# Yaglom limits can depend on the starting state

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# The long run is a misleading guide . . .

The long run is a misleading guide to current affairs. In the long run we are all dead. Economists set themselves too easy, too useless a task if in tempestuous seasons they can only tell us that when the storm is past the ocean is flat again.

John Maynard Keynes

- Keynes was a Probabilist: Keynes, John Maynard (1921), Treatise on Probability, London: Macmillan & Co.
- Rather than insinuating that Keynes didn't care about the long run, probabilists might interpret Keynes as advocating the study of evanescent stochastic process: $\mathbb{P}_x\{X_n = y \mid X_n \in S\}$.

# An evanescent process–Gambler's ruin

- Suppose a gambler is pitted against an infinitely wealthy casino.
- The gambler enters the casino with $x > 0$ dollars.
- With each play, the gambler either wins a dollar with probability $b$ where $0 < b < 1/2$ ...
- ... or loses a dollar with probability $a$ where $a + b = 1$.
- The gambler continues to play for as long as possible.
- In the long run the gambler is certainly broke.
- What can be said about her fortune after playing many times given that she still has at least one dollar?

# A quasi-stationary distribution

- Seneta and Vere-Jones (1966) answered this question with the following probability distribution $\pi^*$:

$$\pi^*(y) = \frac{1-\rho}{a} y \left( \sqrt{\frac{b}{a}} \right)^{y-1} \quad \text{for } y = 1, 2, \ldots \tag{1}$$

- where $a = 1 - b$ and $\rho = 2\sqrt{ab}$.

# Limiting conditional distributions

- Let $X_n$ be her fortune after $n$ plays.
- Notice that her fortune alternates between being odd and even.
- For $n$ large, Seneta and Vere-Jones proved that

$$\mathbb{P}_x\{X_n = y \mid X_n \geq 1\} \approx \begin{cases} \frac{\pi^*(y)}{\pi^*(2\mathbb{N})} & \text{for } y \text{ even, } x+n \text{ even,} \\ \frac{\pi^*(y)}{\pi^*(2\mathbb{N}-1)} & \text{for } y \text{ odd, } x+n \text{ odd.} \end{cases}$$

- The subscript $x$ means that $X_0 = x$, $\mathbb{N} := \{1, 2, \ldots\}$.
- The probability $\pi$ assigns to the even and odd natural numbers is denoted by $\pi^*(2\mathbb{N})$ and $\pi^*(2\mathbb{N} - 1)$, respectively.

# Gambler's ruin as a Markov chain

- The Seneta–Vere-Jones example has a state space $\mathbb{N}_0 := \{0\} \cup \mathbb{N}$ where $0$ is absorbing.
- The transition matrix between states in $\mathbb{N}$ is

$$P = \begin{bmatrix} 0 & b & 0 & 0 & 0 & \cdots \\ a & 0 & b & 0 & 0 & \cdots \\ 0 & a & 0 & b & 0 & \cdots \\ \vdots & & & & & \end{bmatrix}.$$

- $P$ is irreducible, strictly substochastic, and periodic with period 2.

# Graphic of Gambler's ruin

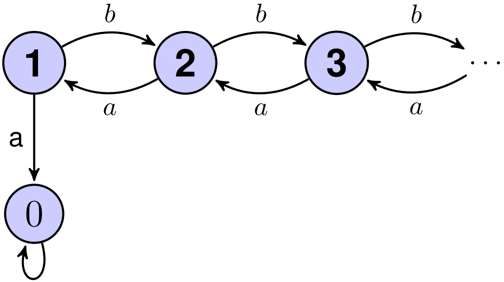



Figure: $P$ restricted to $\mathbb{N}$.

