Wireless communications: from simple stochastic geometry models to practice
I Coverage

B. Błaszczyszyn Inria/ENS

Workshop on Probabilistic Methods in Telecommunication

WIAS Berlin, November 14–16, 2016

Outline

- **COVERAGE**
- **CONNECTIVITY**
- **CAPACITY**

COVERAGE

- Availability of the network for one user (test users) in the space.

COVERAGE

- Availability of the network for one user (test users) in the space.
- Stochastic geometry provides simple models and tools.
- Information theory suggests more adequate coverage models.
- Quantitative results with Poisson process modeling transmitters in the space.
- We shall present the SINR (or shot-noise) coverage model for cellular networks and its relations to Poisson-Dirichlet processes.

CONNECTIVITY

- Multi-hop connecting at least two users (source and destination) distant in space. Existence of routes.

CONNECTIVITY

- Multi-hop connecting at least two users (source and destination) distant in space. Existence of routes.
- Percolation theory provides tools to study macroscopic connectivity.
- First passage percolation to study the speed of message propagation on long routes.
- Mostly qualitative results.
- Comparisons methods for non-Poisson models.
- We shall present some results on connectivity and routing on the SINR graph.

CAPACITY

- Ability to serve simultaneously many users. How many? Quality of service in function to the number of served users.

CAPACITY

- Ability to serve simultaneously many users. How many? Quality of service in function to the number of served users.
- Queueing theory in association with stochastic geometry.
- Space-time models. Simulations required for quantitative results.
- We shall present some model capturing the dependence between the traffic demand and the quality of service in large cellular networks, validated w.r.t. some real data.

COVERAGE

OUTLINE

- Poisson point process,
- Germ-grain coverage models in stochastic geometry,
- SINR (or shot-noise) coverage model,
- Palm and stationary coverage characteristics,
- Relations to Poisson-Dirichlet processes.

Poisson point process

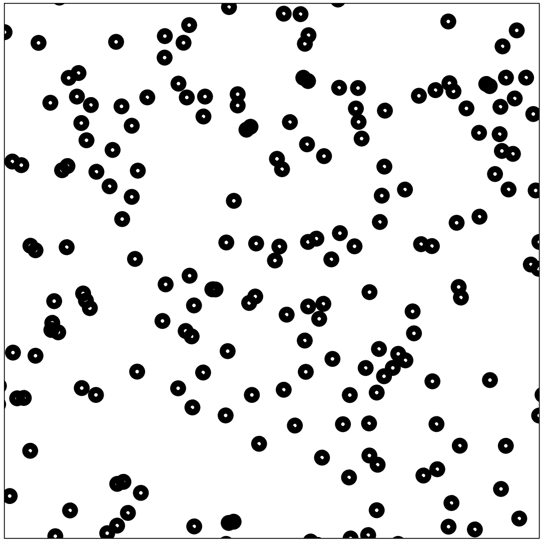
Poisson point process

DEF. Poisson point process Φ of intensity λ on the plane $\mathbb{R}^2$

- Number of Points $\Phi(B)$ of Φ in subset B of the plane is Poisson random variable with parameter $\lambda|B|$, where $|\cdot|$ is the Lebesgue measure on the plane; i.e.,

$$\mathsf{P}\{\,\Phi(B) = k\,\} = e^{-\lambda|B|}\,\frac{(\lambda|B|)^k}{k!}\,,$$

- Numbers of points of Φ in disjoint sets are independent.



Laplace transform of Poisson process

FACT Laplace transform of the Poisson process

$$\mathcal{L}_\Phi(h) = \mathsf{E}[e^{\int h(x)\,\Phi(\mathrm{d}x)}] = e^{-\lambda \int (1-e^{h(x)})\,\mathrm{d}x},$$

where $h(\cdot)$ is a real function on the plane and $\int h(x)\,\Phi(\mathrm{d}x) = \sum_{X_i \in \Phi} h(X_i)$.

Slivnyak's theorem

THM Conditioning Poisson process on having a point at some location, say at the origin $\mathbf{0}$, does not modify the distribution of other points.

$$\mathsf{P}^{\mathbf{0}}\{\mathbf{\Phi} \setminus \mathbf{0} \in \mathbf{\Gamma}\} = \mathsf{P}\{\mathbf{\Phi} \in \mathbf{\Gamma}\}\,,$$

where $\mathbf{\Gamma}$ is some subset of realizations of $\mathbf{\Phi}$ (configurations of points).

Slivnyak's theorem

THM Conditioning Poisson process on having a point at some location, say at the origin $\mathbf{0}$, does not modify the distribution of other points.

$$\mathsf{P}^{\mathbf{0}}\{\Phi \setminus 0 \in \Gamma\} = \mathsf{P}\{\Phi \in \Gamma\}\,,$$

where Γ is some subset of realizations of Φ (configurations of points).

- More formally, $\mathsf{P}^{\mathbf{0}}$ is called Palm probability and defined
$$\mathsf{P}^{\mathbf{0}}\{\Phi \in \Gamma\} = \frac{1}{\lambda|B|}\mathsf{E}\Big[\sum_{X_i \in \Phi \cap B} \mathbf{1}(\Phi - X_i \in \Gamma)\Big],$$
with any B: $0 < |B| < \infty$.
- Under $\mathsf{P}^{\mathbf{0}}$, the origin $\mathbf{0} \in \Phi$ is called the typical point of Φ.

Poisson process as a limit

- Random independent thinning of points of arbitrary point process (pp) converges to Poisson pp, provided the retention probability goes to 0, and the process is rescaled to preserve constant intensity.

Poisson process as a limit

- Random independent thinning of points of arbitrary point process (pp) converges to Poisson pp, provided the retention probability goes to 0, and the process is rescaled to preserve constant intensity.
- Random independent displacement of points of pp converges to Poisson pp, provided ...
  cf e.g. [Daley&Vere-Jones 1988]

Poisson process as a limit

- Random independent thinning of points of arbitrary point process (pp) converges to Poisson pp, provided the retention probability goes to 0, and the process is rescaled to preserve constant intensity.
- Random independent displacement of points of pp converges to Poisson pp, provided ...
  cf e.g. [Daley&Vere-Jones 1988]

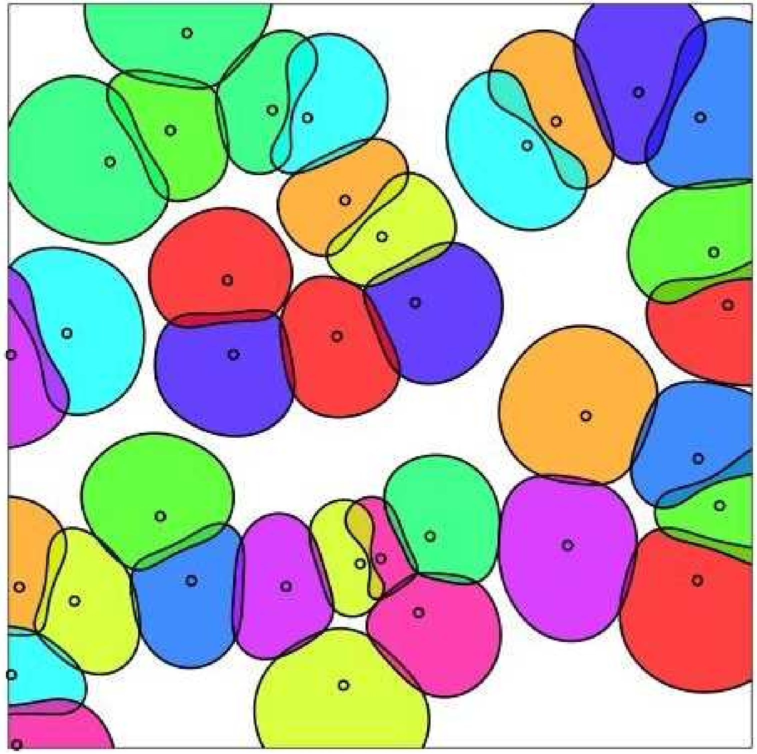
In wireless network context: Arbitrary homogeneous network of transmitters with strong random propagation effects is perceived at a given location as an equivalent Poisson network without shadowing.

