The Shannon Capacity of an Energy-harvesting Transmitter Over an Additive Noise Channel

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Joint work with Varun Jog

Introduction: Energy harvesting (EH)

- Harvest ambient energy that would otherwise be lost; e.g., solar, thermal, electromagnetic

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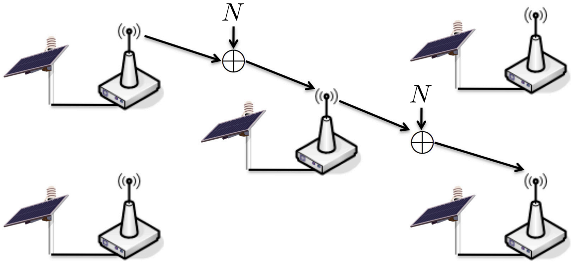
- Harvest ambient energy that would otherwise be lost; e.g., solar, thermal, electromagnetic
- Can use EH for communication:



EH channel model

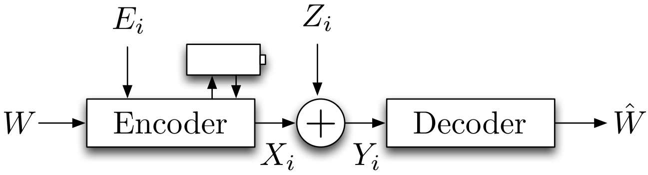


Figure: EH communication system block diagram

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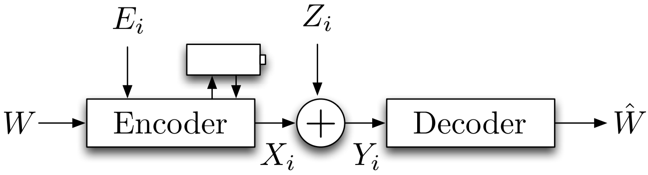

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Challenges: New power constraints!

- Unpredictability of energy
- Presence of a battery

Outline

1. Problem setup
2. The set $\mathcal{S}_n(\sigma, \rho)$
3. Volume based capacity bounds

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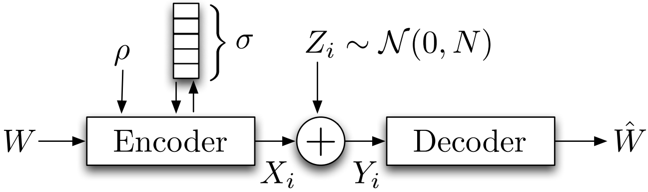
AWGN channel with a finite battery



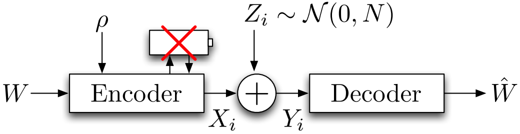
Question

What is the channel capacity of a (σ, ρ) energy constrained AWGN channel?

No battery, $\sigma = 0$

The Information Capacity of Amplitude- and Variance-Constrained Scalar Gaussian Channels*

JOEL G. SMITH



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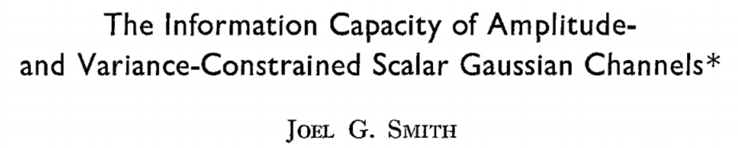
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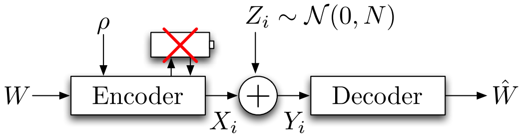

- Smith shows that

$$\text{Capacity} = \sup_{p(x) \text{ supported on } [-\sqrt{\rho}, \sqrt{\rho}]} I(X;Y)$$

- $p^*(x)$ is discrete!

