Random geometry of wireless cellular networks

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PRAWDOPODOBIEŃSTWO I STATYSTYKA DLA NAUKI I TECHNOLOGII
Sesja Specjalna pod auspicjami Komitetu Matematyki PAN

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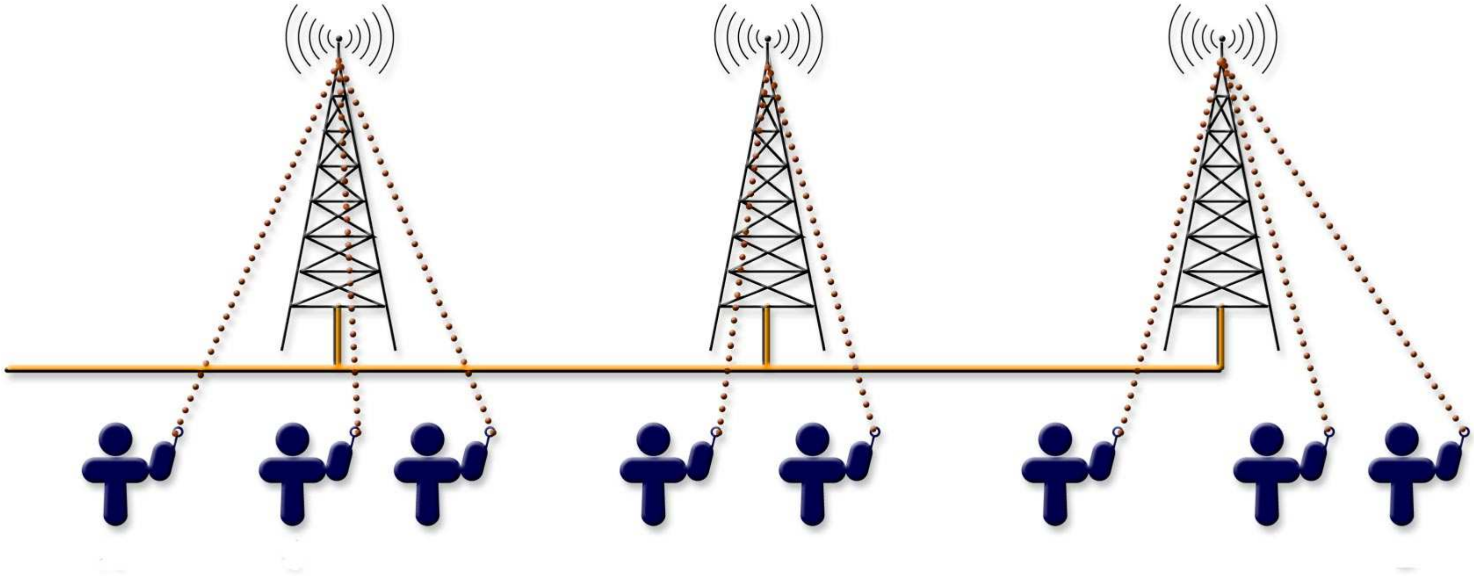
Cellular Network

Infrastructure of base stations (BS) provided by an operator.

Individual users talk to these stations and listen to them



send/receive data (Internet, VOD, mobile TV, etc).

Technology and geometry

- Radio communications between users and BS's share some part of the electromagnetic spectrum.
- Successive "generations" of cellular networks (1G,...,4G, GSM, CDMA, HSDPA, LTE, etc) use different technologies to "separate" individual communications (in time and/or frequency, and/or coding).

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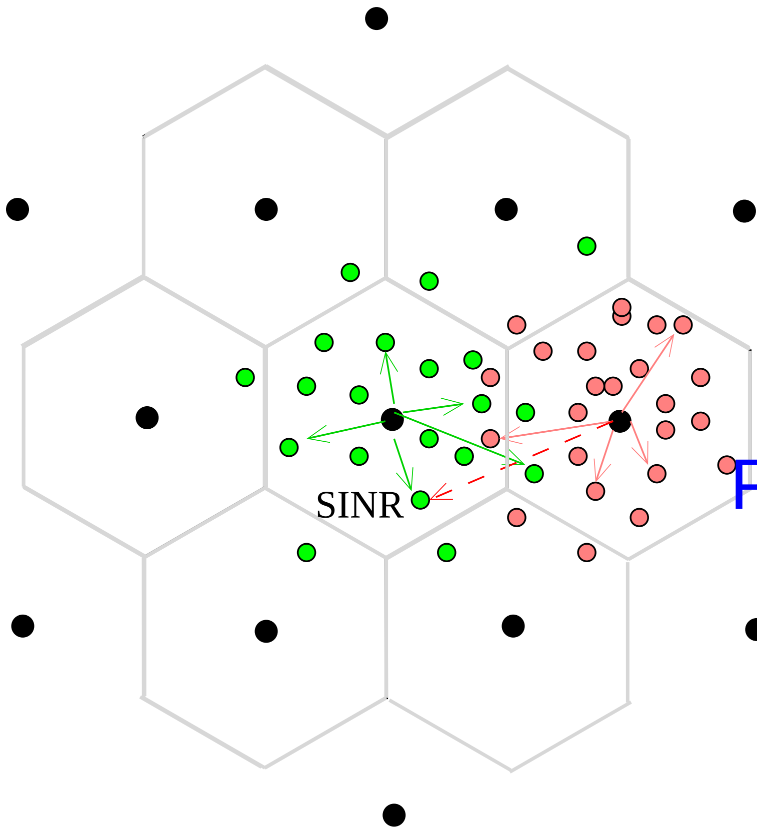
- Performance of the (radio part) of a cellular network depends very much on its geometry (relative location of BS's and users).

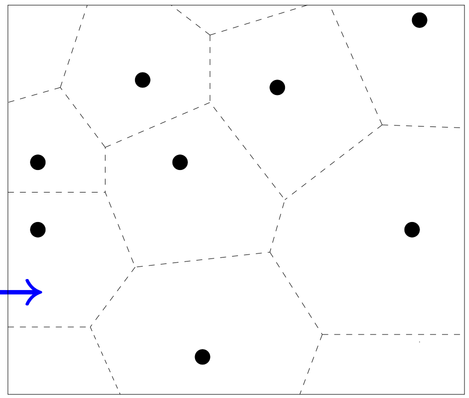
Geometry and dynamics



←Honeycomb

vs

Poisson-Voronoi→



geometry: (static) pattern of BS with their path-loss fields,
dynamics: arrivals/departures/mobility of users

Questions: SINR based QoS prediction ⇒ capacity models ⇒ operator dimensioning tools.

Shadowing and network geometry

- Shadowing — signal power loss due to reflection, diffraction, and scattering. Modeled by random field with log-normal marginals with mean 1 and some variance.
- Impacts geometry of cellular networks:
  Serving BS $\equiv$ with smallest path-loss $\not\equiv$ the closets one.
- Problems:
  - Is believed to degrade QoS (?) $\Rightarrow$ Not always!
  - How it harms the "perfect" honeycomb? $\Rightarrow$ Makes it more Poisson-like. Poisson analysis may be useful!

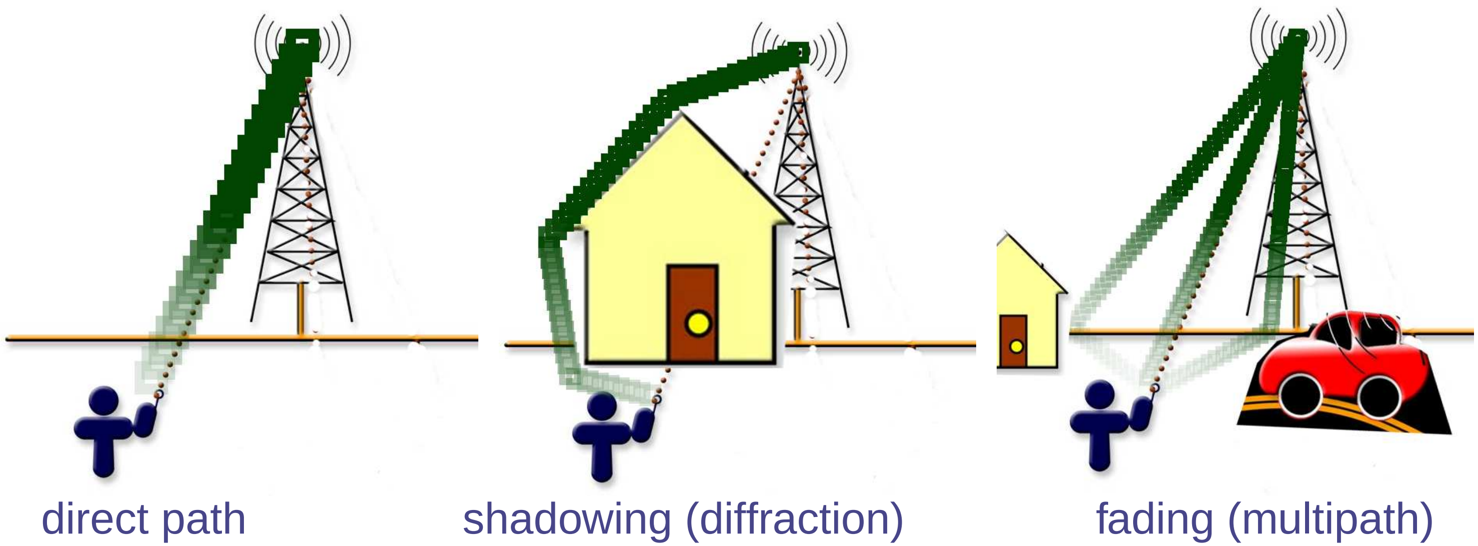
OUTLINE of the remaining part of the talk