see Dominic Schuhmacher's talk

Germ-grain coverage models
in stochastic geometry

General germ-grain coverage model

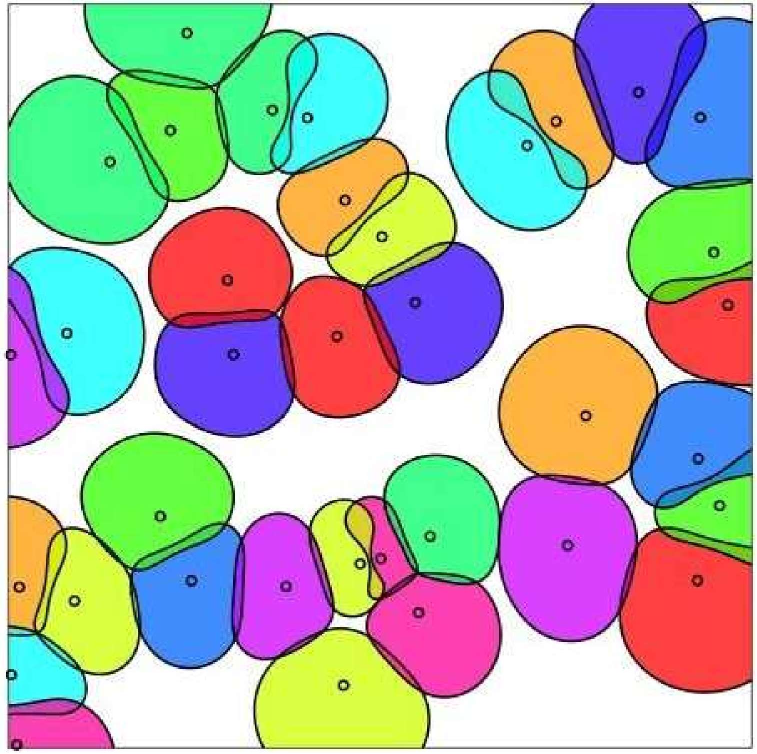
Germ-Grain (GG) coverage model $\{(X_i, \mathcal{C}_i)\}$, where $\{X_i\}$ are germs forming a point process Φ on $\mathbb{R}^d$, and $\mathcal{C}_i = \mathcal{C}_i(X_i, \Phi)$ are, possibly dependent, random closed subsets of $\mathbb{R}^d$, called grains.



Germ – communicating device
Grain – its coverage region

General germ-grain coverage model

Germ-Grain (GG) coverage model $\{(X_i, \mathcal{C}_i)\}$, where $\{X_i\}$ are germs forming a point process Φ on $\mathbb{R}^d$, and $\mathcal{C}_i = \mathcal{C}_i(X_i, \Phi)$ are, possibly dependent, random closed subsets of $\mathbb{R}^d$, called grains.

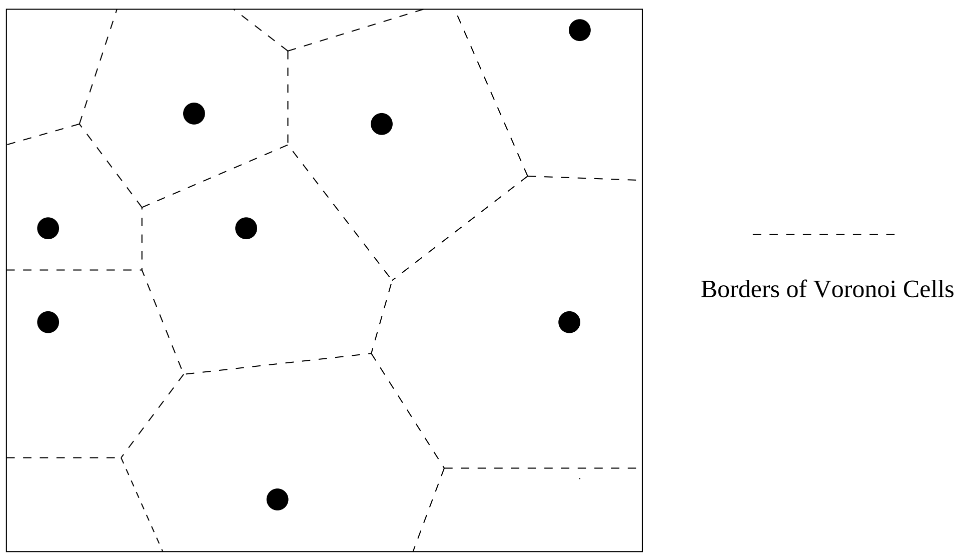


Germ – communicating device
Grain – its coverage region

Voronoi tessellation and Boolean Model are special cases of GG coverage model.

Voronoi tessellation (VT)

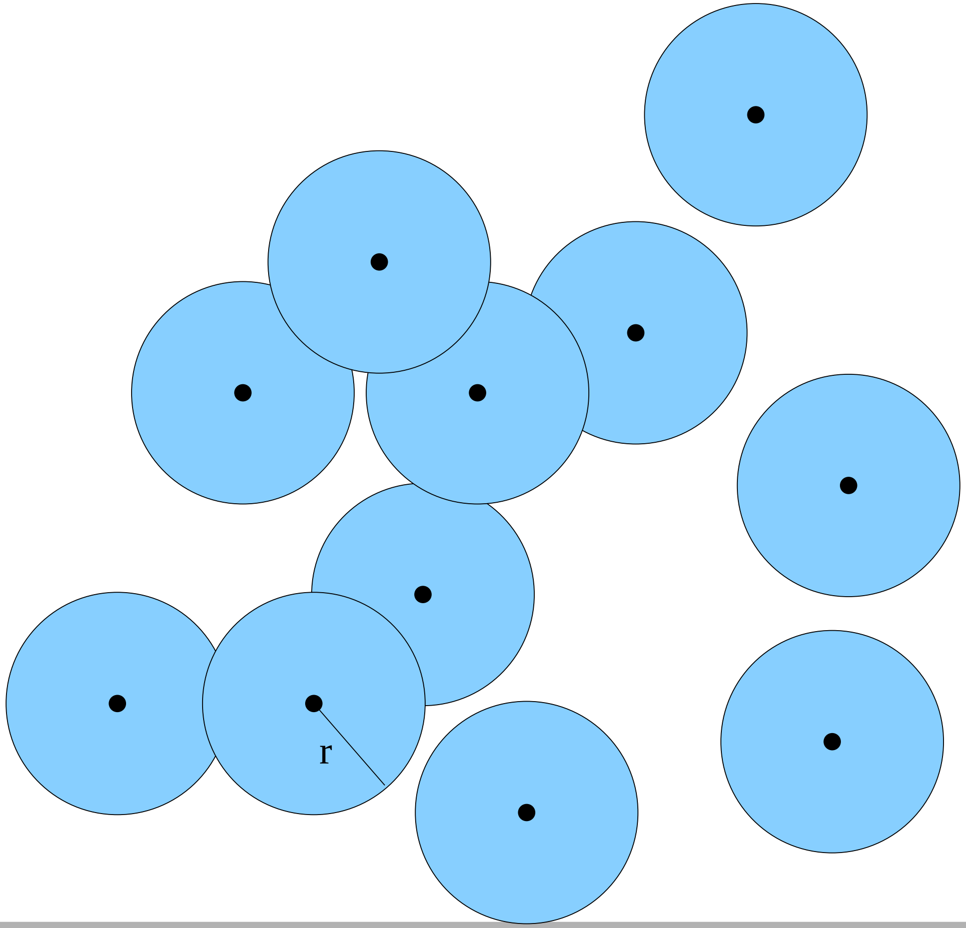
$$\mathcal{C}_i = \{y \in \mathbb{R}^d : |y - x| \leq |y - X_i| \; \forall X_i \in \Phi\}$$



Boolean model (BM)

$$\mathcal{C}_i = X_i \oplus G_i = \{X_i + y : y \in G_i\},$$

where, given $\Phi = \{X_i\}$, G_i are i.i.d. random closed (compact) sets in $\mathbb{R}^d$.



Coverage probabilities

Let $\{(X_i, \mathcal{C}_i)\}$ be a general stationary GG model. In particular, $\Phi = \{X_i\}$ is a stationary point process. One considers two types of coverage characteristics:

Coverage probabilities

Let $\{(X_i, \mathcal{C}_i)\}$ be a general stationary GG model. In particular, $\Phi = \{X_i\}$ is a stationary point process. One considers two types of coverage characteristics:

Coverage by the typical grain

$p(x) := \mathsf{P}^0\{x \in \mathcal{C}_0\}$ where $x \in \mathbb{R}^d$ and $\mathcal{C}_0 = \mathcal{C}(0, \Phi)$ the grain attached to the typical point $X_0 = 0$ of Φ considered under its Palm distribution P^0.

Coverage probabilities

Let $\{(X_i, \mathcal{C}_i)\}$ be a general stationary GG model. In particular, $\Phi = \{X_i\}$ is a stationary point process. One considers two types of coverage characteristics:

Coverage by the typical grain

$p(x) := \mathsf{P}^0\{x \in \mathcal{C}_0\}$ where $x \in \mathbb{R}^d$ and $\mathcal{C}_0 = \mathcal{C}(0, \Phi)$ the grain attached to the typical point $X_0 = 0$ of Φ considered under its Palm distribution P^0.

Stationary coverage

$p := \mathsf{P}\left\{0 \in \bigcup_i \mathcal{C}_i\right\}$ arbitrary location 0 covered by the union.