# Facts from Seneta and Vere-Jones

- The $z$-transform of the return time to $1$ is given in Seneta and Vere-Jones:

$$F_{11}(z) = \left( \frac{1 - \sqrt{1 - 4abz^2}}{2} \right).$$

- Hence the convergence parameter of $P$ is $R = 1/\rho$ where $\rho = 2\sqrt{ab}$.
- Moreover $F_{11}(R) = 1/2$ so $P$ is $R$-transient.
- Using Stirling's formula as $n \to \infty$: for $y - x$ even

$$P^{2n}(x,y) \quad \sim \quad \frac{xy}{\sqrt{\pi} n^{3/2}} \left(2\sqrt{ab}\right)^n \left(\sqrt{\frac{a}{b}}\right)^{x-1} \left(\sqrt{\frac{b}{a}}\right)^{y-1}.$$

- Denote the time until absorption by $\tau$ so $P_x(\tau = n) = f_{x0}^{(n)}$.
- If $n - x$ is even then from Feller Vol. 1

$$f_{x0}^{(n)} \quad \sim \quad \frac{x \cdot 2^{n+1}}{(2\pi)^{1/2}(n)^{3/2}} b^{\frac{1}{2}(n-x)} a^{\frac{1}{2}(n+x)}.$$

# Define the kernel $Q$

- It will be convenient to introduce a chain with kernel $Q$ on $\mathbb{N}_0$ with absorption at $\delta$
- defined for $x \geq 0$ by $Q(x, y) = P(x + 1, y + 1)$

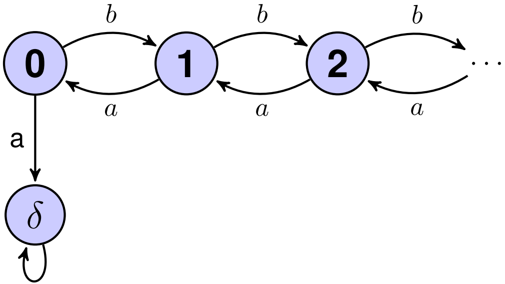



Figure: $Q$ is $P$ relabelled to $\mathbb{N}_0$.

# Our example

- The kernel $K$ of our example has state space $\mathbb{Z}$.
- For $x > 0$, $K(x, y) = Q(x, y), K(-x, -y) = Q(x, y)$,
- $K(0, 1) = K(0, -1) = b/2, K(0, \delta) = a$.
- Folding over the two spoke chain gives the chain with kernel $Q$.

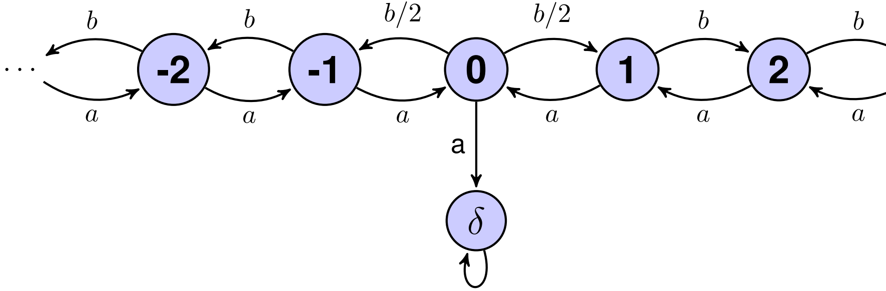
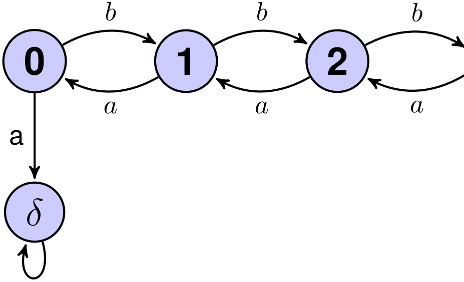



