Segment recombinations and random sharing models

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Motivation

- Take a homogeneous Poisson point process (PPP) on $\mathbb{R}$ and shift all the points randomly independently by equally distributed amounts. The result is again a homogeneous PPP.

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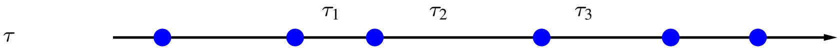
- What's if the shifts depend on the neighbouring points?

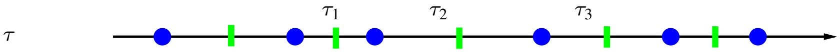
Middle-point divisions

Take a homogeneous PPP on the line:



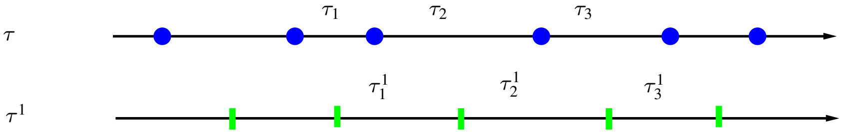
Middle-point divisions

Take the middle points of the segments:



Middle-point divisions

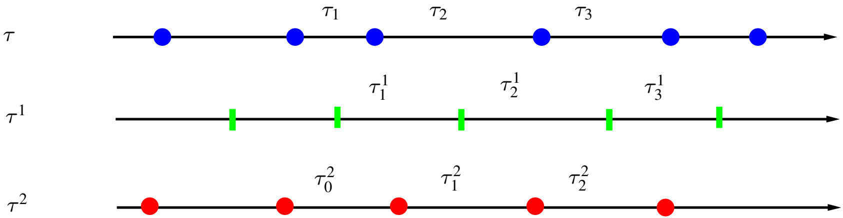
The middle points of each segment do not form a PPP anymore



because the independence of the segment lengths is broken!

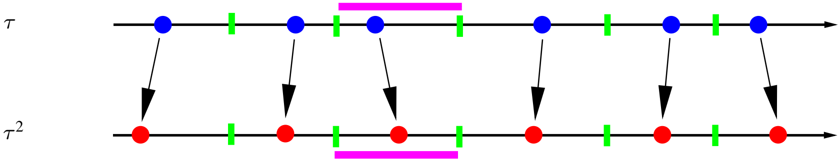
Iterations

Iterate the same procedure



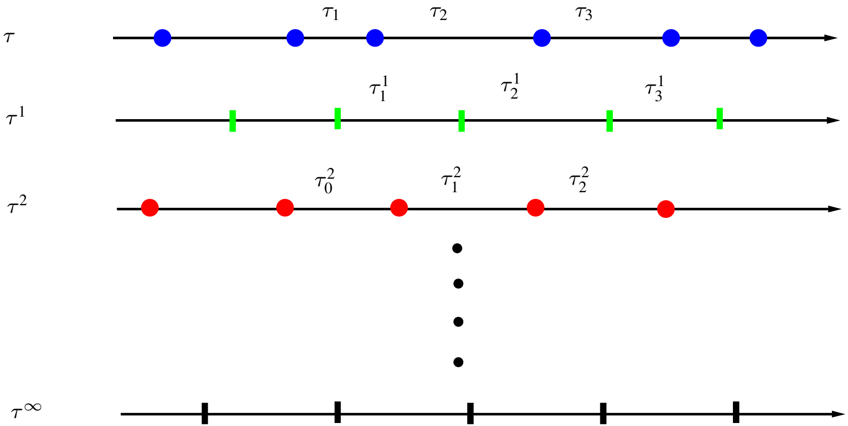
1D adjustment process

Notice that every second generation moves each point to the centre of its 1D Voronoi cell



Limiting configuration

Limiting configuration is a grid (Hasegawa & Tanemura'76)



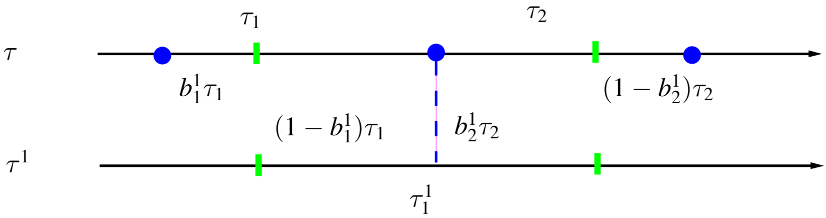
Recursion

Above we have

$$\tau_k^n = \frac{1}{2}\,\tau_k^{n-1} + \frac{1}{2}\,\tau_{k+1}^{n-1}, \quad n = 1, 2, \ldots, \; (\tau^0 \stackrel{\text{def}}{=} \tau).$$