Stationary coverage number

More generally, denote by $\mathcal{N}$, the number of grains covering the origin $\mathbf{0}$

$$\mathcal{N} := \sum_i \mathbf{1}(\mathbf{0} \in \mathcal{C}_i)$$

and its (stationary) distribution by

$$p_k := \mathsf{P}\{\mathcal{N} \geq k\}\,.$$

p_k is called stationary k-coverage probability
Obviously, $p = p_1 = \mathsf{P}\{0 \in \bigcup_i \mathcal{C}_i\}$ stationary coverage probability.

Exercise: coverage in Poisson-VT

Typical cell coverage

$$p(x) := \mathsf{P}^0\Big\{|x - 0| \le |x - X_i| \ \forall 0 \ne X_i \in \Phi\Big\}$$

$$\overset{\text{Slivnyak}}{=} \mathsf{P}\{\Phi(B_x(|x|)) = 0\}$$

$$\overset{\text{Poisson definition}}{=} e^{-\lambda \kappa_d |x|^d},$$

where $B_a(r) = \{y : |y - a| \le r\}$ and $\kappa_d = |B_0(1)|$ and λ is the intensity of Poisson Φ.

Exercise: coverage in Poisson-VT

Typical cell coverage

$$p(x) := \mathsf{P}^0\Big\{|x-0| \le |x - X_i| \;\forall 0 \ne X_i \in \Phi\Big\}$$

$$\text{Slivnyak} = \mathsf{P}\{\Phi(B_x(|x|)) = 0\}$$

$$\text{Poisson definition} = e^{-\lambda \kappa_d |x|^d},$$

where $B_a(r) = \{y : |y - a| \le r\}$ and $\kappa_d = |B_0(1)|$ and λ is the intensity of Poisson Φ.

Stationary coverage: (Almost) trivially

$p_k := \mathsf{P}\Big\{\#\{i : 0 \in \mathcal{V}_i\} \ge k\Big\} = 1$ for $k = 1$ and 0 for $k \ge 2$.

Indeed, VT is a partition of $\mathbb{R}^d$ modulo boundaries of the cells, on which 0 lies with probability $\mathsf{P} = 0$.

Exercise: coverage in Poisson-BM

Typical grain coverage

By the Slivnyk's theorem and the independence of grains G_i $p(x) := \mathsf{P}^0\{x \in 0 \oplus G_0\} = \mathsf{P}\{x \in G_0\}$ is given directly by the generic grain G distribution.

Exercise: coverage in Poisson-BM

Stationary coverage: $\mathcal{N}$ is Poisson$(\lambda \mathsf{E}[|\check{G}|])$, where $\check{G} = \{-y : y \in G\}$.

Exercise: coverage in Poisson-BM

Stationary coverage: $\mathcal{N}$ is Poisson$(\lambda \mathsf{E}[|\check{G}|])$, where $\check{G} = \{-y : y \in G\}$. Indeed:

$$p_k := \mathsf{P}\Big\{\#\{i : 0 \in X_i \oplus G_i\} \geq k\Big\}$$
$$= \mathsf{P}\{\Phi'(\mathbb{R}^d) \geq k\},$$

where points whose grains cover $\mathbf{0}$,

$$\Phi' = \{X_i \in \Phi : 0 \in X_i \oplus G_i\},$$

form an independent thinning of points of Φ.

Exercise: coverage in Poisson-BM

Stationary coverage: $\mathcal{N}$ is Poisson$(\lambda \mathsf{E}[|\check{G}|])$, where $\check{G} = \{-y : y \in G\}$. Indeed:

$$p_k := \mathsf{P}\Big\{\#\{i : 0 \in X_i \oplus G_i\} \geq k\Big\}$$
$$= \mathsf{P}\{\Phi'(\mathbb{R}^d) \geq k\},$$

where points whose grains cover 0,

$$\Phi' = \{X_i \in \Phi : 0 \in X_i \oplus G_i\},$$

form an independent thinning of points of Φ.

Φ' is a non-homogeneous Poisson process w intensity at x equal to $\lambda'(x) = \lambda \mathsf{P}\{0 \in x \oplus G\} = \lambda \mathsf{P}\{x \in \check{G}\}$.

Exercise: coverage in Poisson-BM

The total intensity of points whose grains cover $\mathbf{0}$ is

$$\int_{\mathbb{R}^d} \boldsymbol{\lambda}'(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} = \boldsymbol{\lambda} \int_{\mathbb{R}^d} \mathsf{P}\{\boldsymbol{x} \in \check{\boldsymbol{G}}\}\,\mathrm{d}\boldsymbol{x}$$

$$= \boldsymbol{\lambda}\mathsf{E}\left[\int_{\mathbb{R}^d} \mathbf{1}(\boldsymbol{x} \in \check{\boldsymbol{G}})\,\mathrm{d}\boldsymbol{x}\right]$$

$$= \boldsymbol{\lambda}\mathsf{E}[|\check{\boldsymbol{G}}|]\,.$$

Exercise: coverage in Poisson-BM

The total intensity of points whose grains cover $\mathbf{0}$ is

$$\begin{aligned}\int_{\mathbb{R}^d} \lambda'(x)\,\mathrm{d}x &= \lambda \int_{\mathbb{R}^d} \mathsf{P}\{x \in \check{G}\}\,\mathrm{d}x \\ &= \lambda \mathsf{E}\left[\int_{\mathbb{R}^d} 1(x \in \check{G})\,\mathrm{d}x\right] \\ &= \lambda \mathsf{E}[|\check{G}|]\,.\end{aligned}$$

Consequently

$$p_k = \sum_{n=k}^{\infty} e^{-\lambda \mathsf{E}[|\check{G}|]} \frac{(\lambda \mathsf{E}[|\check{G}|])^n}{n!}\,.$$

In particular $p_0 = e^{-\lambda \mathsf{E}[|\check{G}|]}$.

Factorial moments of $\mathcal{N}$

Back to the general GG model. For $n \geq 1$, the k-th factorial moment of (an integer valued rv) $\mathcal{N}$ is defined as

$$\mathsf{E}[\mathcal{N}^{(k)}] := \mathsf{E}\left[\mathcal{N}\,(\mathcal{N}-1)^{+}\ldots(\mathcal{N}-k+1)^{+}\right].$$

Factorial moments of $\mathcal{N}$

Back to the general GG model. For $n \geq 1$, the k-th factorial moment of (an integer valued rv) $\mathcal{N}$ is defined as

$$\mathsf{E}[\mathcal{N}^{(k)}] := \mathsf{E}\Big[\mathcal{N}\,(\mathcal{N}-1)^{+}\ldots(\mathcal{N}-k+1)^{+}\Big].$$

FACT Factorial moments characterize the distribution of the random variable. In particular, for $k \geq 1$

$$p_k = \sum_{n=k}^{\infty}(-1)^{n-k}\binom{n-1}{k-1}n!\mathsf{E}[\mathcal{N}^{(n)}],$$

$$\mathsf{P}\{\mathcal{N}=k\} = \sum_{n=k}^{\infty}(-1)^{n-k}\binom{n}{k}n!\mathsf{E}[\mathcal{N}^{(n)}],$$

$$\mathsf{E}[z^{\mathcal{N}}] = \sum_{n=0}^{\infty}(z-1)^{n}n!\mathsf{E}[\mathcal{N}^{(n)}], \quad z \in [0,1].$$

Campbell's formula (Little's law, mass transport principle)

$$
\begin{aligned}
\mathsf{E}[\mathcal{N}^{(1)}] &= \mathsf{E}[\mathcal{N}] \\
&= \mathsf{E}\Big[\sum_{X_i \in \Phi} 1(0 \in C_i)\Big] \\
\text{Campbell} &= \int_{\mathbb{R}^d} \mathsf{P}^x\{0 \in C_x\}\, \lambda \mathrm{d}x \\
\text{symmetry} &= \int_{\mathbb{R}^d} \mathsf{P}^0\{x \in C_0\}\, \lambda \mathrm{d}x \\
&= \int_{\mathbb{R}^d} p(x)\, \lambda \mathrm{d}x = \lambda \mathsf{E}^0[|\mathcal{C}_0|]\,,
\end{aligned}
$$

where $p(x)$ is the typical grain coverage probability.

Higher-order extensions

For $n \geq 1$, quite similarly

$$\mathsf{E}[\mathcal{N}^{(n)}] = \mathsf{E}\Big[\sum_{\substack{X_{i_1}, X_{i_2}, \ldots, X_{i_n} \in \Phi \\ \text{distinct}}} \mathbf{1}\Big(0 \in \bigcap_{j=1}^{n} C_{i_j}\Big)\Big]$$

higher-order Campbell $$= \int_{\mathbb{R}^d} \mathsf{P}^{x_1,\ldots,x_n}\Big(0 \in \bigcap_{j=1}^{n} C_{x_j}\Big)\, \lambda^{(n)}(\mathrm{d}(x_1 \ldots x_n))$$

where $\mathsf{P}^{x_1,\ldots,x_n}$ is n-fold Palm distribution of Φ and $\lambda^{(n)}(\cdot)$ is n-fold factorial moment measure of Φ.