- Foundations of wireless communications
- Geometry of cellular networks
- Poisson pp — basic facts
- Poisson pp as a model for BS
- When everything is similar to Poisson — a convergence result
- When shadowing improves performance — heavy tails in action

Foundations of wireless communications

Signal propagation

Maxwell's electromagnetic field equations? — Not needed. Simple statistical models relating received power P_{rec} to the emitted one P_{em}.



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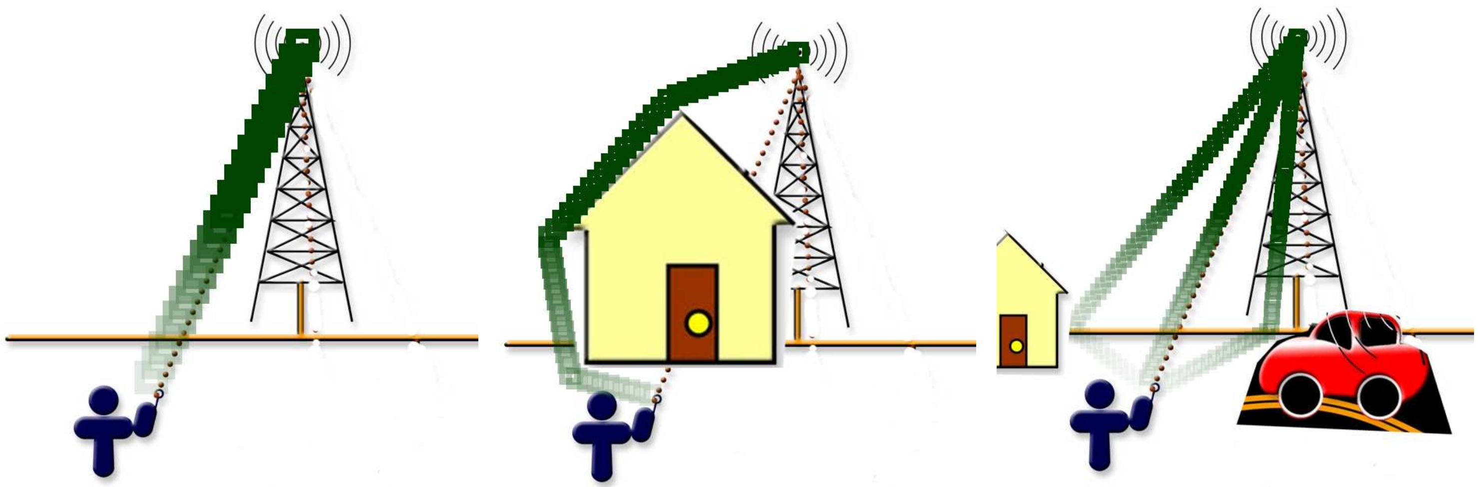

direct path   shadowing (diffraction)   fading (multipath)

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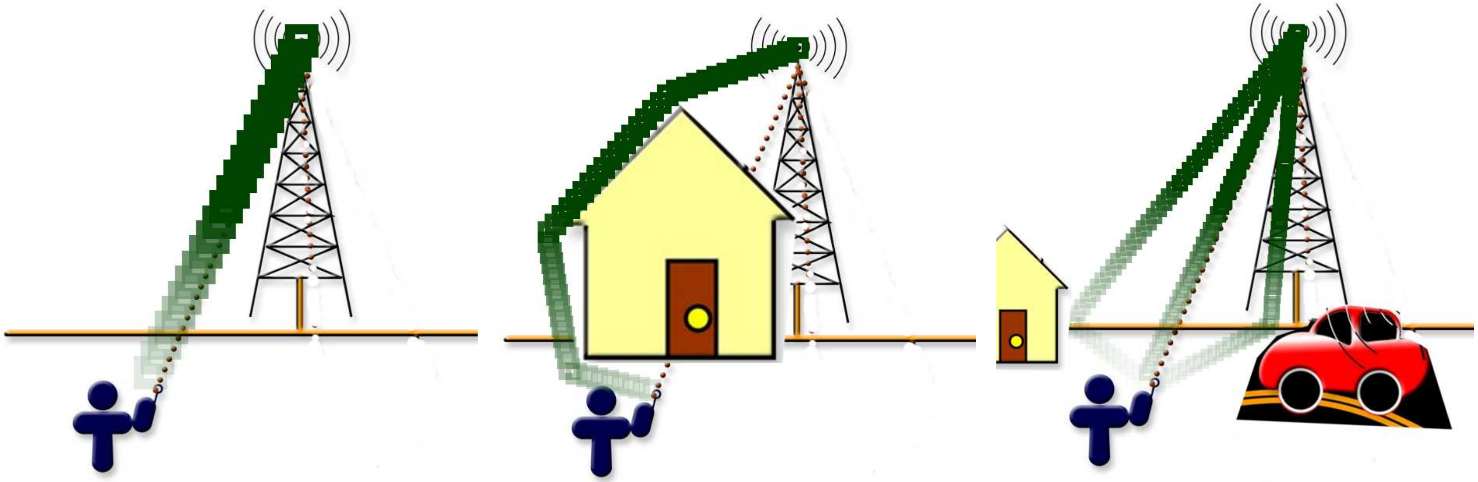
- Random path-loss: $P_{\text{rec}} \approx S \cdot F \cdot P_{\text{em}} \cdot (\text{distance})^{-\beta}$, S, F random variables for shadowing and fading.
  Typically $S \sim$ log-normal, $F \sim$ exponential (Rayleigh fading).

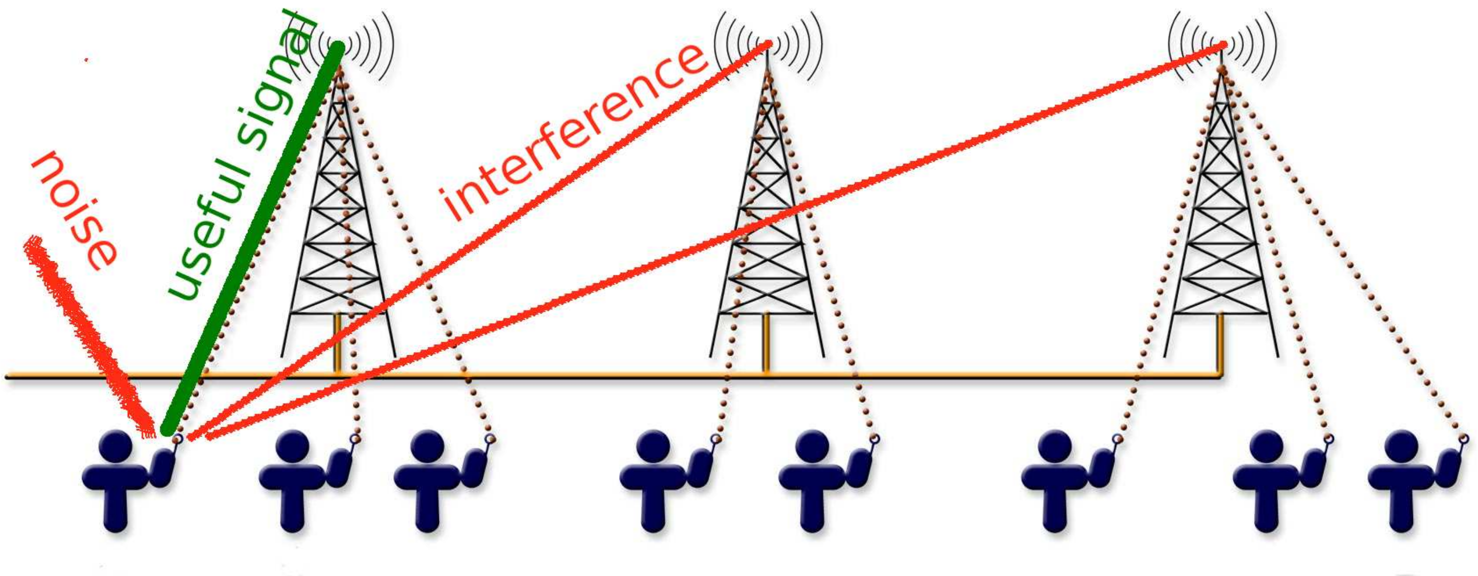


direct path shadowing (diffraction) fading (multipath)