Figure: $K$ restricted to $\mathbb{Z}$.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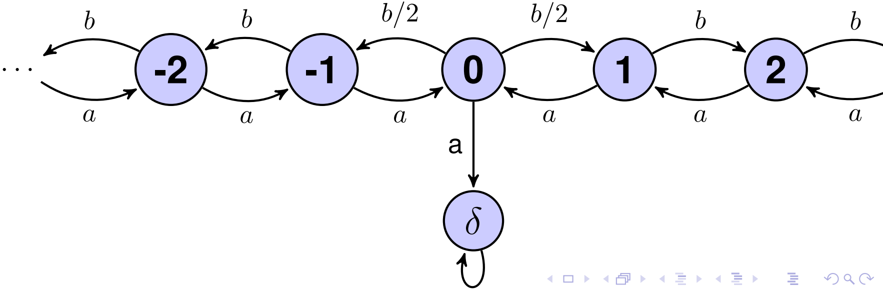
...
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# Yaglom limit of our example

- Define a family $\sigma_\xi$ of $\rho$-invariant qsd's for $K$
- indexed by $\xi \in [-1, 1]$ and given by

$$\sigma_\xi(0) = \frac{1-\rho}{a} \tag{2}$$

$$\sigma_\xi(y) = \sigma_\xi(0) \frac{[1 + |y| + \xi\, y]}{2} \left(\sqrt{\frac{b}{a}}\right)^{|y|} \quad \text{for } y \in \mathbb{Z} \tag{3}$$

- For $x, y \in 2\mathbb{Z}$,

$$\lim_{n\to\infty} \frac{K^{2n}(x,y)}{K^{2n}(x,2\mathbb{Z})} = \frac{1+\rho}{\rho} \sigma_{\xi(x)}(y) \text{ where } \frac{\rho}{1+\rho} = \sigma_{\xi(x)}(2\mathbb{Z}).$$

- where $\xi(x) = \dfrac{x}{1+|x|}$ for $x \in \mathbb{Z}$.
- Notice the limit depends on $x$!

# Definition of Periodic Yaglom limits

- For periodic chains, define $k = k(x, y) \in \{0, 1, 2, \ldots d - 1\}$ so that $K^{nd+k}(x, y) > 0$ for $n$ sufficiently large.
- We can partition S into d sets labeled $S_0, \ldots, S_{d-1}$ so that the starting state $x \in S_0$ and that $K^{nd+k}(x, y) > 0$ for $n$ sufficiently large if $y \in S_k$.
- Theorem A of Vere-Jones implies that for any $y \in S_k$, $[K^{nd+k}(x, y)]^{1/(nd+k)} \to \rho$.
- We say that we have a periodic Yaglom limit if for some $k \in \{0, \ldots, d - 1\}$

$$\mathbb{P}_x\{X_{nd+k} = y \mid X_{nd+k} \in S\} = \frac{K^{nd+k}(x, y)}{K^{nd+k}(x, S)} \to \pi_x^k(y) \quad (4)$$

where $\pi_x^k$ is a probability measure on $S$ with $\pi_x^k(S_k) = 1$.

# Asymptotics of Periodic Yaglom limits

## Proposition

- *If* $\pi_x^k$ *is the periodic Yaglom limit for some* $k \in \{0, 1, \ldots, d-1\}$*, then there are periodic Yaglom limits for all* $k \in \{0, 1, \ldots, d-1\}$*.*
- *Moreover, there is a* $\rho$ *invariant qsd* $\pi_x$ *such that* $\pi_x^k(y) = \pi_x(y)/\pi_x(S_k)$ *for* $y \in S_k$ *for each* $k \in \{0, 1, \ldots, d-1\}$*.*
- *We conclude* $\dfrac{K^{nd+k}(x,y)}{K^{nd+k}(x,S)} \to \dfrac{\pi_x(y)}{\pi_x(S_k)}$ *for all* $k \in \{0, 1, \ldots, d-1\}$ *where* $x \in S_0$ *by definition and* $y \in S_k$*.*

# Periodic ratio limits

- We say that we have a periodic ratio limit if for $x, y \in S_0$

$$\lim_{n\to\infty} \frac{K^{nd}(y, S_0)}{K^{nd}(x, S_0)} = \lambda(x, y) = \frac{h(y)}{h(x)}.$$