Higher-order extensions

For $n \geq 1$, quite similarly

$$\mathsf{E}[\mathcal{N}^{(n)}] = \mathsf{E}\Big[\sum_{\substack{x_{i_1},x_{i_2},\ldots,x_{i_n}\in\Phi \\ \text{distinct}}} \mathbf{1}\Big(0 \in \bigcap_{j=1}^{n} C_{i_j}\Big)\Big]$$

higher-order Campbell $$= \int_{\mathbb{R}^d} \mathsf{P}^{x_1,\ldots,x_n}\Big(0 \in \bigcap_{j=1}^{n} C_{x_j}\Big)\,\lambda^{(n)}(\mathrm{d}(x_1\ldots x_n))$$

where $\mathsf{P}^{x_1,\ldots,x_n}$ is n-fold Palm distribution of Φ and $\lambda^{(n)}(\cdot)$ is n-fold factorial moment measure of Φ.
In case of Poisson Φ of intensity $\lambda(\cdot)$,

$$\mathsf{P}_{\Phi}^{x_1,\ldots,x_n} = \mathsf{P}_{\Phi+\sum_{j=1}^{n}\delta_{x_j}} \qquad \text{(Slivnyak's Thm)}$$

and $$\lambda^{(n)}(\mathrm{d}(x_1\ldots x_n)) = \lambda(\mathrm{d}x_1)\ldots\lambda(\mathrm{d}x_n).$$

Stationary coverage via moment expansion

COR

$$p_k = \sum_{n=k}^{\infty} (-1)^{n-k} \binom{n-1}{k-1} n! \int_{\mathbb{R}^d} \mathsf{P}^{x_1,\ldots,x_n}\Big(0 \in \bigcap_{j=1}^{n} C_x\Big) \times \lambda^{(n)}(\mathsf{d}(x_1 \ldots x_n))$$

and similarly for $\mathsf{P}\{\mathcal{N} = k\}$, $\mathsf{E}[z^{\mathcal{N}}]$.

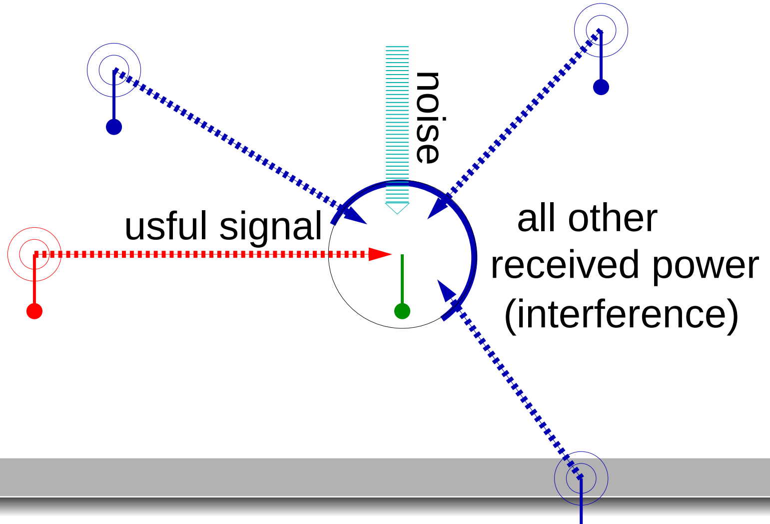
Coverage model for wireless communications and its relations to a Poisson-Dirichlet process

SINR

SINR=Signal-to-Interference-and-Noise Ratio

$$\text{SINR} = \frac{\text{POWER of TAGGED RECEIVED SIGNAL}}{\text{POWER of ALL OTHER RECEIVED SIGNALS (and/or) NOISE}}$$

SINR characterizes the capacity of the communication channel; i.e., the number of bits/second that can be reliably sent in this channel.
Formalization on the ground of information theory.



SINR coverage model

SINR (Signal-to-Interference-and-Noise Ratio) cell:

$$C_i = C_i(\tau) = \left\{ y \in \mathbb{R}^2 : \frac{S_i/\ell(|y - X_i|)}{W + \gamma \sum_{j \neq i} S_j/\ell(|y - X_j|)} \geq \tau \right\}$$

- $\Phi = \{X_i\}$ hom. Poisson p.p. on $\mathbb{R}^2$ of int. λ; locations of wireless transmitters (extension to $\mathbb{R}^d$ straightforward)
- $\tilde{\Phi} = \{(X_i, S_i)\}$ independently marked Φ, $S_i \sim S \geq 0$, $\mathsf{E}[S^{2/\beta}] < \infty$; random signal propagation effects, “shadowing”, “fading”
- $W \geq 0$, r.v. independent of $\tilde{\Phi}$; “noise” power
- $\ell(r) = (Kr)^\beta$, ($K \geq 0$, $\beta > 2$) “path-loss” function,
- $\tau, \gamma \geq 0$ parameters.

SINR coverage model

$$\bigcup_i C_i \quad \text{or} \quad \{C_i\}$$

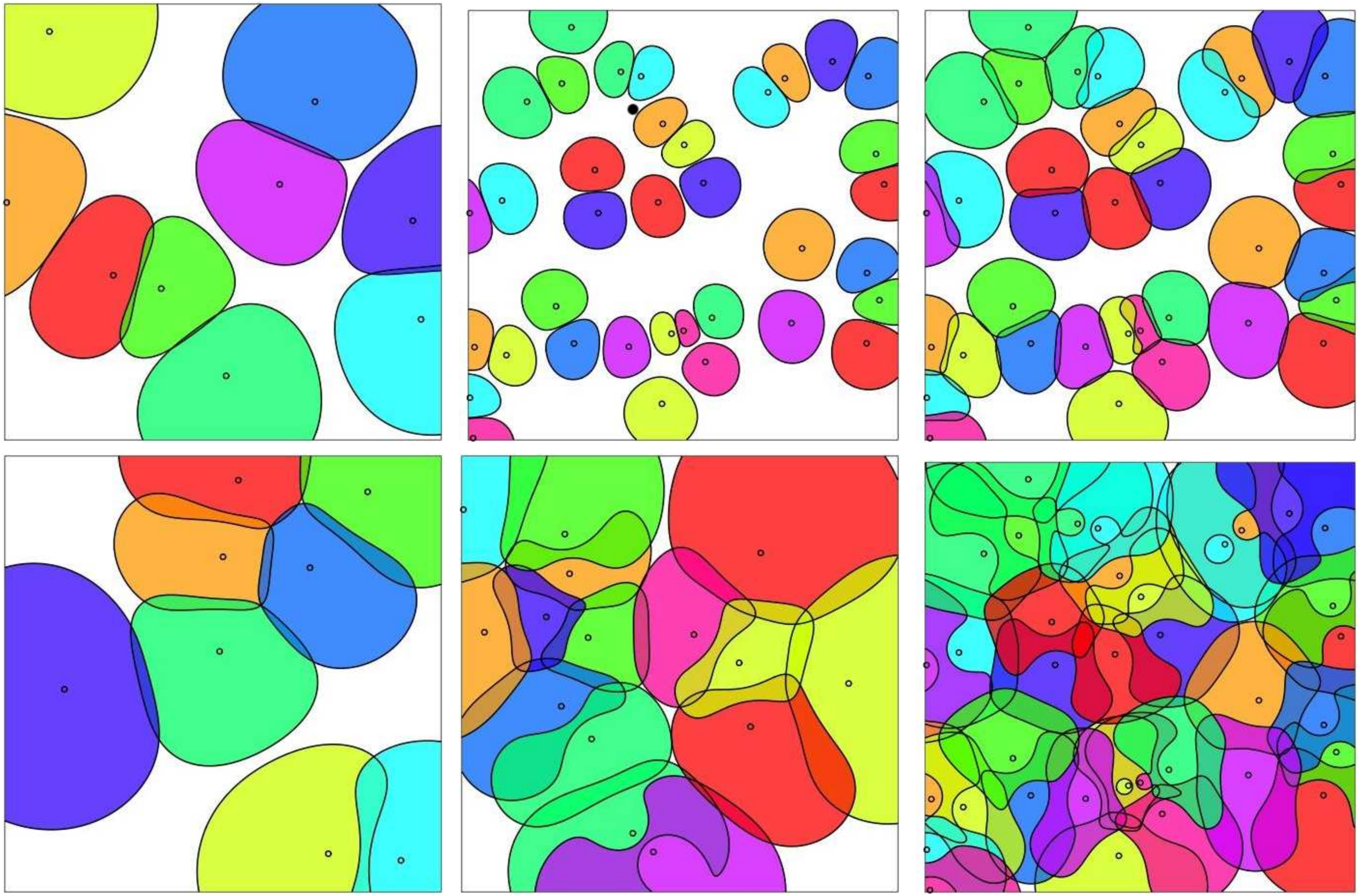
SINR coverage model Baccelli, BB (2001), shot-noise coverage model in Chiu, Stoyan, Kendall, Mecke (2013), a germ grain model with dependent grains.