Signal detection and processing

Basic radio channel characteristic:
Signal-to-Interference-and-Noise Ratio SINR $=$

$$\frac{\overbrace{\text{power of the signal from serving BS}}^{\text{useful signal}}}{\text{noise power} + \underbrace{\text{total power received from non-serving BS}}_{\text{interference}}}$$



Signal coding

- constant bit-rate (CBR) coding (e.g. for voice, mobile TV)
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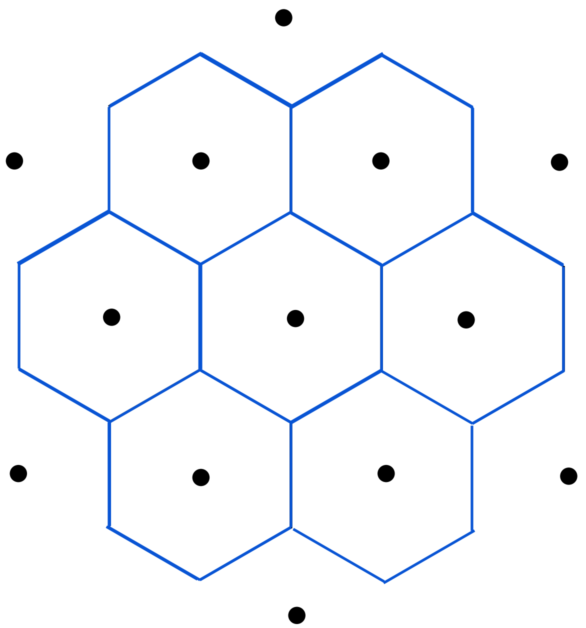
More recent information theoretic concepts (not considered in this presentation): MIMO (multiple antenna systems), broadcast and MAC channels (joint coding to and from several users), interference cancellation.

Geometry of cellular networks

Honeycomb model for BS placement



Traditionally considered as optimal placement of BS.

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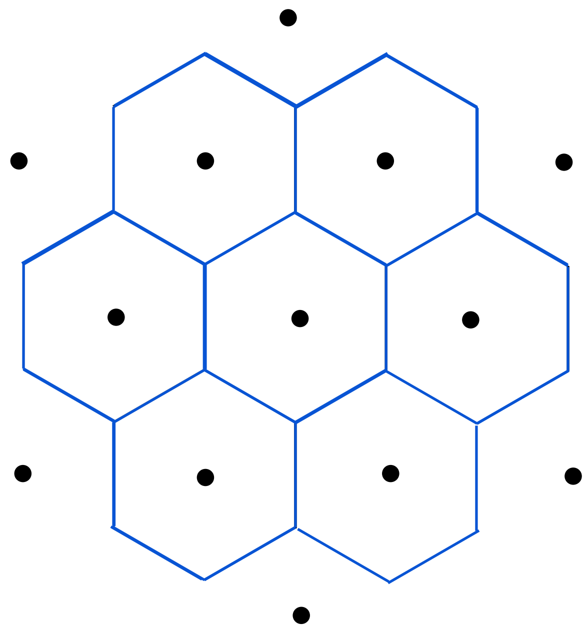

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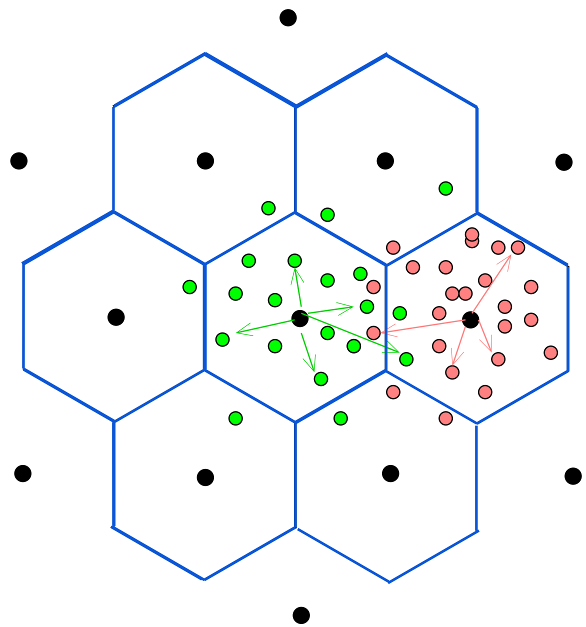

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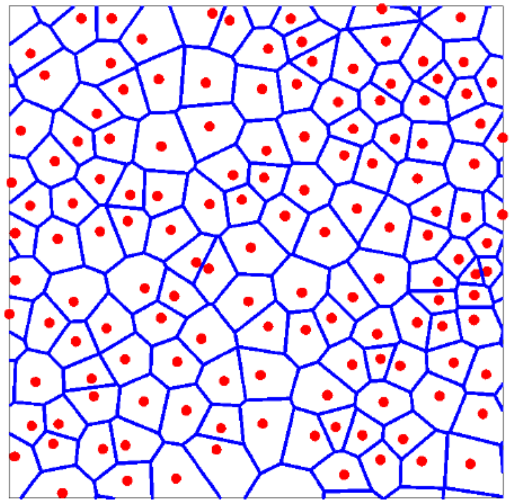
Why? Hexagons can tile the plane. Have the smallest ratio of perimeter to area (compared to equilateral triangles and squares, which tile the plane too) $\longrightarrow$ minimizes the relative number of users next to the cell boundary (which are more difficult to serve due to smaller SINR).

Problems with the Honeycomb

- Real deployment of BS is never a honeycomb (due to obvious architectural constrains). Often "looks like a random pattern".



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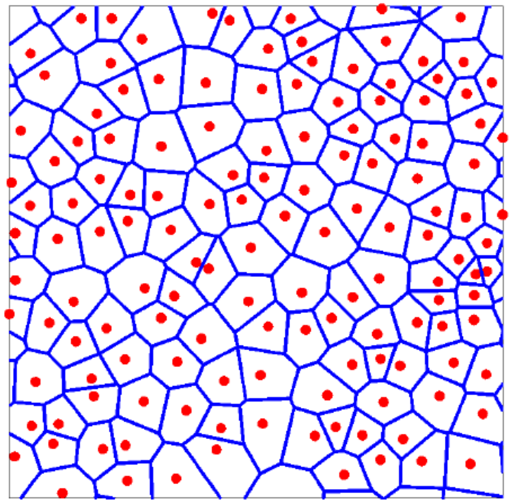

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- Honeycomb cellular networks may be also hard for analytic evaluation.
  E.g. distribution function of SINR of the "typical" user in the network?

"Wireless cell" is not (always) Voronoi

Voronoi Cell of $\boldsymbol{X}$ in $\{\boldsymbol{X}_i\}$: set of "locations" closer to $\boldsymbol{X}$ than to any point of $\{\boldsymbol{X}_i\}$;
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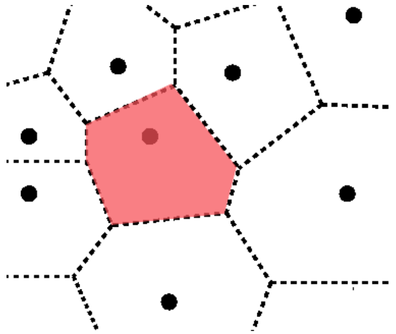

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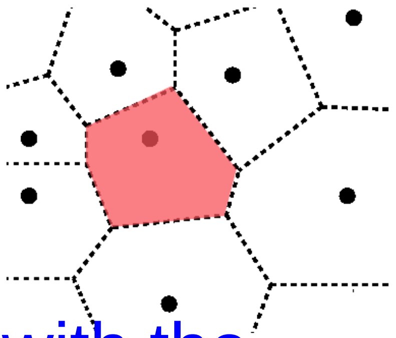

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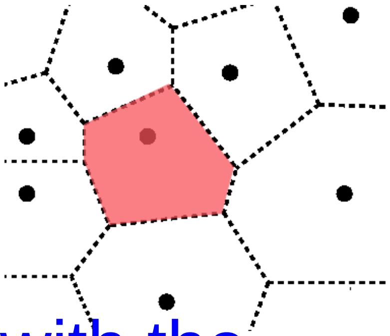

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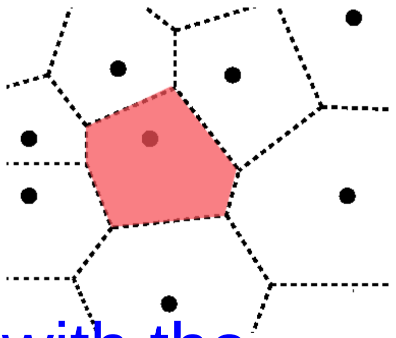

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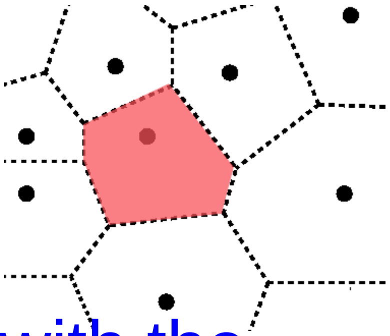

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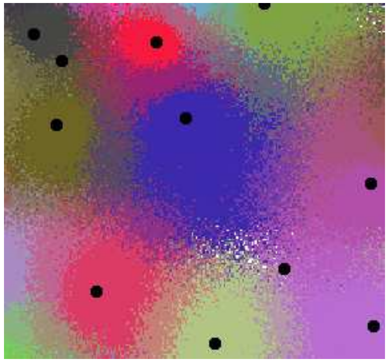

Shadowing randomly "perturbs" geometry.