Infinite battery, $\sigma = \infty$

Information-Theoretic Analysis of an Energy Harvesting Communication System

Omur Ozel   Sennur Ulukus

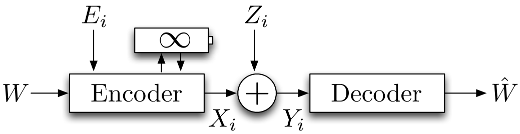

Figure: Infinite battery EH transmitter

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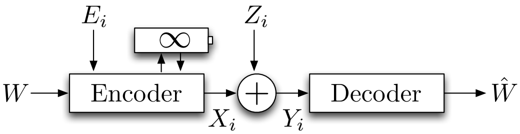

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If $\mathbb{E}(E_i) = P$, capacity is $\frac{1}{2}\log\left(1 + \frac{P}{N}\right)$

(σ, ρ) power constraints

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Energy centered view:

$$\underbrace{\sum_{i=k+1}^{\ell} x_i^2}_{\text{Energy consumed}} \leq \underbrace{\sigma + (\ell - k)\rho}_{\text{Battery + Harvested energy}} \quad \text{for all } 0 \leq k < \ell \leq n$$

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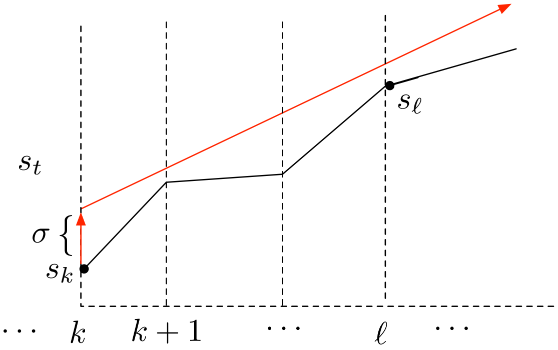
If $s_t = \sum_{i=1}^{t} x_i^2$,

$s_\ell \leq s_k + \sigma + (\ell - k)\rho$

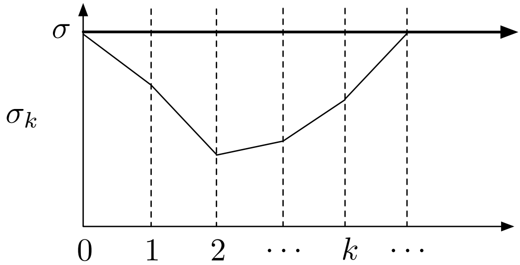


(σ, ρ) power constraints

Battery centered view:

Begin with a *fully charged battery* at time 0, i.e $\sigma_0 = \sigma$. Battery charge at all times must be non-negative, i.e.,

$$\sigma_{k+1} = \min(\underbrace{\sigma}_{\text{Battery capacity}}, \underbrace{\sigma_k + \rho - x_k^2}_{\text{Remaining energy}}) \geq 0, \quad \forall k \geq 0$$



(σ, ρ) power constraints

- Both views are equivalent,

$$\sigma_{k+1} = \min(\sigma,\ \sigma + \rho - x_k^2,\ \cdots,\ \sigma + k\rho - \sum_{i=1}^{k} x_i^2)$$

(σ, ρ) power constraints

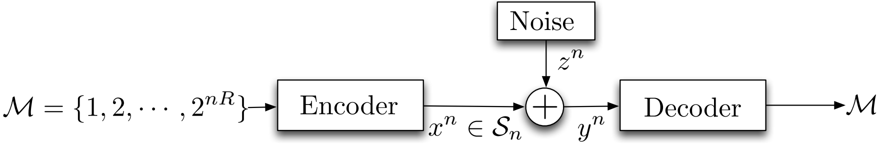
- Both views are equivalent,

$$\sigma_{k+1} = \min(\sigma,\ \sigma + \rho - x_k^2,\ \cdots,\ \sigma + k\rho - \sum_{i=1}^{k} x_i^2)$$

- Let $\mathcal{S}_n(\sigma, \rho) \subseteq \mathbb{R}^n$ be the set of all $(x_1, x_2, ..., x_n)$ satisfying the (σ, ρ) power constraints

Capacity in terms of $\mathcal{S}_n(\sigma, \rho)$

$(2^{nR}, n)$ code:



- Capacity C^* is supremum of all achievable rates R

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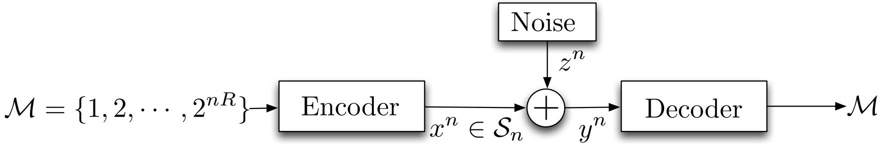