More generally, proportions can be independent random variables with a common distribution G on $[0, 1]$:

$$\tau_k^n = (1 - b_k^n)\tau_k^{n-1} + b_{k+1}^n \tau_{k+1}^{n-1}, \quad b_k^m \sim G \text{ i.i.d.}$$



Question:

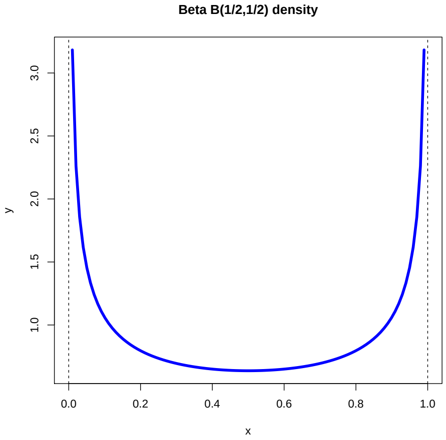
Is there a distribution G such that for $\tau \sim \mathrm{PPP}$,

$$\tau^1 \stackrel{\mathcal{D}}{=} \tau?$$

Answer:

Yes!

G is Beta distribution $\mathrm{B}(r, 1-r)$ for some $r \in (0, 1)$



A General Question:

Assume that τ is a renewal process, so that τ_k are i.i.d. with a common distribution F on $(0, \infty)$. Is there G such that $\tau^1 \stackrel{\mathcal{D}}{=} \tau$?

Result

Theorem

$\tau^1 \stackrel{\mathcal{D}}{=} \tau$ *iff one of the following alternatives is true:*

1. *F is degenerate (τ is a grid) and G is degenerate concentrated on some $b \in [0, 1]$.*
2. *$F = \Gamma(\alpha, \gamma)$ and $G = \mathsf{B}(r\alpha, (1 - r)\alpha)$ for some constants $\alpha > 0$, $\gamma > 0$ and $r \in [0, 1]$,*

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Corollary

- $\tau \sim \mathrm{PPP}$ *and* $G = \mathrm{B}(r, 1 - r)$ *for some* $r \in [0, 1]$
- τ *is renewal process with* $\Gamma(2, \lambda)$ *inter-point distances (every second point in a PPP) and* $G = \mathrm{Unif}(0, 1)$.

Gamma measure

Definition

An independently scattered random measure Λ on $\mathbb{R}$ with Gamma-distributed increments:

$$\Lambda([a,b]) \sim \Gamma(\alpha(b-a),\lambda) \quad \text{for some } \alpha,\ \lambda > 0$$

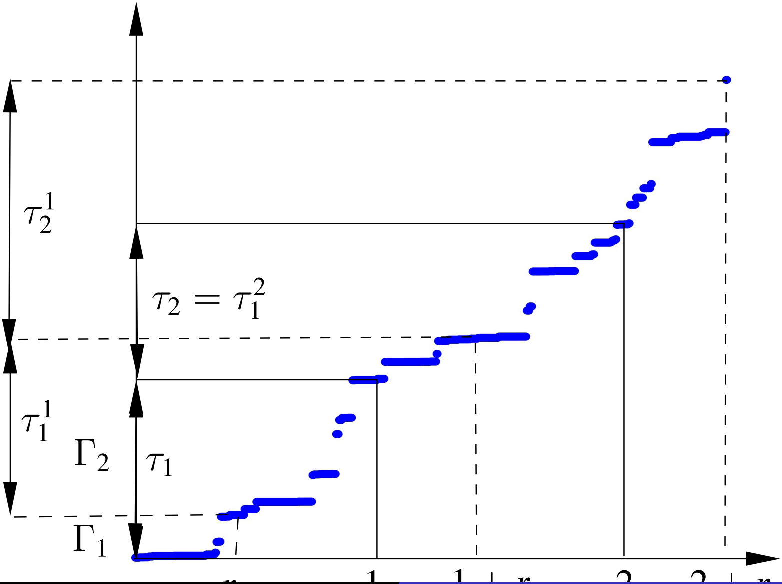
is called the Gamma random measure. $\xi(t) = \Lambda[0,t]$ is called the Gamma process.