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Enough arguments to adopt a stochastic modeling of the BS!

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A bit of formalism:

Random point patterns (random elements in the space of locally finite subsets of some space, here plane) are called point processes (pp). Usually considered as random (purely atomic) measures (in point process theory). Can be also seen as random closed sets (in stochastic geometry).

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Point processes exhibiting some "point repulsion" are potentially suitable to model BS locations:

Gibbs pp (with appropriate conditional density), Hard-core models (e.g. arising from random sequential packing of balls), Determinantal pp (arise in physics, random matrix theory, combinatorics), Zeros of Gaussian Analytic Functions, Perturbed Lattices, etc.

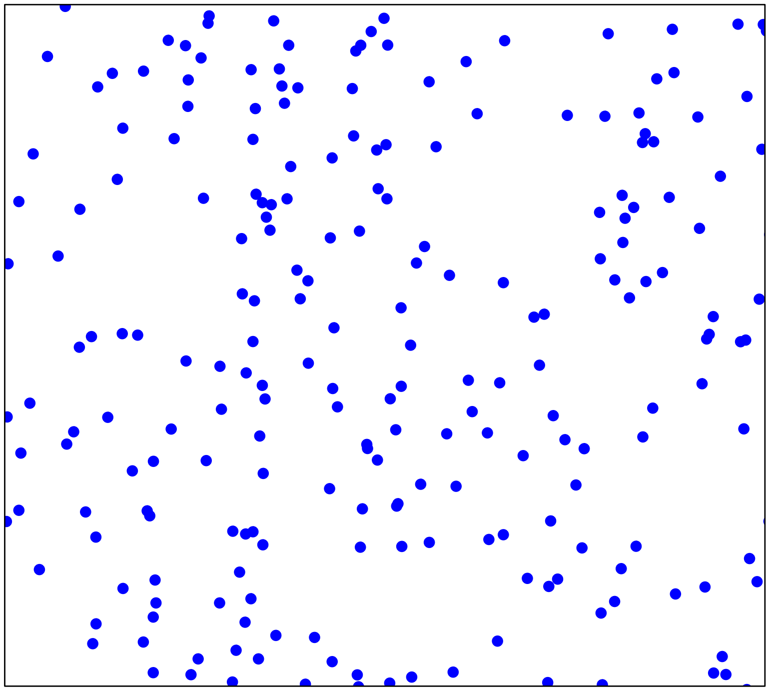
:-(Usually not amenable to explicit quantitative analysis of SINR!

And if we assume

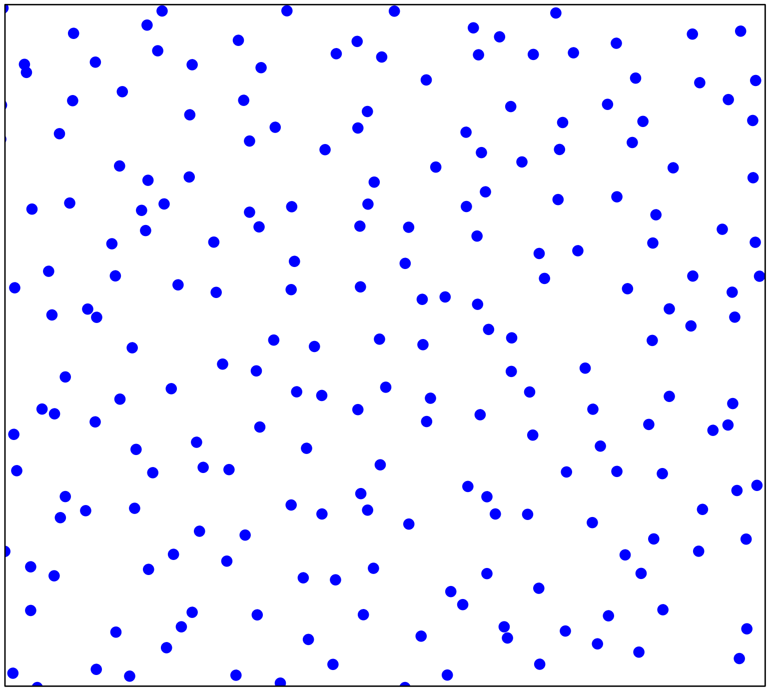
"completely random" network?

Completely random point pattern $\equiv$ Poisson pp.

Poisson pp versus a “perturbed Honeycomb”



Poisson pp



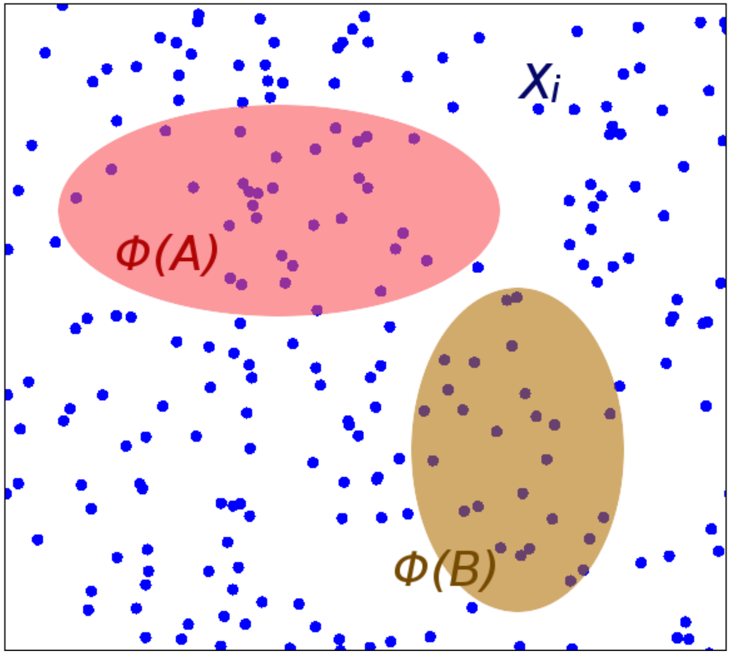
a perturbed Honeycomb

Poisson pp is easy to work with but exhibits more clustering.

Poisson point process — basic facts

Poisson pp

Definition: $\boldsymbol{\Phi} = \{\boldsymbol{X_i}\}$ is a Poisson pp of intensity measure $\boldsymbol{\Lambda}$ in a (Polish) space $\mathbb{E}$ if



(1) number of points of $\boldsymbol{\Phi}$ in any set $\boldsymbol{A}$, $\boldsymbol{\Phi(A)}$, is Poisson random variable with mean $\boldsymbol{\Lambda(A)}$.

(2) numbers of points of $\boldsymbol{\Phi}$ in disjoint sets, $\boldsymbol{\Phi(A_1), \ldots \Phi(A_n)}$, $\boldsymbol{n \geqslant 1}$ are independent random variables.

