not\Rightarrow \beta_1 \leq \beta'_1$.

(5) $\Rightarrow$: idem (3); $\Leftarrow$: Proposition 1.

(6-8) Proposition 2 and $\Rightarrow$: idem (3); $\Leftarrow$: Proposition 1.

3.2 Families of service curves

We now discuss the fact that a system may admit several service curves of one or several types. Let $(\beta_i)_{i \in J}$ be a (possibly infinite) family of functions from $\mathcal{F}$, let $T$ be a particular type of service curve. The fact that a system $S$ admits all the functions $\beta_i$ as $T$-service curve can be also written $S \subseteq \bigcap_{i \in J} S_T(\beta_i)$.

Theorem 2. (Families of curves). (1) Let $I$ and $J$ be finite sets and $(\beta_i)_{i \in I}$ and $(\beta'_j)_{j \in J}$ be two families in $\mathcal{F}_1$. Then, $\bigcap_{i \in I} S_{\text{simple}}(\beta_i) = \bigcap_{i \in I} S_{\text{simple}}(\beta'_i)$ if $\{\beta \in \mathcal{F} \mid \exists i \in I, \beta \leq \beta_i\} = \{\beta \in \mathcal{F} \mid \exists j \in J, \beta \leq \beta'_j\}$.

(2) $\bigcap_{i \in I} S_{\text{wstrict}}(\beta_i) = S_{\text{wstrict}}(\sup_{i \in I} (\beta_i))$.
(3) $\bigcap_{i \in I} S_{\text{strict}}(\beta_i) = S_{\text{strict}}(\sup_{i \in I} (\beta_i))$.
(4) $\bigcap_{i \in I} S_{\text{ecn}}(\beta_i) = S_{\text{ecn}}(\sup_{i \in I} (\beta_i))$.

Proof. (1) First set $I = \{0, \ldots, k\}$ and consider $\cap_{i \in I} S_{\text{simple}}(\beta_i)$ such that functions $\beta_i$ are pairwise non-comparable. Then, there exists $t_1, \ldots, t_k$ such that $\forall i \in I \setminus \{0\}$, $\beta_0(t_k) > \beta_i(t_k)$ and one can assume without loss of generality that $0 < t_1 < \cdots < t_k$.

Set $A(t) = \{\begin{array}{cl} 0 & \text{if } t = 0, \\ \beta_0(t_k) - \beta_0(t_1) & \text{if } t_1 < t_k < t < t_{i+1}, \\ \alpha & \text{if } t > t_{k-1}. \end{array}$

Now, for all $i \in I$, $A \ast \beta_i(t_k) = \inf_j (\beta_j(t_k) + A(t_k - t_j)) = \inf_j (\beta_j(t_k) + \beta_0(t_k) - \beta_0(t_j))$. Then, $A \ast \beta_i(t_k) = \beta_0(t_k)$ and $\forall i \in \{0, \ldots, k\}, A \ast \beta_i(t_k) \leq \beta_i(t_k) + \beta_0(t_k) - \beta_0(t_k)$. Thus $\cap_{i \in I} S_{\text{simple}}(\beta_i) \subseteq \cap_{i \in I \setminus \{0\}} S_{\text{simple}}(\beta_i)$. Figure 5 illustrates this.

Fig. 5. Non-inclusion of families of curves.

Come back to $\cap_{i \in I} S_{\text{simple}}(\beta_i) = \cap_{j \in J} S_{\text{simple}}(\beta'_j)$ where the $\beta_i$’s are pairwise non-comparable and so are the $\beta'_j$’s. We also have for all $j \in J$, $\cap_{i \in I} S_{\text{simple}}(\beta_i) \cap S_{\text{simple}}(\beta'_j) = \cap_{i \in I} S_{\text{simple}}(\beta_i)$, then, by the contraposition of the previous paragraph, $\exists i \in I$, such that $\beta_i' \leq \beta_i$. By symmetry, $\forall i \in I, \exists j$ such that $\beta_i \leq \beta'_j$. As functions in $I$ and $J$ are not two-by-two comparable, then this means that $\forall i \in I, \exists j \in J$ such that $\beta_i = \beta'_j$ and the symmetric.

(2) $\forall i \in I, S_{\text{wstrict}}(\beta_i) \supseteq S_{\text{wstrict}}(\sup_{i \in I} (\beta_i))$, thus $\bigcap_{i \in I} S_{\text{wstrict}}(\beta_i) \supseteq S_{\text{wstrict}}(\sup_{i \in I} (\beta_i))$. Let $(A, B) \in S_{\text{wstrict}}(\sup_{i \in I} (\beta_i))$. Then, $\forall t \in \mathbb{R}_+, \forall i \in I, B(t) \geq A(\sup_{i \in I} (t) - \beta_i(t) - \beta_i'(t), \beta_i'(t) - \beta_i(t))$. Then, $A \ast (\beta_i(t) - \beta_i'(t)) = B(t) - \beta_i'(t)$. Then, $\forall i \in I, \exists j \in J$ such that $\beta_i = \beta'_j$.