SINR coverage model

$$\bigcup_i C_i \quad \text{or} \quad \{C_i\}$$

SINR coverage model Baccelli, BB (2001),
shot-noise coverage model in Chiu, Stoyan, Kendall, Mecke (2013),
a germ grain model with dependent grains.

- When $\gamma = \mathbf{0}$ (no interference) SINR grains (cells) are independent; Boolean Model
- When $\boldsymbol{W} = \mathbf{0}$ (no noise) and $\boldsymbol{\beta} \to \infty$ (“strong path-loss”) SINR cells converge to Voronoi cells,
- Playing with $\boldsymbol{W} \to \mathbf{0}$ and $\boldsymbol{\beta} \to \infty$ SINR becomes Johnson-Mehl.



OUTLINE of the remaining part

- Palm and stationary coverage characteristics of the model,
- Poisson-Dirichlet processes,
- Relations to the coverage model.

Palm coverage probabilities

Coverage by the typical cell

- Without loss of generality $\gamma = 1$.
- Under Palm $\mathsf{P}^{\mathbf{0}}$, cell C_0 of $X_0 = 0$, $x \in \mathbb{R}^2$, $|x| = r$,

$$\mathsf{P}^{\mathbf{0}}\{x \in C_0\} = \mathsf{P}^{\mathbf{0}}\left\{ S_0 \geq \tau W \ell(r) + \tau \ell(r) \sum_{i \neq 0} \frac{S_i}{\ell(|y - X_i|)} \right\}$$

with S_0, W and $\sum_{i\neq 0}(\ldots)$ being independent.

Coverage by the typical cell

- Without loss of generality $\gamma = 1$.
- Under Palm P^0, cell C_0 of $X_0 = 0$, $x \in \mathbb{R}^2$, $|x| = r$,

$$\mathsf{P}^0\{x \in C_0\} = \mathsf{P}^0\left\{ S_0 \geq \tau W \ell(r) + \tau \ell(r) \sum_{i \neq 0} \frac{S_i}{\ell(|y - X_i|)} \right\}$$

with S_0, W and $\sum_{i\neq 0}(\ldots)$ being independent.

- The Laplace transform $\mathcal{L}_I$ of $I = \sum_{i\neq 0}(\ldots)$ (Poisson shot-noise) is well known. In particular for $\ell(r) = (Kr)^\beta$

$$\mathcal{L}_I(\xi) = \exp\{-\lambda K^{-2} \xi^{2/\beta} \pi \Gamma(1 - 2/\beta) \mathsf{E}[S^{\frac{2}{\beta}}]\}$$

Coverage by the typical cell

- Without loss of generality $\gamma = 1$.
- Under Palm P^0, cell C_0 of $X_0 = 0$, $x \in \mathbb{R}^2$, $|x| = r$,

$$\mathsf{P}^0\{x \in C_0\} = \mathsf{P}^0\left\{ S_0 \geq \tau W \ell(r) + \tau \ell(r) \sum_{i \neq 0} \frac{S_i}{\ell(|y - X_i|)} \right\}$$

with S_0, W and $\sum_{i \neq 0}(\ldots)$ being independent.

- The Laplace transform $\mathcal{L}_I$ of $I = \sum_{i \neq 0}(\ldots)$ (Poisson shot-noise) is well known. In particular for $\ell(r) = (Kr)^\beta$

$$\mathcal{L}_I(\xi) = \exp\{-\lambda K^{-2} \xi^{2/\beta} \pi \Gamma(1 - 2/\beta) \mathsf{E}[S^{\frac{2}{\beta}}]\}$$

- $\mathsf{P}^0\{x \in C_0\}$ can be numerically calculated using "standard" techniques for arbitrary distribution of S.

Coverage by the typical cell; exponential S

Assume S exponential (mean 1 without loss of generality). With $|x| = r$

$$\mathsf{P}^0\{x \in C_0\}$$
$$= \mathcal{L}_W\Big(\tau(Kr)^\beta\Big) \times \mathcal{L}_I\Big(\tau(Kr)^\beta\Big)$$
$$= \mathcal{L}_W\Big(\tau(Kr)^\beta\Big) \times \exp\Big\{-\lambda r^2 \tau^{2/\beta} \pi \Gamma(1 - 2/\beta)\Gamma(1 + 2\beta)/\beta\Big\}$$

Baccelli, BB (2003), cf also Zorzi, Pupolin (1994) for an early idea.

Coverage by the typical cell; exponential S

Assume S exponential (mean 1 without loss of generality).
With $|x| = r$

$$\mathsf{P}^0\{x \in C_0\}$$

$$= \mathcal{L}_W\left(\tau (Kr)^\beta\right) \times \mathcal{L}_I\left(\tau (Kr)^\beta\right)$$

$$= \mathcal{L}_W\left(\tau (Kr)^\beta\right) \times \exp\left\{-\lambda r^2 \tau^{2/\beta} \pi \Gamma(1 - 2/\beta)\Gamma(1 + 2\beta)/\beta\right\}$$

Baccelli, BB (2003), cf also Zorzi, Pupolin (1994) for an early idea.
This very simple observation inspired amazing amount of subsequent works in the engineering literature...

Stationary coverage probabilities

Coverage of the typical location

- SINR coverage probability

$$\mathcal{P} = \mathsf{P}\{0 \in \bigcup_i C_i\}\,.$$

More generally, k-coverage probability ($k \geq 1$)

$$\mathcal{P}^{(k)} = \mathsf{P}\{\mathcal{N} \geq k\}\,,$$

where $\mathcal{N} := \sum_i \mathbf{1}(0 \in C_i)$ is the number of cells covering 0.

Coverage of the typical location

- SINR coverage probability

$$\mathcal{P} = \mathsf{P}\{0 \in \bigcup_i C_i\}\,.$$

More generally, k-coverage probability ($k \geq 1$)

$$\mathcal{P}^{(k)} = \mathsf{P}\{\mathcal{N} \geq k\}\,,$$

where $\mathcal{N} := \sum_i \mathbf{1}(0 \in C_i)$ is the number of cells covering 0.

- Model invariance: $\mathcal{P}^{(k)}$ depend only on β, W and

$$a := \frac{\lambda \pi \mathsf{E}[(S)^{\frac{2}{\beta}}]}{K^2}\,.$$

In case $W = 0$, $\mathcal{P}^{(k)}$ depend only on β. (To be explained).

Special functions I

For $n \geq 1$, define functions of $x \geq 0$

$$\mathcal{I}_{n,\beta}(x) = \frac{2^n \int_0^\infty u^{2n-1} e^{-u^2 - u^\beta x \Gamma(1-2/\beta)^{-\beta/2}} du}{\beta^{n-1}(\Gamma(1-2/\beta)\Gamma(1+2/\beta))^n (n-1)!}.$$

In particular

$$\mathcal{I}_{n,\beta}(0) = \frac{2^{n-1}}{\beta^{n-1}(C'(\beta))^n},$$

where $C'(\beta) = \Gamma(1-2/\beta)\Gamma(1+2/\beta)$.

Special functions II

For $n \geq 1$, define functions of $(x_1, \ldots, x_i) \geq 0$

$$\mathcal{J}_{n,\beta}(x_1, \ldots, x_n)$$

$$= \frac{(1 + \sum_{j=1}^{n} x_j)}{n} \int_{[0,1]^{n-1}} \frac{\prod_{i=1}^{n-1} v_i^{i(2/\beta+1)-1} (1 - v_i)^{2/\beta}}{\prod_{i=1}^{n} (x_i + \eta_i)} dv_1 \ldots dv_{n-1},$$

where

$$\begin{cases} \eta_1 & = v_1 v_2 \ldots v_{n-1} \\ \eta_2 & = (1 - v_1) v_2 \ldots v_{n-1} \\ \eta_3 & = (1 - v_2) v_3 \ldots v_{n-1} \\ & \ldots \\ \eta_n & = 1 - v_{n-1}. \end{cases}$$

Stationary coverage probabilities

- The SINR k-coverage probability $\mathcal{P}^{(k)} = \mathcal{P}^{(k)}(\tau)$ is equal to

$$\mathcal{P}^{(k)} = \sum_{n=k}^{\lceil 1/\tau \rceil} (-1)^{n-k} \binom{n-1}{k-1} \tau_n^{-2n/\beta} \mathsf{E}[\mathcal{I}_{n,\beta}(W a^{-\beta/2})] \mathcal{J}_{n,\beta}(\tau_n) \,,$$

where $\tau_n := \tau_n(\tau) = \frac{\tau}{1-(n-1)\tau}$; Keeler, BB, Karray (2013).