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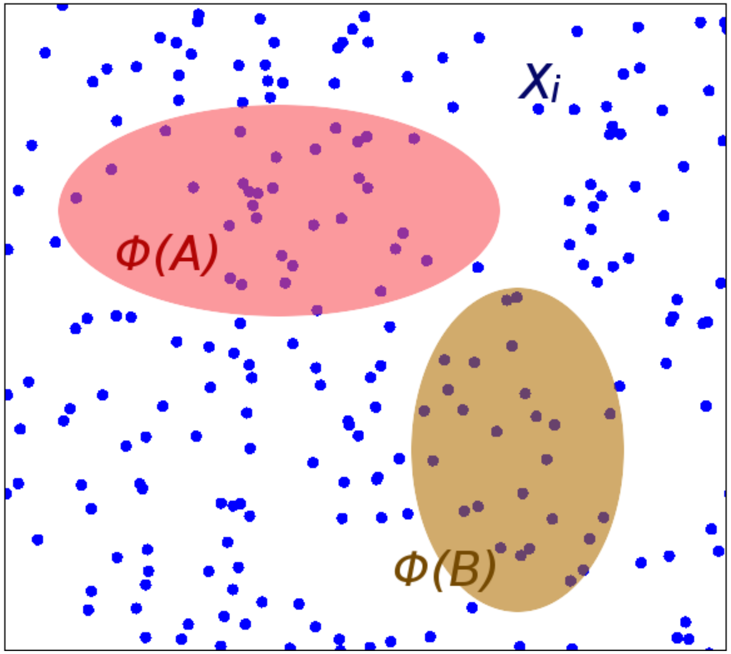

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(2) numbers of points of Φ in disjoint sets, $\Phi(A_1), \ldots \Phi(A_n)$, $n \geqslant 1$ are independent random variables.

"Complete independence" (2) characterizes Poisson pp (provided it is simple, without fixed atoms).

Poisson pp and random mapping of points

Consider probability kernel $\boldsymbol{p(x,\cdot)}$ from $\mathbb{E}$ to some (Polish) space $\mathbb{E}'$. Given pp $\boldsymbol{\Phi}$ on $\mathbb{E}$, consider independent, mapping of each $\boldsymbol{X} \in \boldsymbol{\Phi}$ to $\mathbb{E}'$ with distribution $\boldsymbol{p(X,\cdot)}$.

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Displacement Theroem: Independent mapping of points of Poisson pp of intensity $\boldsymbol{\Lambda}$ on $\mathbb{E}$ to $\mathbb{E}'$, with kernel $\boldsymbol{p}(\boldsymbol{x}, \cdot)$ is Poisson pp on $\mathbb{E}'$ with intensity $\boldsymbol{\Lambda}'(\cdot) = \int_{\mathbb{E}} \boldsymbol{p}(\boldsymbol{x}, \cdot)\, \boldsymbol{\Lambda}(\mathrm{d}\boldsymbol{x})$.

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Corollary:

- Independent thinning (removing of points) of Poisson pp remains Poisson.
- Independent marking, i.e. attaching “auxiliary” random elements $K_i \in \mathbb{K}$ to Poisson points $\{X_i\}$ makes $\{(X_i, K_i)\}$ Poisson on $\mathbb{E} \times \mathbb{K}$.

Application: — K_i radio channel conditions from BS X_i.

Poisson pp as a weak limit

A “spatial” extension of the “classical” Poisson theorem for the convergence of binomial variables to Poisson.

Poisson Theorem: Independent thinning of arbitrary pp converges weakly to Poisson pp, provided the retention probability goes to 0, and the process is rescaled to preserve constant intensity.

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Application: The latter result will be used to show that “increasing variability” of the radio channel conditions makes any given network geometry (including the Honeycomb) “is perceived” by a given user as Poisson pp...

Linear and extremal “shot-noise”

$\Phi = \{X_i\}$ — pp on $\mathbb{E}$, f — real function o $\mathbb{E}$. Define:

- $I = I_{\Phi,f} = \int f(x)\Phi(\mathrm{d}x) = \sum_{X_i\in\Phi} f(X_i)$ — shot-noise,
- $Z = Z_{\Phi,f} = \sup\{f(X_i) : X_i \in \Phi\}$ — extremal shot-noise

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Fact: For Poisson pp of intensity $\Lambda(\cdot)$

- $\mathcal{L}_\Phi(f) = \mathsf{E}[e^{-I}] = \exp\{-\int_{\mathbb{E}}(1 - e^{-f(x)})\,\Lambda(\mathrm{d}x)\}$,
- $\mathsf{P}\{\,Z \leqslant t\,\} = \exp\{-\int_{\mathbb{E}} 1(f(x) > t)\,\Lambda(\mathrm{d}x)\}$.

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Application: Z — signal from the strongest BS, I — interference.
Question: Distribution of SINR?
$L/(w + I)$ for $w > 0$ can be evaluated as well.

“Typical point” of Poisson pp

- Consider stationary pp $\mathbf{\Phi}$ on $\mathbb{R}^{d}$ of intensity $\boldsymbol{\lambda} \in (\mathbf{0}, \infty)$ (mean number of points per unit of volume).
  Stationary (homogeneous) Poisson pp: $\mathbf{\Lambda}(\mathrm{d}\boldsymbol{x}) = \boldsymbol{\lambda}\mathrm{d}\boldsymbol{x}$.

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- “Typical pint” of Φ is supposed to be a point uniformly sampled from Φ.
- Formalized in Palm theory. One defines Palm distribution P^0 of Φ: $\mathsf{P}^0\{\,\Gamma\,\} = 1/\lambda\,\mathsf{E}\Big[\int_{[0,1]^d} 1(\Phi - x \in \Gamma)\,\Phi(\mathrm{d}x)\Big]$
where E corresponds to the stationary distribution of Φ.
P^0 is the distribution of points of Φ “seen by an observer sitting on its typical point”.

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  where E corresponds to the stationary distribution of $\mathbf{\Phi}$.
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Theorem [Slivnyak-Mecke characterization of Poisson pp]

$$\mathsf{P}^{\mathbf{0}}\{\,\mathbf{\Gamma}\,\} = \mathsf{P}\{\,\mathbf{\Phi} + \boldsymbol{\delta}_{\mathbf{0}} \,\in \mathbf{\Gamma}\} \text{ iff } \mathbf{\Phi} \text{ is Poisson pp.}$$

“Typical Poisson point sees stationary configuration.”

Neighbours of the typical point

For a stationary point process Φ of intensity λ on $\mathbb{R}^2$ define

$K(r) = \frac{1}{\lambda}\mathsf{E}^0[\Phi(\{x : |x| \leqslant r\}) - 1]$ (Riplay's K function)

expected number of "additional" points of Φ within the distance r of its typical point.

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Define also $L(r) = \sqrt{K(r)/\pi}$ (Riplay's L function).

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Fact: For Poisson pp $K(r) = \pi r^2$, $L(r) = r$ (Slivnyak-Mecke).

Neighbours of the typical point

For a stationary point process Φ of intensity λ on $\mathbb{R}^2$ define

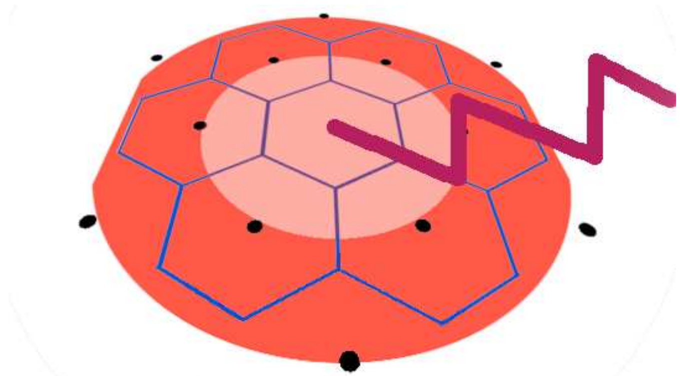
$K(r) = \frac{1}{\lambda}\mathsf{E}^0[\Phi(\{x : |x| \leqslant r\}) - 1]$ (Riplay's K function)

expected number of "additional" points of Φ within the distance r of its typical point.

Define also $L(r) = \sqrt{K(r)/\pi}$ (Riplay's L function).

Fact: For Poisson pp $K(r) = \pi r^2$, $L(r) = r$ (Slivnyak-Mecke).

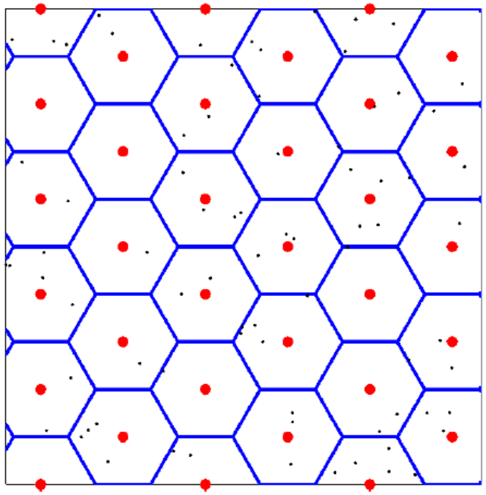
Application: Estimators of Riplay's K and L functions, are used to compare (regularity/clustering in) observed point patterns ... to be used for BS patterns.