(3) Idem 2., except: Let $(A, B) \in \bigcap_{i \in I} S_{\text{strict}}(\beta_i)$, then $\exists s < t \in \mathbb{R}_+, \forall i \in I, B(t) \geq A(\beta_i(t) - \beta_i'(t), \beta_i'(t) - \beta_i(t))$, so $B(t) \geq B(s) + \sup_{i \in I} (\beta_i(t) - \beta_i'(t))$ and $(A, B) \in S_{\text{wstrict}}(\sup_{i \in I} (\beta_i')) = S_{\text{wstrict}}(\sup_{i \in I} (\beta_i'))$.

(4) Idem 3., replacing $B$ by $C$.

3.3 Hierarchy

The next hierarchy between the different notions of service curves is often considered as folklore, but the cases of equality have never been investigated (e.g. $S_{\text{ecn}}(\beta) = S_{\text{strict}}(\beta)$ is claimed with no assumption on $\beta$ in Le Boudec and Thiran (2001) and Wandelier (2006)).

Theorem 3. (Hierarchy). For all $\beta \in \mathcal{F}$, we have the following inclusions:

$S_{\text{ecn}}(\beta) \subseteq S_{\text{strict}}(\beta) \subseteq S_{\text{wstrict}}(\beta) \subseteq S_{\text{simple}}(\beta)$.

The equalities require:
by definition of the start of those periods. We suppose that $t^\beta_0 > 0$. Then, let $(A, B) \in S_{\text{strict}}(\beta)$. Let $s < t$ in the same backlogged period. Then, $t - s < T$ and $B(t) \geq B(s) = B(s) + \beta(t - s)$.

Otherwise $\beta_1$ is not a delay: let $A = \delta_0$. There exists $s > 0$ such that $0 < \beta(s) < \infty$. Let $t_0 = \sup \{t \geq 0 \mid \beta(t) = 0\} \leq s$. Define $B(t) = \beta(t)$ in $t < (s - t_0)/2$ or $t > s$, and $B(s)$ if $t \in [(s - t_0)/2, s]$. We have $(A, B) \in S_{\text{strict}}(\beta) \setminus S_{\text{strict}}(\beta)$. Indeed, $B(s) - B((s - t_0)/2) = 0 < \beta(t)(s + t_0)/2$.

$S_{\text{strict}}(\beta) = S_{\text{simple}}(\beta)$: if $\beta_1 = \delta_\beta$, then $(A, B) \in S_{\text{strict}}(\beta)$.

Moreover, $B(t) = A(\text{start}(t)) + C(t) - C(\text{start}(t))$, $\forall s \in [\text{start}(t), t]$, $B(t) \leq A(s) + C(t) - C(s)$, and $\forall s < \text{start}(t)$, $C(s) - A(s) \geq C(\text{start}(t)) - A(\text{start}(t))$ so $B(t) \leq A(s) + C(t) - C(s)$. Then, $(A, B) \in S_{\text{vconc}}(\beta)$.

On the other hand, if $\beta(\beta(t)) = \infty$, take $(\delta_{t_0}, \beta \ast \delta_\beta) \in S_{\text{strict}}(\beta)$. In order to make the computations possible, $C(t)$ must be finite. But, one must have $C(t \pm t_0) = \beta(t)$ and $C(t + t_0) - C(0) \geq \beta(t + t_0)$ and $C(t) - C(t + t_0) \geq +\infty$.

$S_{\text{strict}}(\beta) = S_{\text{simple}}(\beta)$: if $\beta_1 = \delta_\beta$, $T \in \{\text{simple}, \text{strict}, \text{weakly strict}\}$ for any arrival process, equality cases are more frequent: for simple and weakly strict service curves, the equality does not hold, there is no chance to express the equality does not hold, there is no chance to express the equality does not hold, there is no chance to express the equality does not hold, there is no chance to express the equality does not hold, there is no chance to express the equality does not hold, there is no chance to express the equality does not hold, there is no chance to express the equality does not hold, there is no chance to express the equality does not hold, there is no chance to express the equality does not hold, there is no chance to express the equality does not hold, there is no chance to express the equality does not hold, there is no chance to express the equality does not hold, there is no chance to express the equality does not hold, there is no chance to express the equality does not hold, there is no chance to express the equality does not hold, there is no chance to express the equality does not hold, there is no chance to express the equality does not hold, there is no chance to express the equality does not hold, there is no chance to express the equality does not hold, there is no chance to express the equality does not hold, there is no chance to express the equality does not hold, there is no chance to express the equality does not hold, there is no chance to express the equality does not hold, there is no chance to express the equality does not hold, there is no chance to express the equality does not hold, there is no chance to express the equality does not hold, there is no chance to express the equality does not hold, there is no chance to express the equality does not hold, there is no chance to express the equality does not hold, there is no chance to express the equality does not hold, there is no chance to express the equality does not hold, there is no chance to express the equality does not hold, there is no chance to express the equality does not hold, there is no chance to express the equality does not hold, there is no chance to express
one type of service curve as a combination of service curves of another type.