Λ is purely atomic and it can be represented as

$$\Lambda([a,b]) \overset{\mathcal{D}}{=} \int_{[a,b]} \int_{(0,\infty)} y\,\Phi(dx\,dy) = \sum_{(x_i,y_i)\in\Phi} y_i\,\mathbb{I}\{x_i \in [a,b]\},$$

where Φ is a PPP on $\mathbb{R} \times (0,\infty)$ driven by intensity measure $\alpha dx\, y^{-1} e^{-\lambda y} dy$ [Ferguson & Klass'72].

Proof: sufficiency

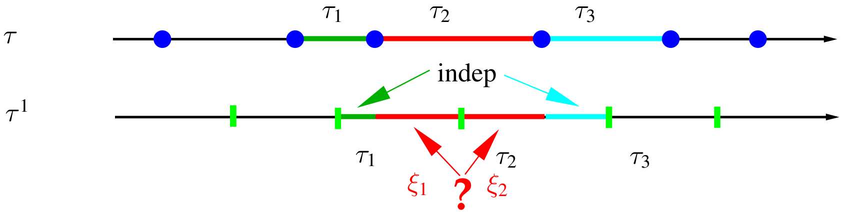


Proof: sufficiency

- $\tau_k \stackrel{\mathcal{D}}{=} \Lambda([k-1,k]) \sim \Gamma(\alpha,\lambda)$ i.i.d.
- $\beta_1 = \Gamma_1/(\Gamma_1+\Gamma_2) \sim \Gamma(r\alpha)/\Gamma(\alpha) = \mathsf{B}(r\alpha,(1-r)\alpha)$. Similarly other b_k i.i.d.

Proof: necessity

Formal proof: via a joint ch.f. *Idea*: shape vs. size result:
$b_1 = \xi_1/(\xi_1 + \xi_2) \perp\!\!\!\perp \tau_2 = (\xi_1 + \xi_2) \Longrightarrow \xi_i \sim \Gamma$.

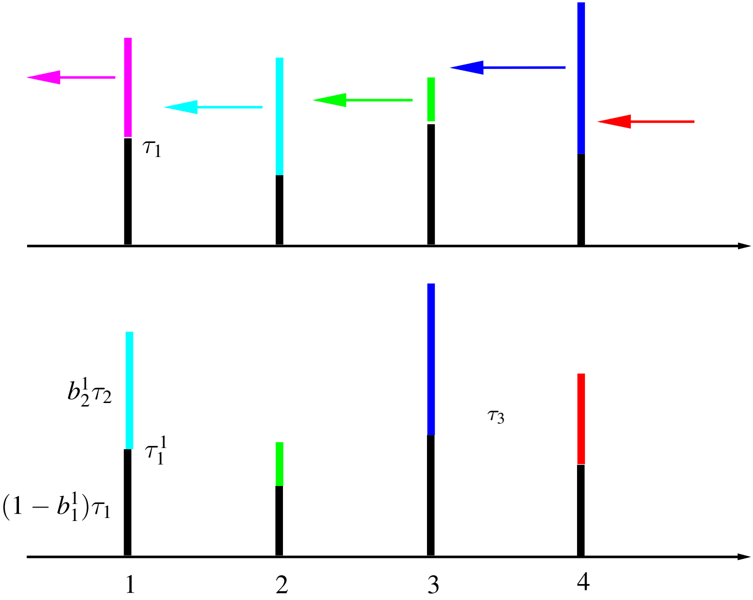


Random sharing

The same recursion

$$\tau_k^n = (1 - b_k^n)\,\tau_k^{n-1} + b_{k+1}^n\,\tau_{k+1}^{n-1}, \quad n = 1, 2, \ldots, \; (\tau^0 \stackrel{\text{def}}{=} \tau).$$

allows for another interpretation: τ_k^n is a wealth of household k at time n, each time a random b-share of the wealth is passed to the left neighbour.