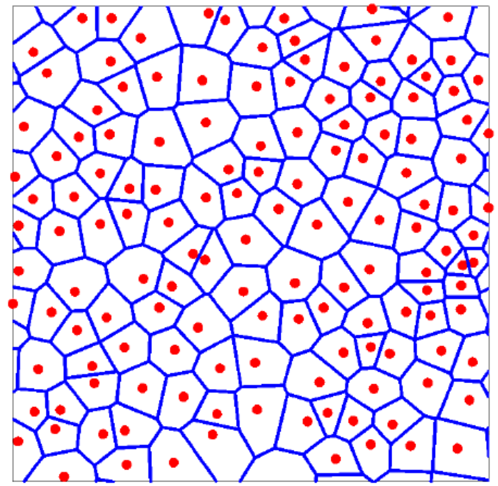
L function for the Honeycomb

Poisson pp as a model for BS

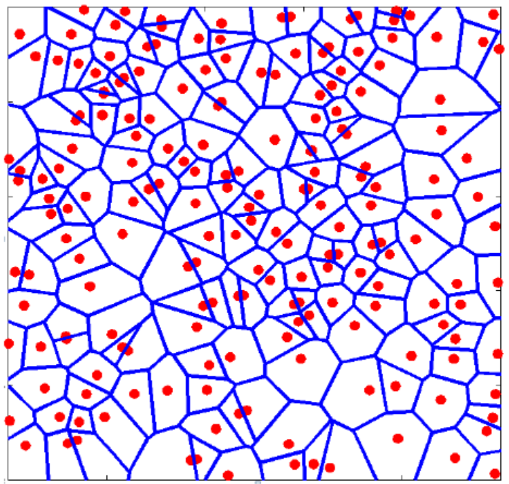
Poisson or not Poisson



Grid Model



Actual 4G Deployment



Poisson Point Process (PPP)

Figure lifted from a talk by Harpreet S. Dhillon of University of Texas at Austin.

Poisson or not Poisson

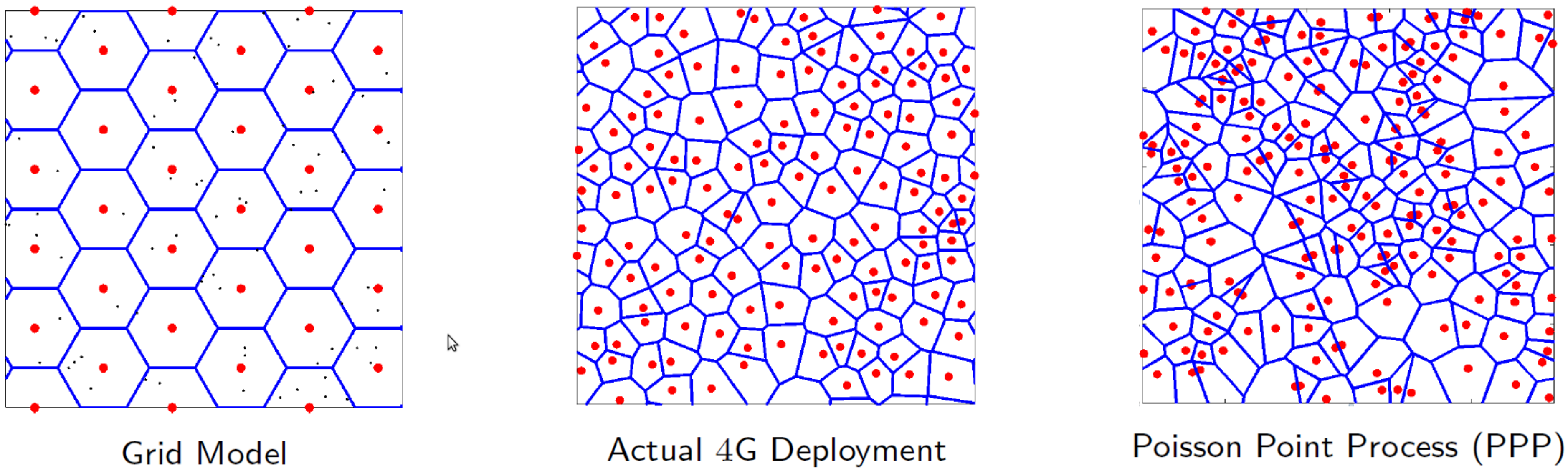
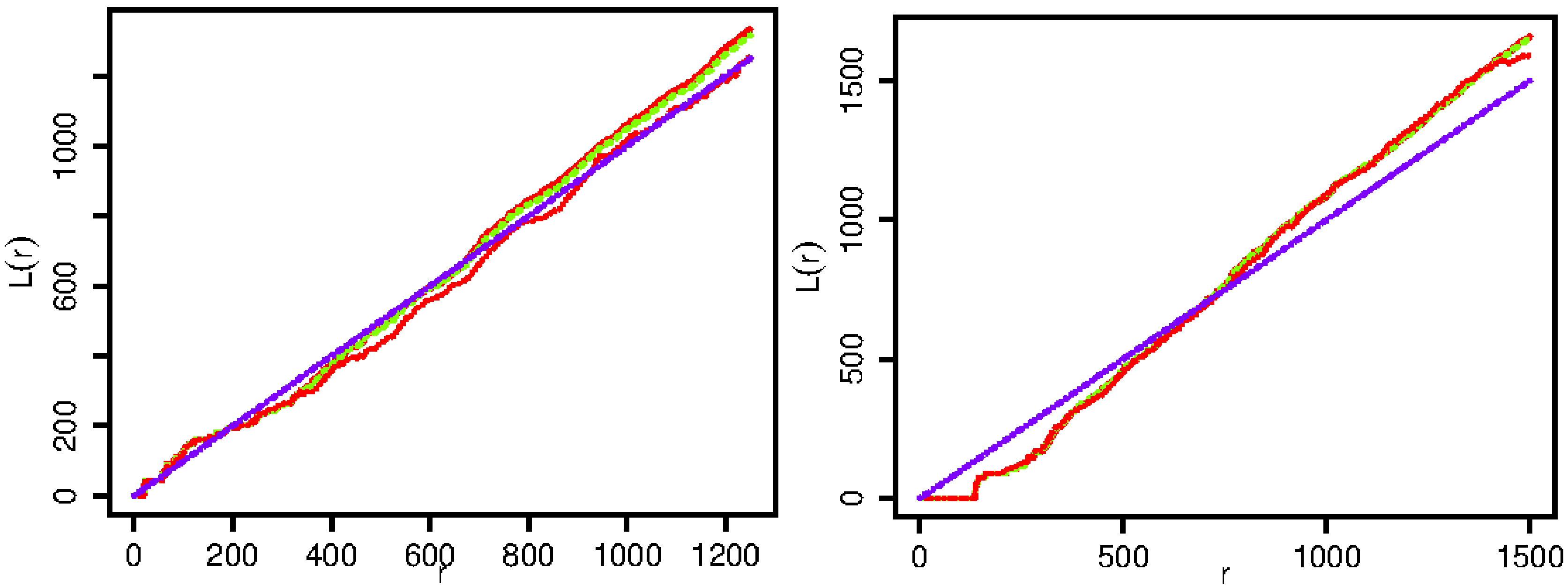


Figure lifted from a talk by Harpreet S. Dhillon of University of Texas at Austin.

More (counter-) arguments for Poisson modeling of real network deployment in

- C.-H. Lee, C.-Y. Shih, and Y.-S. Chen, “Stochastic geometry based models for modeling cellular networks in urban areas”, *Wireless Networks*, 2012.

Comparison of Riplay’s function



”Poisson-like network” more “regular” network

Empirical Riplay’s L function for real positioning of BS in some big European city

thanks to M. Jovanovic and M.K. Karray [Orange Labs]

User in Poisson network

- Consider one user located at, say, the origin of the plane $\mathbb{R}^{\mathbf{2}}$.

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User in Poisson network

- Consider one user located at, say, the origin of the plane $\mathbb{R}^2$.
- Locations of BS on the plane represented by homogeneous Poisson pp $\Phi = \{X_i \in \mathbb{R}^2\}$ on the plane.
- Usual path-loss model: the received power of a signal originating from a base station at X_i is

$$P_{X_i} = \frac{S_{X_i}}{\ell(|X_i|)} = \frac{S_{X_i}}{(K|X_i|)^\beta},$$

where $\ell(x) = (K|x|)^\beta$ is a deterministic path-loss function, with constants $K > 0$ and $\beta > 2$, and the random variable S_i represents shadowing.

Path-loss to serving BS

The smallest path-loss (to a serving BS):

$$L^* = \min_{X_i \in \Phi} \frac{(K|X_i|^{\beta})}{S_{X_i}}.$$

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Fact: The distribution function of L^* admits a simple expression in Poisson model

$$\mathsf{P}(L^* \geqslant t) = \exp\{-\lambda\pi\mathrm{E}[S^{2/\beta}]t^{2/\beta}/K^2\}\,,$$

for a general distribution of S with $\mathsf{E}[S^{2/\beta}] < \infty$.
proof: extremal shot-noise distribution.