**Theorem 4.** (No translation with families.) Let $T$ and $T'$ be two different types of service curves among $vcn$, $simple$, $strict$ ($weak$), $strict$. Then 
\[ S_T^\beta(\lambda_i)_{i \in F^I}, (\lambda_j')_{j \in F^J}, \bigcap_{i \in I} S_T^\beta(\lambda_i) = \bigcap_{j \in J} S_T^\beta(\lambda_j'), \]
except for the equality cases defined in Theorem 3.

**Proof.** Translation between variable capacity node and strict, and between weakly strict and strict service curves: consider $(\beta_i)_{i \in F^I}$ and $(\beta_j')_{j \in F^J}$, and suppose that $S_T^\beta(\lambda_i) = S_T^{\beta'}(\lambda_i)$. Then, from Theorem 3, $S_T^\beta(\lambda_i) \subseteq S_T^{\beta'}(\lambda_i)$ and from Theorem 1, $\beta_i \leq \beta_j$. Now, $(\delta_0, \beta_i') \in S_T^\beta(\lambda_i') = S_T^{\beta'}(\lambda_i')$, then $\beta' \geq \beta$. In conclusion, one must have $\beta = \beta'$.

Between simple and weakly strict service curves: consider $(\beta_i)_{i \in F^I}$ and $(\beta_j')_{j \in F^J}$, and suppose that $\sup_{i \in I} S_{\text{simple}}(\beta_i) = S_{\text{wstrict}}(\beta')$. Then, $\beta' \leq \sup_{i \in I} \beta_i$. On the other hand, $S_{\text{wstrict}}(\beta') = \cap_{i \in I} S_{\text{simple}}(\beta_i)$, then $\beta' \geq \sup_{i \in I} \beta_i$. Moreover, $S_{\text{wstrict}}(\sup_{i \in I} \beta_i) = \cap_{i \in I} S_{\text{simple}}(\beta_i) = S_{\text{wstrict}}(\beta')$. Then, all inequalities are in fact equalities and then one must have $S_{\text{simple}}(\sup_{i \in I} \beta_i) = \cap_{i \in I} S_{\text{simple}}(\beta_i)$ and $\beta' = \sup_{i \in I} \beta_i = \delta_0$ or 0.

\[ \Box \]

4. CONCLUSION

We hope that this study casts new light on the main service curve definitions used in envelope-based models. We have shown that there exist strong gaps in the hierarchy, like the impossibility to express a strict service as a family of simple services, even if we allow infinite families. As a consequence, providing an unified framework to achieve tight performance analyses for all the corresponding models, may turn out to be very difficult. As a matter of fact, while some tight analyses of simple curve models exclusively rely on $(\min, +)$ algebra (window flow control or multimedia traffic smoothing in Chang (2000); Le Boudec and Thiran (2001)), some tight analyses of strict curve models completely avoid any reference to $(\min, +)$ worst case performances in some acyclic networks in Bouillard models completely avoid any reference to $(\min, +)$ algebra (window flow control or multimedia traffic smoothing in Chang (2000); Le Boudec and Thiran (2001)).

Nevertheless further research is required about the notion of service curves in order to fill those gaps or take into account additional features.

One can investigate the possibility to mix different definitions, as suggested with adaptative service curves (mixing strict service and simple service) in Le Boudec and Thiran (2001), or try to introduce a full range of intermediate definitions between the classical definitions exposed in the article.

One should also include maximum service curves in the study, like maximum simple service curves defined in Network Calculus (Le Boudec and Thiran, 2001, Chapter 1, pages 42-48) or upper service curves intensively used in Real-Time Calculus (Wandeler (2006)). Their use introduces new connections between the different notions. For instance, a constant rate server can be seen as a server with a minimum and a maximum simple service curves both equal to $\lambda_r(t) = rt$. In this case, it is easy to check that this server admits $\lambda_r$ as a strict service curve. In contrast a system defined exclusively by minimum service curves can only ensure either $\lambda_r$ or $\delta_0$ as strict services.

Beyond the expressiveness issue, other important aspects include the robustness of the definitions with regard to flow multiplexing. For instance, a server with fixed priorities offering a simple minimum service, does not guarantee any non null simple service to the flow with lowest priority.

Ultimately such studies will be useful to adjust both the modeling power (to be relevant) and the computational complexity of analyses (to be effective), in Network Calculus and its extensions.

REFERENCES