Stationary coverage probabilities

- The SINR k-coverage probability $\mathcal{P}^{(k)} = \mathcal{P}^{(k)}(\tau)$ is equal to

$$\mathcal{P}^{(k)} = \sum_{n=k}^{\lceil 1/\tau \rceil} (-1)^{n-k}\binom{n-1}{k-1}\tau_n^{-2n/\beta}\mathsf{E}[\mathcal{I}_{n,\beta}(Wa^{-\beta/2})]\mathcal{J}_{n,\beta}(\tau_n)\,,$$

where $\tau_n := \tau_n(\tau) = \frac{\tau}{1-(n-1)\tau}$; Keeler, BB, Karray (2013).

- For $\tau \geq 1$ we have $\lceil 1/\tau \rceil = 1$. Thus $\mathcal{P}^{(k)} = 0$ for all $k \geq 2$ and

$$\mathcal{P} = \mathcal{P}^{(1)} = \frac{2(\tau)^{-2/\beta}}{\Gamma(1+\frac{2}{\beta})}\int_0^\infty ue^{-u^2\Gamma(1-2/\beta)}\mathcal{L}_W\left(a^{-\beta/2}u^\beta\right)du\,.$$

Dhillon et al. (2012).

Mapping on 1D and an invariance property

- Denote powers received at $\mathbf{0}$ by

$$\Theta := \left\{ Y_i := S_i/\ell(|X_i|), X_i \in \Phi \right\}.$$

Θ is inhomogeneous Poisson pp on $(0, \infty)$ with intensity measure $2a/\beta\, t^{-1-2/\beta}\, dt$. (Recall, $a = \frac{\lambda \pi \mathsf{E}[(S)^{\frac{2}{\beta}}]}{K^2}$.)

Mapping on 1D and an invariance property

- Denote powers received at $\mathbf{0}$ by

$$\Theta := \left\{ Y_i := S_i/\ell(|X_i|), X_i \in \Phi \right\}.$$

  Θ is inhomogeneous Poisson pp on $(0, \infty)$ with intensity measure $2a/\beta\, t^{-1-2/\beta}\, dt$. (Recall, $a = \frac{\lambda\pi \mathsf{E}[(S)^{\frac{2}{\beta}}]}{K^2}$.)
- k-coverage probabilities and all functionals of Θ (and W) depend ony on β and a (and W), but are invariant w.r.t. the distribution of S.

Mapping on 1D and an invariance property

- Denote powers received at $\mathbf{0}$ by

$$\Theta := \left\{ Y_i := S_i/\ell(|X_i|), X_i \in \Phi \right\}.$$

  Θ is inhomogeneous Poisson pp on $(0,\infty)$ with intensity measure $2a/\beta\, t^{-1-2/\beta}\, dt$. (Recall, $a = \frac{\lambda\pi \mathsf{E}[(S)^{\frac{2}{\beta}}]}{K^2}$.)
- k-coverage probabilities and all functionals of Θ (and W) depend ony on β and a (and W), but are invariant w.r.t. the distribution of S.
- This invariance helpful in various proofs, where for mathematical convenience S is often assumed exponential or deterministic, with the results generalized to arbitrary S by appropriate modification of λ.

Poisson-Dirichlet processes

Poisson Dirichlet processes

- Let $\Theta_\alpha = \{Y_i\}$ by Poisson process on $(0, \infty)$ with intensity $t^{-1-\alpha}\,dt$, with $\alpha \in [0, 1)$.

Poisson Dirichlet processes

- Let $\Theta_\alpha = \{Y_i\}$ by Poisson process on $(0, \infty)$ with intensity $t^{-1-\alpha}\,dt$, with $\alpha \in [0, 1)$. Denote

$$\left\{V_i' := \frac{Y_i}{\sum_j Y_j}, Y_i \in \Theta_\alpha\right\};$$

$\{V_i'\}$ is called Poisson-Dirichlet PD$(\alpha, 0)$ pp.

Poisson Dirichlet processes

- Let $\Theta_\alpha = \{Y_i\}$ by Poisson process on $(0, \infty)$ with intensity $t^{-1-\alpha}\,dt$, with $\alpha \in [0, 1)$. Denote

$$\left\{ V_i' := \frac{Y_i}{\sum_j Y_j}, Y_i \in \Theta_\alpha \right\};$$

  $\{V_i'\}$ is called Poisson-Dirichlet PD$(\alpha, 0)$ pp.

- The same construction with Θ_α replaced by Θ^θ of intensity $\theta t^{-1} e^{-t}\,dt$, with $\theta > 0$, leads to PD$(0, \theta)$; Kingman (1975).

Poisson Dirichlet processes

- Let $\Theta_\alpha = \{Y_i\}$ by Poisson process on $(0, \infty)$ with intensity $t^{-1-\alpha}\,dt$, with $\alpha \in [0, 1)$. Denote

$$\left\{V_i' := \frac{Y_i}{\sum_j Y_j}, Y_i \in \Theta_\alpha\right\};$$

  $\{V_i'\}$ is called Poisson-Dirichlet PD$(\alpha, 0)$ pp.

- The same construction with Θ_α replaced by Θ^θ of intensity $\theta t^{-1} e^{-t}\,dt$, with $\theta > 0$, leads to PD$(0, \theta)$; Kingman (1975).

- Both belong to a two-parameter family PD(α, θ), $\alpha \in [0, 1)$, $\theta > -\alpha$, whose Poisson construction is slightly more involved; Pitman, Yor (1997).

Size biased representation of $\mathbf{PD}(\alpha, \theta)$

- Let

$$V_1 = U_1, \quad V_i = (1 - U_1) \ldots (1 - U_{i-1}) U_i, \quad i \geq 2,$$

where $U_1, U_2, \ldots$ are independent random variables on $(0, 1)$ with $U_i \sim \text{Beta}(1 - \alpha, \theta + i\alpha)$;
stick-breaking rule or residual allocation model.

Size biased representation of $\mathbf{PD}(\alpha, \theta)$

- Let

$$V_1 = U_1, \quad V_i = (1 - U_1) \dots (1 - U_{i-1}) U_i, \quad i \geq 2,$$

  where $U_1, U_2, \dots$ are independent random variables on $(0, 1)$ with $U_i \sim$ Beta$(1 - \alpha, \theta + i\alpha)$; stick-breaking rule or residual allocation model.

- $\{V_1, V_2, \dots\}$ considered as a pp is PD(θ, α); Pitman, Yor (1997).

Size biased representation of $\mathbf{PD}(\alpha, \theta)$

- Let

$$V_1 = U_1, \quad V_i = (1 - U_1) \dots (1 - U_{i-1}) U_i, \quad i \geq 2,$$

where $U_1, U_2, \dots$ are independent random variables on $(0, 1)$ with $U_i \sim$ Beta$(1 - \alpha, \theta + i\alpha)$;
stick-breaking rule or residual allocation model.

- $\{V_1, V_2, \dots\}$ considered as a pp is PD(θ, α); Pitman, Yor (1997).

- $(V_1, V_2, \dots)$ considered as a random vector is invariant with respect to size-biased permutation. In fact, it is the only distribution obtained from the stick-breaking model with this property; Pitman (1996). Called also GEM model after Griffith, Engen, McCloskey.

Poisson-Dirichlet and SINR coverage

SIR coverage and $\mathbf{PD}(\alpha, 0)$ process

- Denote $Z_i := \frac{S_i/\ell(|X_i|)}{W + \sum_{j \neq i} S_j/\ell(|X_j|)} = \frac{Y_i}{W + \sum_{j \neq i} Y_j}$.

SIR coverage and PD$(\alpha, 0)$ process

- Denote $Z_i := \frac{S_i/\ell(|X_i|)}{W+\sum_{j\neq i} S_j/\ell(|X_j|)} = \frac{Y_i}{W+\sum_{j\neq i} Y_j}$.
- Recall $\Theta = \{Y_i\}$ is Poisson pp of intensity $2a/\beta t^{-1-2/\beta}\, dt$, on $(0,\infty)$, equal (modulo irrelevant in this context constant $2a/\beta$) to this of Θ_α, with $\alpha = 2/\beta$. Recall, Θ_α gives rise to PD$(\alpha, 0)$ via the same points' normalization $V_i' = \frac{Y_i}{\sum_j Y_j}$.

SIR coverage and PD$(\alpha, 0)$ process

- Denote $Z_i := \frac{S_i/\ell(|X_i|)}{W+\sum_{j\neq i} S_j/\ell(|X_j|)} = \frac{Y_i}{W+\sum_{j\neq i} Y_j}$.
- Recall $\Theta = \{Y_i\}$ is Poisson pp of intensity $2a/\beta t^{-1-2/\beta}\,dt$, on $(0,\infty)$, equal (modulo irrelevant in this context constant $2a/\beta$) to this of Θ_α, with $\alpha = 2/\beta$. Recall, Θ_α gives rise to PD$(\alpha, 0)$ via the same points' normalization $V_i' = \frac{Y_i}{\sum_j Y_j}$.
- $\Psi := \{Z_i\}$ can be easy related to $\Psi' := \{Z_i' := \frac{Y_i}{W+\sum_j Y_j}\}$ via $Z_i' = Z_i/(1+Z_i)$.