SINR to serving BS

SINR corresponding to L^*:

$$\mathrm{SINR}^* = \frac{1/L^*}{W + \sum_{X_i \in \Phi} S_{X_i}/(K|X_i|)^\beta - 1/L^*}.$$

SINR to serving BS

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Fact: $\mathsf{P}\{\,\mathrm{SINR}^* \geqslant t\,\} =$
$\sum_{n=1}^{\lceil 1/t\rceil} (-1)^{n-k}\binom{n-1}{k-1} t_n^{-2n/\beta}\mathcal{I}_{n,\beta}(Wa^{-\beta/2})\mathcal{J}_{n,\beta}(t_n)$,

where $a = \lambda\pi\mathsf{E}[S^{2/\beta}]/K^2$, $t_n = \frac{t}{1-(n-1)t}$,

$\mathcal{I}_{n,\beta}(x) = \frac{2^n\int_0^\infty u^{2n-1}e^{-u^2-u^\beta x\Gamma(1-2/\beta)^{-\beta/2}}du}{\beta^{n-1}(C'(\beta))^n(n-1)!}$, with $C'(\beta) = \frac{2\pi}{\beta\sin(2\pi/\beta)}$,

$\mathcal{J}_{n,\beta}(x) = \int_{[0,1]^{n-1}} \frac{\prod\limits_{i=1}^{n-1} v_i^{i(2/\beta+1)-1}(1-v_i)^{2/\beta}}{\prod\limits_{i=1}^{n-1}(x+\eta_i)} dv_1\ldots dv_{n-1}$, with

$\eta_i := (1-v_i)\prod_{k=i+1}^{n-1} v_k$

cf. H. P. Keeler, B.B., and M.K. Karray, *Proc. of IEEE ISIT*, 2013

Shadowing makes everything is similar to Poisson — a convergence result.

Claim

Hexagonal or any spatially homogeneous network with shadowing of sufficiently large variance is perceived at a given user location as an equivalent Poisson network without shadowing.

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Hexagonal or any spatially homogeneous network with shadowing of sufficiently large variance is perceived at a given user location as an equivalent Poisson network without shadowing.

It means that distribution of the path-loss, interference, SINR, and many more characteristics of the network “measured” by this user can be approximated using the equivalent Poisson network model.

Arbitrary network with log-normal shadowing

- Locations of BS on the plane represented by a locally finite point pattern $\boldsymbol{\phi = \{X_i \in \mathbb{R}^2\}}$; arbitrary, subject to some general condition, to be specified.

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Usually one uses logarithmic standard deviation (log-SD) $v = \sigma 10/\log 10$ (SD of the path-loss expressed in dB) to parametrize the shadowing variance.

User's perception of the network

Denote by

$$\mathcal{N} = \mathcal{N}^{(\sigma)} := \left\{ Y_i^{(\sigma)} = \frac{K(\sigma)^\beta |X_i|^\beta}{S_i^{(\sigma)}} : X_i \in \phi \right\}$$

the path-loss process of the typical user, i.e.; the values of path-loss it has with all BS.

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For convenience, we consider $\boldsymbol{\mathcal{N}}^{(\sigma)}$ as a point process on $\mathbb{R}^+$ having unit-mass atoms located at $\boldsymbol{Y}_i^{(\sigma)}$.

Note: $\boldsymbol{\mathcal{N}}$ is a one-dimensional image of the planar network geometry ϕ perceived by the user at the origin.

Homogeneous network assumption

Let $B_0(r) = \{x : |x| < r\}$ ball of radius r centered at the origin.

(Empirical) homogeneity condition: we require that

$$\frac{\#\{X_i \in B_0(r)\}}{\pi r^2} \to \lambda \quad \text{as } r \to \infty$$

for some constant λ, $0 < \lambda < \infty$;

λ is the (empirical) density of the network.

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This condition is satisfied in particular by any regular lattice (e.g. honeycomb) network model, all reasonable perturbed lattice models and for almost any realization ϕ of a random ergodic point process Φ.

Convergence result

Theorem:

Let ϕ be a given pattern of BS satisfying the homogeneity condition with empirical density λ. Then, the one-dimensional image $\mathcal{N}^{(\sigma)}$ of the network ϕ perceived by the user at the origin converges weakly as $\sigma \to \infty$ to the Poisson point process on $\mathbb{R}^+$ with the intensity measure $\Lambda(t) = \frac{\lambda\pi}{K^2} t^{\frac{2}{\beta}}$. This image is characteristic for a planar Poisson network of BS with intensity λ.

Proof idea

Increasing by $\boldsymbol{\Delta}$ the variance of the log-normal shadowing corresponds on the logarithmic scale, to adding to all path-loss values $\boldsymbol{Y_i^{(\sigma)}}$ received by given user (i.e. points of $\boldsymbol{\mathcal{N}^{(\sigma)}}$) independent Gaussian terms. Indeed

$$\boldsymbol{S_i^{(\sigma+\Delta)} = \exp(-(\sigma+\Delta)^2/2 + (\sigma+\Delta)Z_i)}$$
$$\boldsymbol{=_{\text{distr.}} S_i^{(\sigma)} \times \exp(-\sigma\Delta - \Delta^2/2 + \Delta Z_i')}.$$

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For $\boldsymbol{\mathcal{N}^{(\sigma)}}$, on the logarithmic scale, one can use the Poisson convergence result for successive translations of points. Sufficient condition for such a result:

$$\boldsymbol{\sup_i \mathsf{P}(Y_i^{(\sigma)} \in [0,t]) \to 0}$$

and

$$\boldsymbol{\mathsf{E}\left[\mathcal{N}^{(\sigma)}([0,t])\right] \to \Lambda(t)}$$

can be verified; cf. B.B., H. P. Keeler and M.K. Karray, *Proc. of IEEE Infocom*, 2013.

Statistical confirmation

- How large σ should be to use Poisson approximation?
- In a given (simulated or real-data scenario), one can compare the empirical distribution of L^* and SINR* to theoretical distribution (just presented) in Poisson model.
- The values of L^* and SINR* are measured by users and reported to the operator in a real network. Operators usually have data regarding the empirical distribution of L^* and SINR*.

L^*, honeycomb versus Poisson

- We assume some value of path-loss exponent β from 2.5 to 5.