τ_1
$b_2^1\tau_2$
τ_1^1
$(1-b_1^1)\tau_1$
τ_3
1
2
3
4

General random sharing model

Fix a random probability distribution π on $I \subseteq \mathbb{Z}$.
Previously π was the pair $(1-G, G)$ concentrated on $I = \{0, 1\}$, $\pi_1 \stackrel{\mathcal{D}}{=} b$, $\pi_0 \stackrel{\mathcal{D}}{=} 1-b$.

At every moment $n = 1, 2, \ldots$ each household k

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At every moment $n = 1, 2, \dots$ each household k

1. draws independently a realisation $\pi^n(k) = (\pi_i^n(k))_{i \in I}$ of π and
2. passes proportion $\pi_i^n(k)$ of its wealth τ_k^{n-1} to household $k - i$ (leaving proportion $\pi_0(k)$ to itself).

Theorem

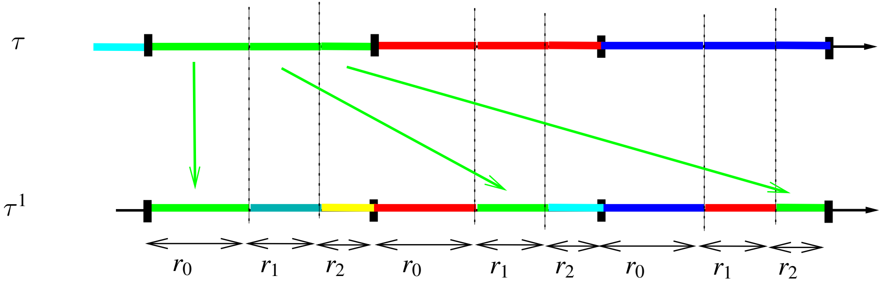
If one of the following alternatives is true:

1. $F = \Gamma(\alpha, \gamma)$ *and* π *is Dirichlet* $\mathrm{Dir}((r_i\alpha)_{i\in I})$ *for some* α, $\gamma > 0$ *and constants* r_i *such that* $\sum_{i\in I} r_i = 1$.
2. F *is degenerate (equal wealths) and* G *is degenerate (non-random proportions sharing)*

then $\tau^1 \stackrel{\mathcal{D}}{=} \tau$.

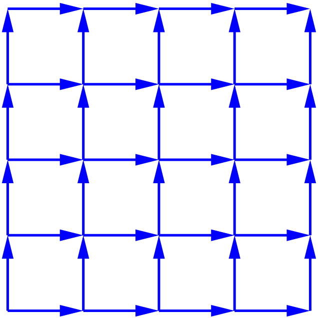
Proof

Redistribution of the Gamma-measure:

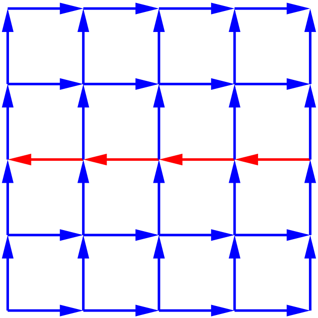


Latices

The same proof works for lattices with one type of vertices



Is the ballance condition enough?



Fundamental implication

- Half-wealth sharing leads to equal wealths (socialism)

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- Beta-wealth sharing of gamma-distributed wealth leaves its distribution unchanged (stable capitalism)
- Question: How attractive is capitalism? (e.g. does beta-sharing of somehow distributed wealths leads to gamma-distributed independent wealths?)

Evolution

The recursion can be written as a linear operator acting on sequences: $\tau^n = B_n \tau^{n-1}$, where B_n is a double-infinite matrix with elements $\pi^n_{i+k}(k)$ at the place (k,i) (the proportion of the wealth of the household i which k receives at time n, ($k, i \in \mathbb{Z}$). E.g.,

$$B_n = \begin{bmatrix} & & & \dots & & \\ \dots & 1-b_1^n & b_2^n & 0 & 0 & \dots \\ \dots & 0 & 1-b_2^n & b_3^n & 0 & \dots \\ \dots & 0 & 0 & 1-b_3^n & b_4^n & \dots \\ \dots & & & \dots & & \end{bmatrix}, \quad n = 1, 2, \dots$$