SIR coverage and PD$(\alpha, 0)$ process

- Denote $Z_i := \frac{S_i/\ell(|X_i|)}{W+\sum_{j\neq i} S_j/\ell(|X_j|)} = \frac{Y_i}{W+\sum_{j\neq i} Y_j}$.
- Recall $\Theta = \{Y_i\}$ is Poisson pp of intensity $2a/\beta t^{-1-2/\beta}\,dt$, on $(0,\infty)$, equal (modulo irrelevant in this context constant $2a/\beta$) to this of Θ_α, with $\alpha = 2/\beta$. Recall, Θ_α gives rise to PD$(\alpha, 0)$ via the same points' normalization $V_i' = \frac{Y_i}{\sum_j Y_j}$.
- $\Psi := \{Z_i\}$ can be easy related to $\Psi' := \{Z_i' := \frac{Y_i}{W+\sum_j Y_j}\}$ via $Z_i' = Z_i/(1+Z_i)$.
- Consequently, for $W = 0$ the SIR k-coverage probability $\mathcal{P}^{(k)} = \mathsf{P}\Big\{V_{(k)}' > \tau/(1+\tau)\Big\}$, where $V_{(1)}' > V_{(2)}' > \dots$ are ordered points of the PD$(2/\beta, 0)$.

Factorial moments of the SINR process

$$M'^{(n)}(t'_1, \ldots, t'_n) \quad := \mathsf{E}\Big[\sum_{\substack{(Z'_1,\ldots,Z'_n)\in(\Psi')^{\times n} \\ \text{distinct}}} \prod_{j=1}^{n} \mathbb{1}(Z'_j > t'_j)\Big]$$

Factorial moments of the SINR process

$$M'^{(n)}(t'_1,\dots,t'_n) \quad := \mathsf{E}\Big[\sum_{\substack{(Z'_1,\dots,Z'_n)\in(\Psi')^{\times n}\\ \text{distinct}}} \prod_{j=1}^{n} \mathbb{1}(Z'_j > t'_j)\Big]$$

We have

$$M'^{(n)}(t'_1,\dots t'_n)$$

$$= n!\left(\prod_{i=1}^{n} \hat{t}_i^{-2/\beta}\right)\mathcal{I}_{n,\beta}((W)a^{-\beta/2})\mathcal{J}_{n,\beta}(\hat{t}_1,\dots,\hat{t}_n),$$

when $\sum_{i=1}^{n} t'_n < 1$ and $M'^{(n)}(t'_1,\dots t'_n) = 0$ otherwise,
where $\hat{t}_i = \hat{t}_i(t'_1,\dots,t'_n) := \frac{t'_i}{1-\sum_{j=1}^{n} t'_j}$;

Observe factorization of the noise contribution to the factorial moment measures; BB, Keeler (2014).

Densities of the SINR process

For $\sum_{i=1}^{n} t'_n < 1$

$$\mu'^{(n)}(t'_1, \ldots t'_n) := (-1)^n \frac{\partial^n M'^{(n)}(t'_1, \ldots t'_n)}{\partial t'_1 \ldots \partial t'_n}$$

$$= \bar{\mathcal{I}}_{n,\beta}((W)a^{-\beta/2}) \underbrace{c_{n,2/\beta,0} \left(\prod_{i=1}^{n} (t'_i)^{-(2/\beta+1)}\right) \left(1 - \sum_{j=1}^{n} (t'_j)\right)^{2n/\beta - 1}}_{\text{density of PD}(2/\beta,0)\text{, Handa (2009)}}$$

where

$$c_{n,\alpha,\theta} = \prod_{i=1}^{n} \frac{\Gamma(\theta + 1 + (i-1)\alpha)}{\Gamma(1-\alpha)\Gamma(\theta + i\alpha)},$$

and

$$\bar{\mathcal{I}}_{n,\beta}(x) = \frac{\mathcal{I}_{n,\beta}(x)}{\mathcal{I}_{n,\beta}(0)};$$

BB, Keeler (2014).

Factorial moment expansions

Expansions of general characteristics ϕ of the SINR process

$$\mathsf{E}[\phi(\Psi')] = \phi(\emptyset) + \sum_{n=1}^{\infty} \int_{(0,1)^n} \phi_{t'_1,\ldots,t'_n}\, \mu'^{(n)}(t'_1,\ldots,t'_n)\, dt'_n \ldots dt'_1$$

where

$$\phi_{t'_1} = \phi(\{t'_1\}) - \phi(\emptyset)$$

$$\phi_{t'_1,t'_2} = \frac{1}{2}\Big(\phi(\{t'_1,t'_2\}) - \phi(\{t'_1\}) - \phi(\{t'_2\}) + \phi(\emptyset)\Big)$$

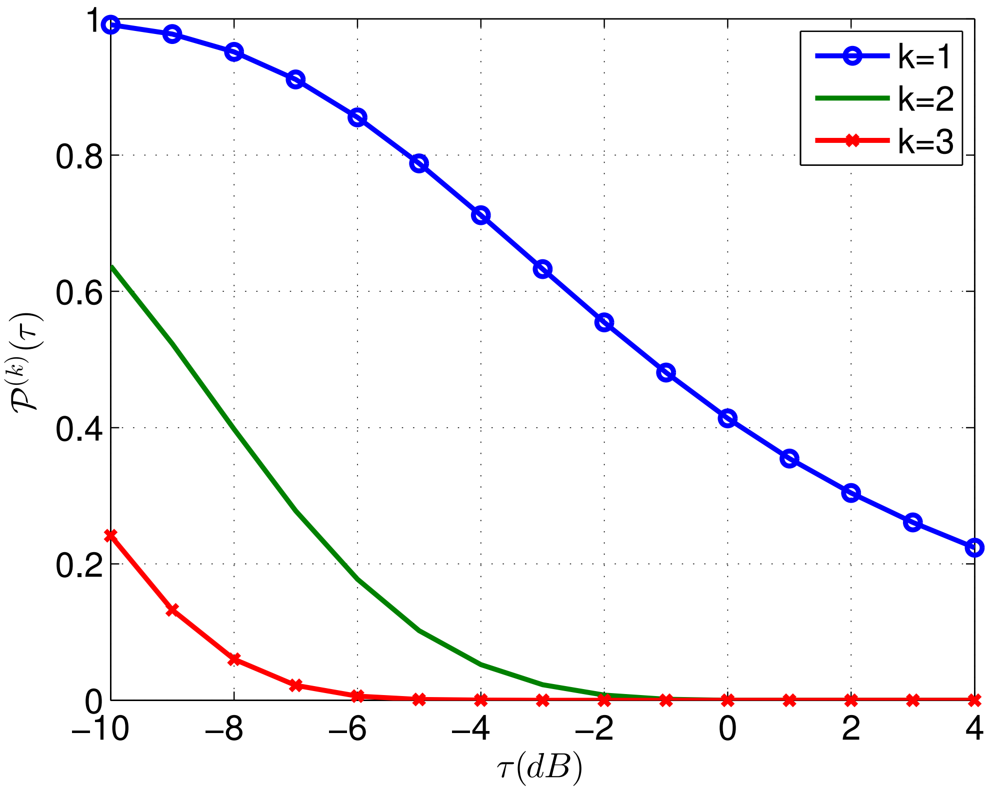
$$\ldots$$

$$\phi_{t'_1,\ldots,t'_n} = \frac{1}{n!}\sum_{k=0}^{n}(-1)^{n-k} \sum_{\substack{t'_{i_1},\ldots,t'_{i_k} \\ \text{distinct}}} \phi(\{t'_{i_1},\ldots,t'_{i_k}\})\,.$$