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Theorem

$$C^* = \lim_{n\to\infty} \frac{1}{n} \underbrace{\sup_{p(x^n) \text{ supported on } \mathcal{S}_n} I(X^n; Y^n)}_{C_n} := \lim_{n\to\infty} \frac{C_n}{n}$$

Outline

$\mathcal{S}_n(\sigma, \rho)$: Shape

Constraints

$$\sum_{i=k+1}^{l} x_i^2 \leq \sigma + (k - l)\rho \text{ for all } 0 \leq k < l \leq n$$

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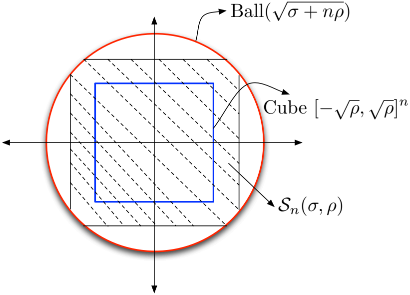

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- How fast does the volume of $\mathcal{S}_n(\sigma, \rho)$ grow with n?

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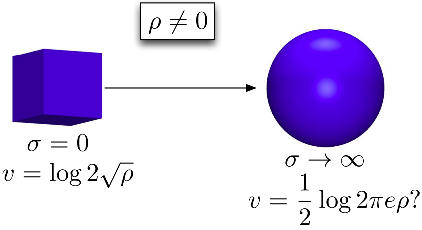

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- Let $\nu_n(b)$ be "volume density of sequences at state b." Then

$$Volume(\mathcal{S}_n) = \int_{b=0}^{\sigma} \nu_n(b) db$$

- How is ν_{n+1} obtained from ν_n?

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- Answer: Via a linear transformation

$$\nu_{n+1}(c) = \int_0^{\sigma} A(b,c) \nu_n(b) db$$

$$A(b,c) = \begin{cases} \frac{1}{\sqrt{b+1-c}} & \text{if } c \neq \sigma \text{ and } c \leq b+1 \\ \delta(c=\sigma) 2\sqrt{b+1-\sigma} & \text{if } c = \sigma \text{ and } \sigma \leq b+1 \\ 0 & \text{otherwise.} \end{cases}$$

Plot of $v(\sigma, 1)$

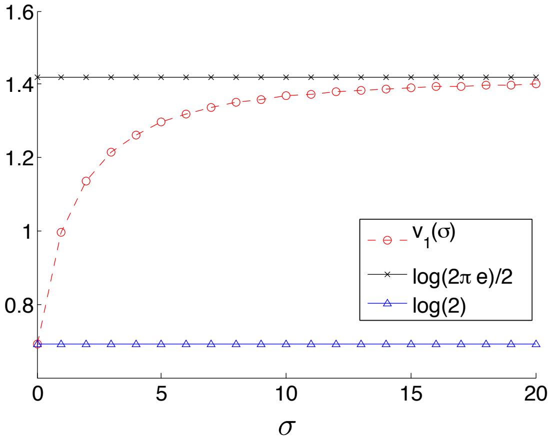

Figure: Plot of $v(\sigma, 1)$

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From volume to capacity

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- Recall capacity of a (σ, ρ) power constrained AWGN channel:

$$C^* = \lim_{n \to \infty} \frac{1}{n} \left[\sup_{p(x^n) \text{ supported on } \mathcal{S}_n} h(Y^n) \right] - \frac{1}{2} \log 2\pi e N$$

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- Recall the Entropy Power Inequality:

$$e^{\frac{2}{n} h(Y^n)} \geq e^{\frac{2}{n} h(X^n)} + e^{\frac{2}{n} h(Z^n)}$$

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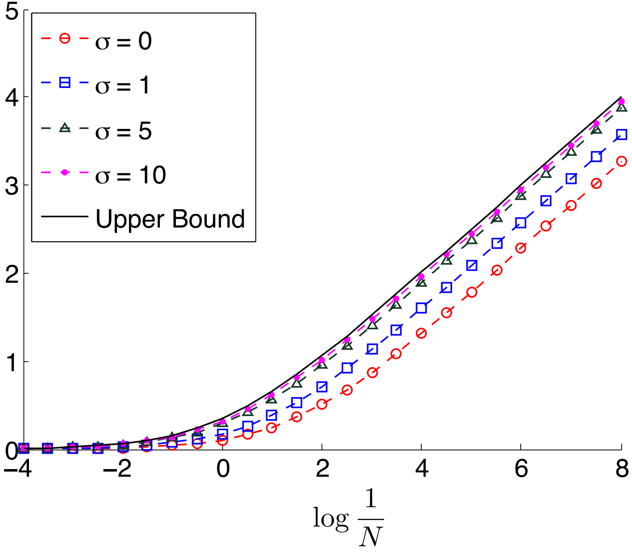
- Now choose $X^n \sim \text{Uniform}(\mathcal{S}_n(\sigma, \rho))$, EPI gives us