Laplace transform

Let $h = (h_i)_{i\in\mathbb{Z}}$ be a non-negative sequence with a compact support: $\sum_i \mathbb{1}_{h_i>0} < \infty$.

$$L_n[h] = \mathbf{E}\, e^{-\langle h,\tau^n\rangle} = \mathbf{E}\, e^{-\langle h,B_n\dots B_1\tau\rangle} = \mathbf{E}\, e^{-\langle h,B_1\dots B_n\tau\rangle}.$$

Uniform starting society

Assume $\tau = \mathbf{1}$ and consider $\xi_n = \langle h, B_1 \dots B_n \mathbf{1} \rangle$. Let $\mathcal{B}_n = \sigma\{B_1, \dots, B_n\}$.

$$\mathbf{E}[\xi_n \,\Big|\, \mathcal{B}_{n-1}] = \langle h, B_1, \dots, B_{n-1}(\mathbf{E}\, B_n)\mathbf{1} \rangle$$

The k-th element of $(\mathbf{E}\, B_n)\mathbf{1}$ is

$$\begin{aligned} \mathbf{E} \sum_i \pi^n_{i+k}(k) &= (\text{average total prop.'ns received by } k) \\ &= (\text{average total prop.'ns sent by } k) = 1 \end{aligned}$$

so that $(\mathbf{E}\, B_n)\mathbf{1} = \mathbf{1}$ and thus ξ_n is a martingale.

Thus $\xi_n = \xi_n(h)$ converges a.s. to some $\xi_\infty(h)$ which is a linear function of a finite-dimensional h. Hence it is $\langle h, \tau^\infty \rangle$ for $\tau_k^\infty = \xi_\infty(\delta_k)$.
Finally, $\exp\{-\xi_n\}$ is a bounded submartingale so it converges in $\mathcal{L}_1$:

$$\mathbf{E}\exp\{-\langle h, \tau^n \rangle\} \to \mathbf{E}\exp\{-\langle h, \tau^\infty \rangle\}$$

so that τ^∞ is a weak limit of τ^n

Convergence

More generally,

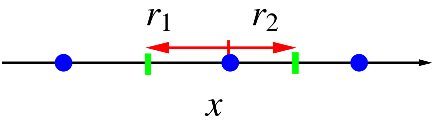
Theorem

If τ^0 is an i.i.d. sequence of realisations of τ with $\mathbf{var}\,\tau < \infty$, then there exists a random sequence τ^∞ (not necessarily i.i.d.) such that τ^∞ is a weak limit of τ^n.

Capitalism is attractive!

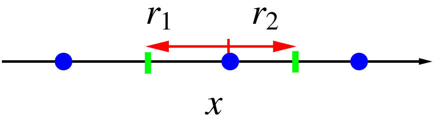
If $\pi \sim \text{Dir}$ then $\{\tau_k^\infty\}$ are independent Gamma-distributed, so the stable capitalism is attractive for i.i.d. starting wealths with finite variance and Dirichlet sharing!

Beyond 1D



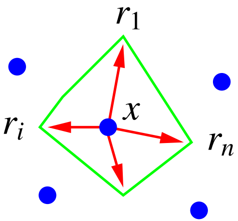
$$x \mapsto x + 1/2(r_1 + r_2)$$

Beyond 1D

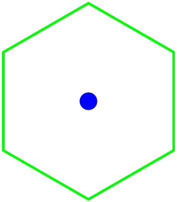


$$x \mapsto x + 2(d_1 r_1 + d_3 r_2)$$
$$(d_1, d_2, d_3) \sim \mathrm{Dir}(1/4, 1/2, 1/4)$$

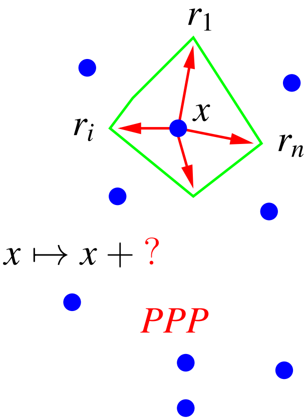
Lloyd algorithm



$$x \mapsto x + n^{-1} \sum_i r_i$$



How to preserve Poisson?



References

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3. A. Muratov and S. Zuyev **On neighbour-dependent shifts preserving renewal process**, arXiv:1308.3351, 2013

Thank you!

Questions?