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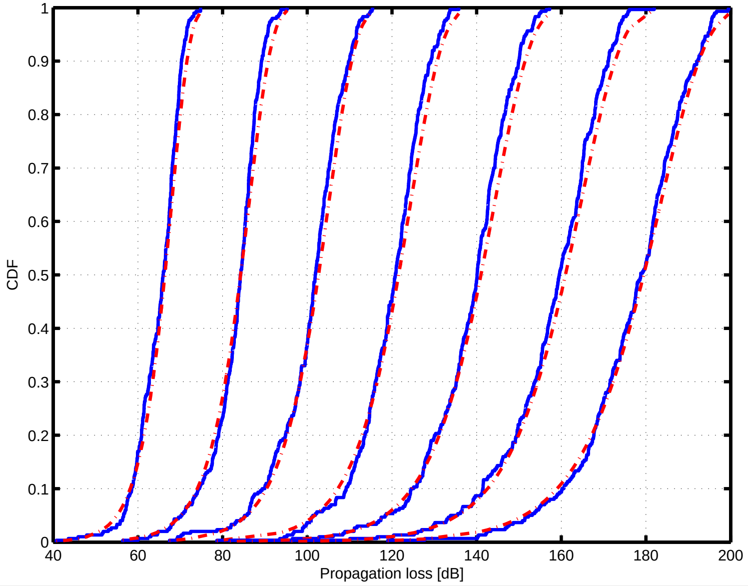
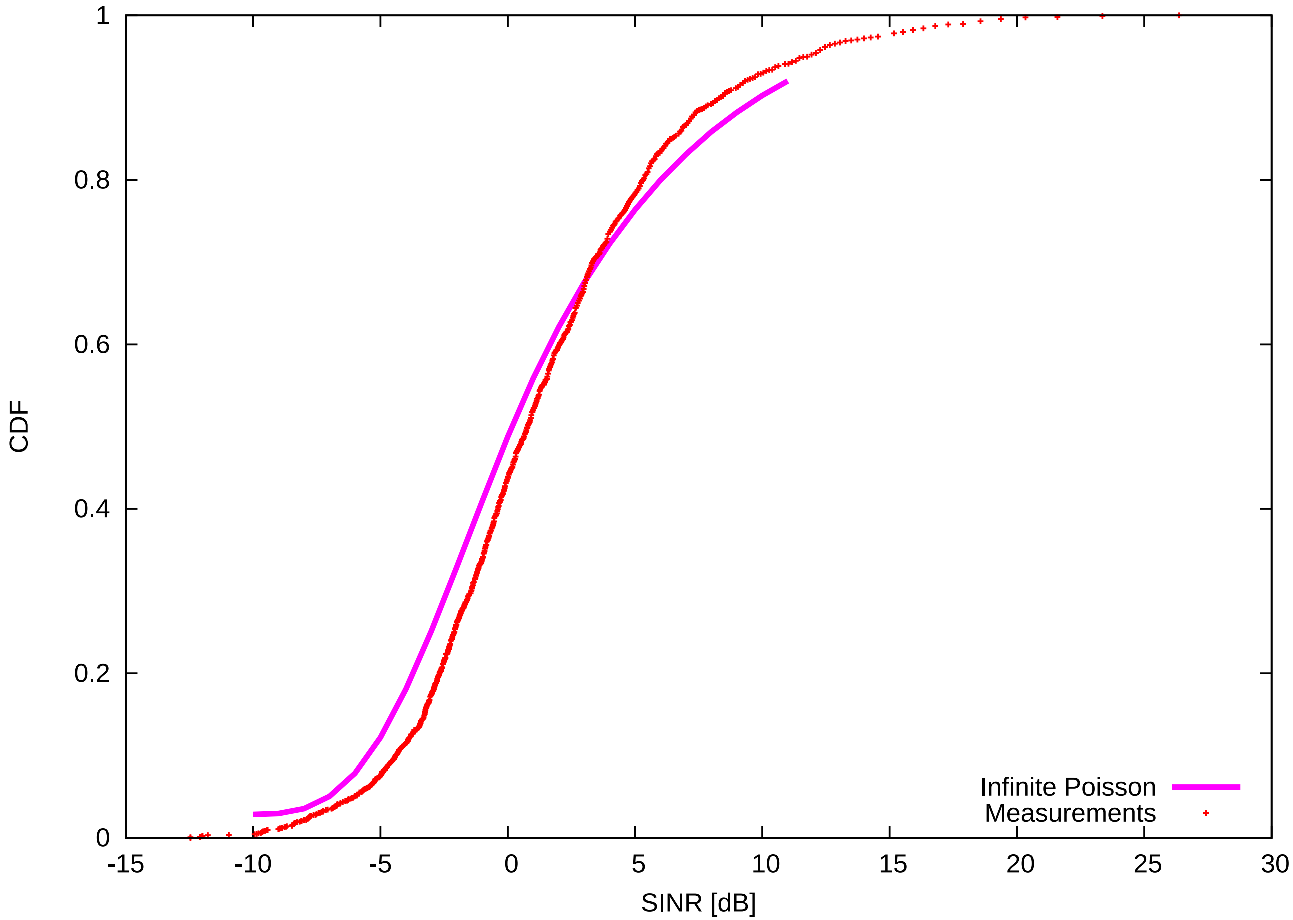
Result: Starting from the log-SD v of 7dB to 15dB (depending on β from 2.5 to 5) the K-S test does not distinguish honeycomb from Poisson at a reasonable confidence level.



SINR* real network data versus Poisson



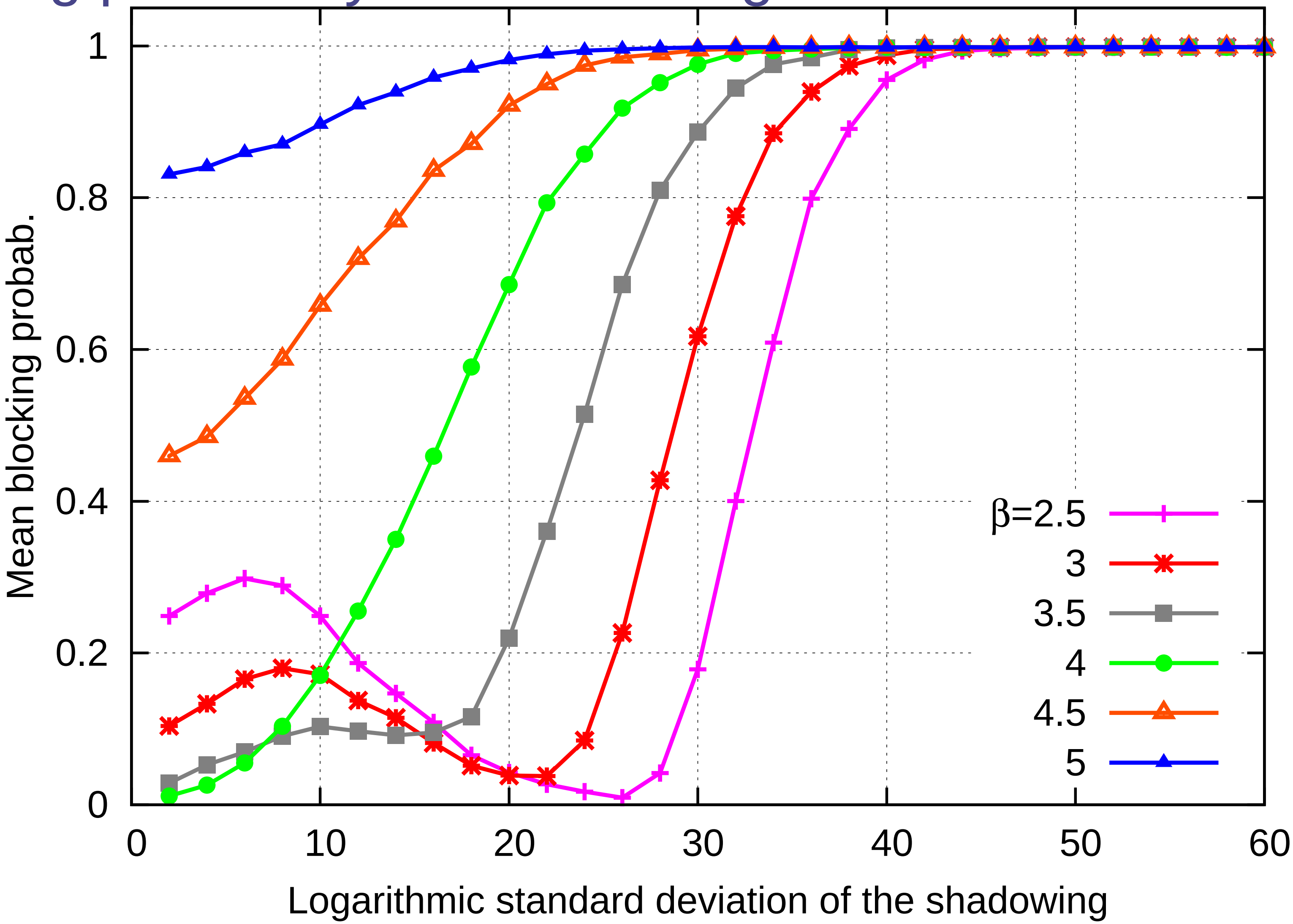
Distribution function of the SINR from the strongest BS

thanks to M. Jovanovic and M.K. Karray [Orange Labs]

When shadowing improves performance — heavy tails in action

Shadowing; a stochastic resonances?

Blocking probability v/s shadowing variance



OFDMA, downlink, hexagonal network of 36 BS, cell radius 0.5 km; log-normal shadowing, path-loss exponent β, traffic 34.6 Erlang per km^2. [BB-Karray (2011)]

Elements of explanation

- User call-blocking depends on
  - Signal from the stronger (serving) BS $\max_i P_i$.
  - signal-to-interference-and-noise ratio from it
    $\text{SINR}^* = \frac{\max_i P_i}{W + \sum_i P_i - \max_i P_i}$.

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  where $P_i = S_i(K|X_i|)^{-\beta}$. Assume log-normal S with $\mathrm{E}[S] = 1$ and increasing variance.
- Signal from the strongest station decreases in variance of S. (No surprise.) Indeed, S converges in Pr and in L1 to 0, when $Var(S) \to \infty$.
- For negligible noise $W = 0$, mean SINR increases in variance of S. (Surprise?) Not really: single big-jump principle of heavy tailed variables! For log-normal (heavy tailed) S_i: $\max_i S_i \sim \sum_i S_i$, hence $\frac{\max_i S_i}{\sum_i S_i - \max_i S_i} \sim \infty$.

Conclusions

CONCLUSIONS

- Deployment of BS is usually not regular (far from the "optimal" Honeycomb"). Often Poisson pp can be used to model it.
- Poisson pp allows to capture explicitly the distribution of the path-loss and SINR of a given user to its serving (strongest) BS, which is a fundamental cellular network characteristic.
- Shadowing impacts geometry of cellular networks. It makes it "even more" Poisson. It can also "separate" the strongest signal from the interference thus increasing SINR (which is a good thing).

thank you