$$\frac{1}{2} \log \left(1 + \frac{\rho}{N} \right) \geq C^* \geq \lim_{n \to \infty} \frac{I(X^n; Y^n)}{n} \geq \frac{1}{2} \log \left(1 + \frac{e^{2v(\sigma,\rho)}}{2\pi e N} \right)$$

Compare capacity bounds



Improving the upper bound

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- $$C \leq \lim_{\epsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} \log \mathrm{Vol}\left(\mathcal{S}_n(\sigma, \rho) \oplus B_n(\sqrt{n(N+\epsilon)})\right) - \frac{1}{2} \log 2\pi e N$$

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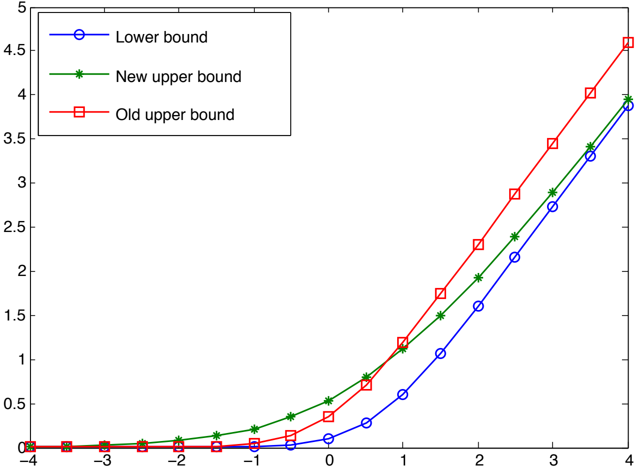
- We can show $I(N)$ is continuous for $N \geq 0$, and so

$$C \leq I(N) - \frac{1}{2} \log 2\pi e N$$

Better capacity bounds



Steiner's formula

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$K_n \subset \mathbb{R}^n$ compact convex set and $B_n \subset \mathbb{R}^n$ the unit ball, then

$$\mathrm{Vol}\left(K_n \oplus tB_n\right) = \sum_{j=0}^{n} \mu_{n-j}(K_n)\epsilon_j t^j$$

where $(\mu_0(K_n), \ldots, \mu_n(K_n))$ are the intrinsic volumes of K_n and ϵ_j the volume of B_j.

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$$I(N) = H(\theta^*) + (1-\theta^*)\log 2\sqrt{\rho} + \frac{\theta^*}{2}\log\frac{2\pi e N}{\theta^*} \ ,$$

where $H(\theta^*) := -\theta^*\log\theta^* - (1-\theta^*)\log(1-\theta^*)$, and

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- Convolution of intrinsic volume sequences and finding the dominant term in the convolution.

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- Cumulant generating function of the intrinsic volume sequence
- We prove that the pointwise limit of $g_n(t)$ as $n \to \infty$ exists, call it $\Lambda(t)$
- If $\Lambda^*(\cdot)$ denotes the convex conjugate dual of $\Lambda(\cdot)$, then

$$I(N) = \sup_{\theta \in [0,1]} \left[-\Lambda^*(1-\theta) + \frac{\theta}{2} \log \frac{2\pi e N}{\theta} \right] .$$

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- (σ, ρ) constraints produce rich geometric structure
- Volume of $\mathcal{S}_n(\sigma, \rho)$ + EPI implies neat lower bound on capacity
- Even small battery provides considerable gains in capacity
- Steiner's formula in the large deviations regime provides refined upper bounds to the capacity.
- The upper and lower bounds match to the first derivative at low noise and to the sixth derivative at high noise.

Figure: **Gorges du Verdon, 25 years ago**

Figure: **With a different kind of Indian**

Figure: **Ten Years Ago**

Figure: **Proving a theorem by the Seine**

Figure: **The Royal Society of Edinburgh**

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Figure: **Freezing in sunny California**

Figure: **Yes, it was windy!**

Figure: **The pig and the Trabant**

Figure: **I dare you to eat it !**

Figure: **These are the types of friends I have !!!**

Thank you!