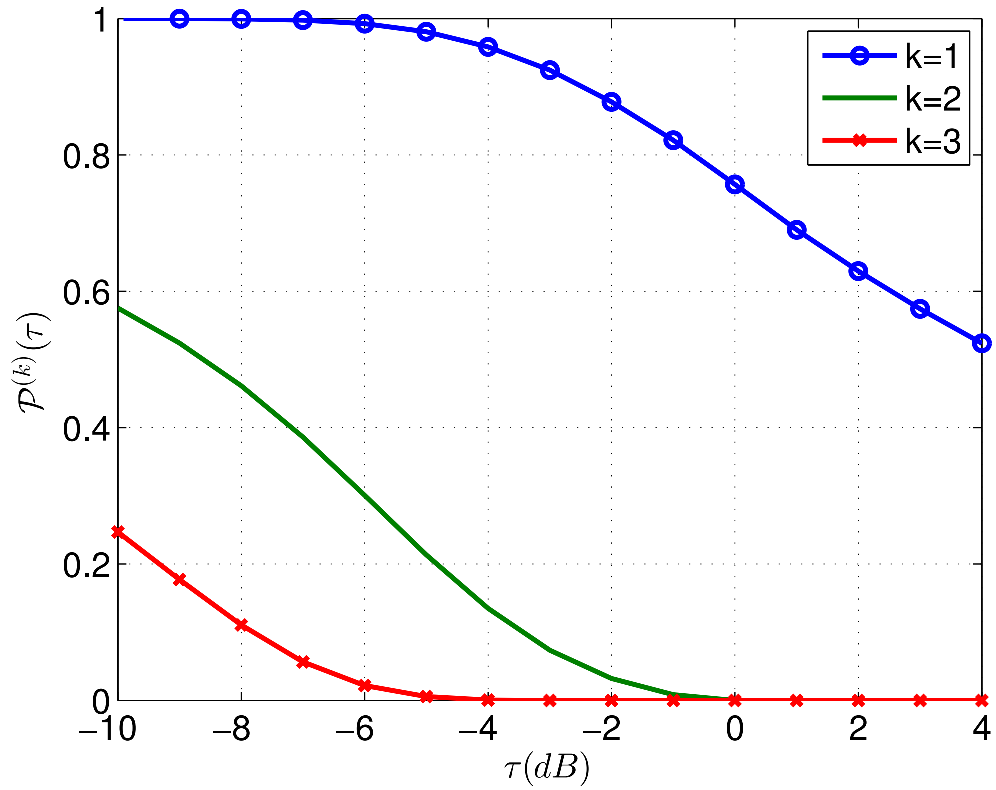
BB (1995).

Numerical examples

SINR k-coverage probability

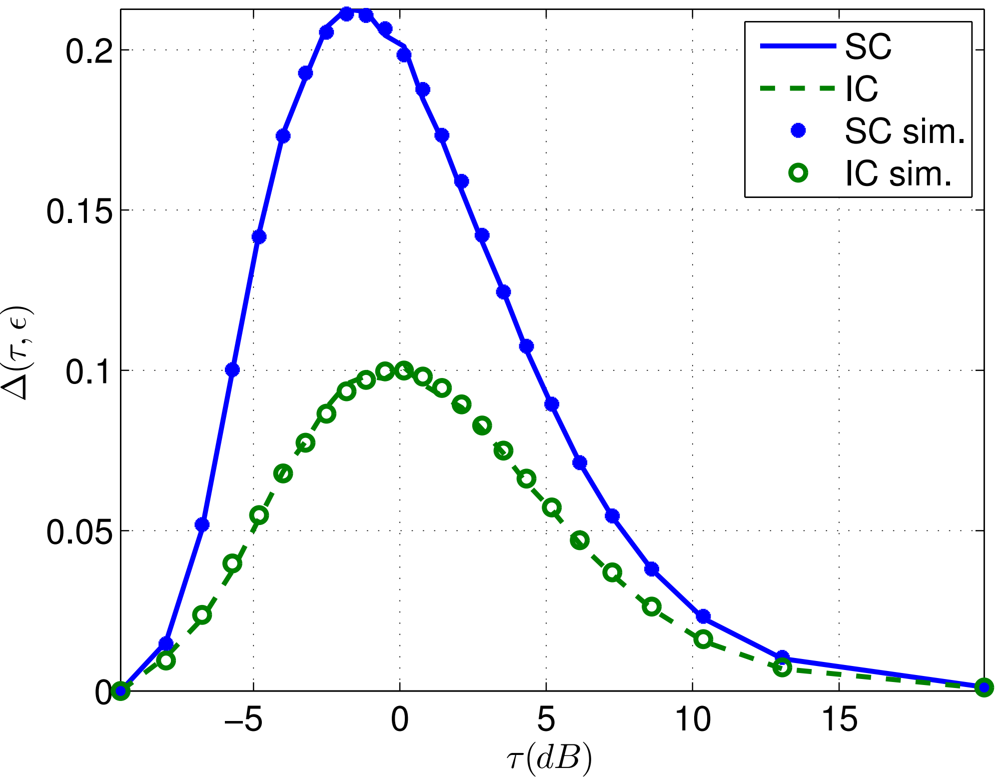


$\beta = 3$

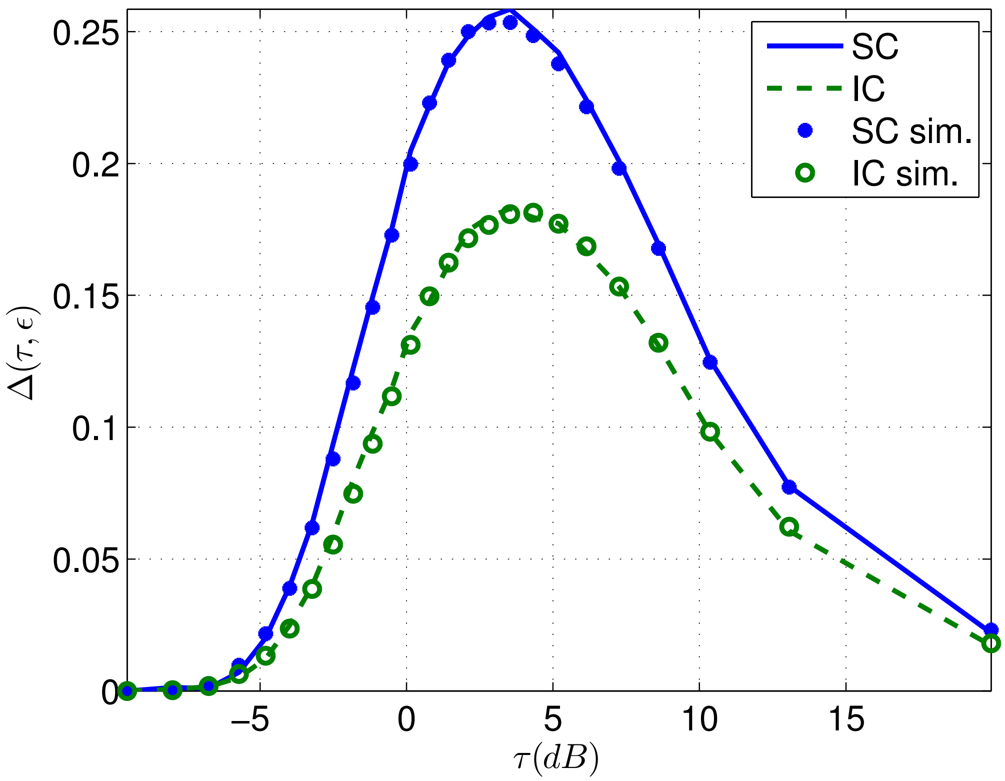


$\beta = 5$

Coverage with interference cancellation and signal combination



$\boldsymbol{\beta = 3}$



$\boldsymbol{\beta = 5}$

The increase of the coverage probability when two strongest signals are combined (SC) or the second strongest signal is canceled from the interference (IC).

Conclusions

- We have seen a Poisson-Dirichlet process in some wireless communication model, where it describes fractions of the SINR spectrum. But Poisson-Dirichlet processes appear in several apparently different contexts.

Conclusions

- We have seen a Poisson-Dirichlet process in some wireless communication model, where it describes fractions of the SINR spectrum. But Poisson-Dirichlet processes appear in several apparently different contexts.
- Two-parameter family of Poisson-Dirichlet processes is used in math/economic models.

Conclusions, cont'd

- In math/physics "our" PD$(\alpha, 0)$ process appears as the thermodynamic (large system) limit in the low temperature regime of Derrida's random energy model (REM). It is also a key component of the so-called Ruelle probability cascades, which are used to represent the thermodynamic limit of the Sherrington-Kirkpatrick model for spin glasses (types of disordered magnets).

Conclusions, cont'd

- In math/physics "our" PD$(\alpha, 0)$ process appears as the thermodynamic (large system) limit in the low temperature regime of Derrida's random energy model (REM). It is also a key component of the so-called Ruelle probability cascades, which are used to represent the thermodynamic limit of the Sherrington-Kirkpatrick model for spin glasses (types of disordered magnets).
- "Our" invariance of the SINR coverage model with respect to the distribution of S can be related to Bolthausen-Sznitman invariance property heavily used to study the Sherrington-Kirkpatrick model; cf Panchenko (2013).

More details in:

- B.B and H. P. Keeler, SINR in wireless networks and the Two-Parameter Poisson-Dirichlet process IEEE Wireless Comm. Letters, 2014.
- B.B. and H. P. Keeler, Studying the SINR process of the typical user in Poisson networks by using its factorial moment measures, IEEE Trans. Inf. Theory, 2015.

Thank you for today.
Tomorrow: Connectivity