- Proposition

  *If we have both periodic Yaglom and ratio limits on $S_0$ then for any $k, m \in \{0, 1, \ldots, d-1\}$, $u \in S^k$ and $y \in S_m$,*

$$K^{nd+d-m+k}(u, y)/K^{nd+d-m+k}(u, S_k) \to \pi_u(y)/\pi_u(S_m).$$

# Theory applied to our example

- Let $S_0 = 2\mathbb{Z}$ and let $x \in S_0$.
- We check that for $y \in 2\mathbb{Z}$,

$$\lim_{n\to\infty} \frac{K^{2n}(x,y)}{K^{2n}(x,2\mathbb{Z})} = \frac{1+\rho}{1}\sigma_{\xi(x)}(y) \text{ where } \sigma_{\xi(x)}(2\mathbb{Z}) = \frac{1}{1+\rho}.$$

- From Proposition 1 we then get for $y \in 2\mathbb{Z} - 1$,

$$\lim_{n\to\infty} \frac{K^{2n+1}(x,y)}{K^{2n}(x,2\mathbb{Z}-1)} = \frac{1+\rho}{\rho}\sigma_{\xi(x)}(y) \text{ where } \sigma_{\xi(x)}(2\mathbb{Z}-1) = \frac{\rho}{1+\rho}.$$

# Checking the periodic Yaglom limit I

- Assume $x, y \geq 1$. Similar to the classical ballot problem, there are two types of paths from $x$ to $y$: those that visit 0 and those that do not. From the reflection principle, any path from $x$ to $y$ that visits 0 has a corresponding path from $-x$ to $y$ with the same probability of occurring.
- Thus, if ${}_{\{0\}}K^n(x,y)$ denotes the probability of going from $x$ to $y$ without visiting zero, we have

$$K^n(x,y) = {}_{\{0\}}K^n(x,y) + K^n(-x,y) = {}_{\{0\}}K^n(x,y) + K^n(x,-y).$$

- From the coupling argument, ${}_{\{0\}}K^n(x,y) = P^n(x,y)$.

# Checking the periodic Yaglom limit II

- For $x, y \geq 0$,

$$Q^n(x,y) = K^n(x,|y|) := K^n(x,y) + K^n(x,-y).$$

- Hence,

$$\begin{aligned} K^n(x,y) &= K^n(x,|y|) - K^n(x,-y) \\ &= K^n(x,|y|) - (K^n(x,y) - {}_{\{0\}}K^n(x,y)) \\ &= \frac{1}{2}({}_{\{0\}}K^n(x,y) + K^n(x,|y|)). \end{aligned}$$

- Similarly,

$$K^n(x,-y) = \frac{1}{2}(K^n(x,|y|) - {}_{\{0\}}K^n(x,y)).$$

# Checking the periodic Yaglom limit III

- For $x, y > 0$ and both even, from (35) in Vere-Jones and Seneta

$$
\begin{aligned}
{}_{\{0\}}K^{2n}(x,y) &= P^{2n}(x,y) \\
&\sim \frac{xy}{\sqrt{\pi}n^{3/2}}\left(2\sqrt{ab}\right)^{2n}\left(\sqrt{\frac{a}{b}}\right)^{x-1}\left(\sqrt{\frac{b}{a}}\right)^{y-1}.
\end{aligned}
$$

- Moreover,

$$
\begin{aligned}
K^{2n}(x,|y|)) &= Q^{2n}(x,y) + Q^{2n}(x,-y) \\
&= P^{2n}(x+1,y+1) + P^{2n}(x+1,-(y+1)) \\
&\sim (x+1)\left(\sqrt{\frac{a}{b}}\right)^{x}(y+1)\left(\sqrt{\frac{b}{a}}\right)^{y}\sqrt{\frac{1}{\pi}}\frac{(4ab)^n}{n^{3/2}}.
\end{aligned}
$$

# Checking the periodic Yaglom limit IV

- Let $\tau_\delta$ be the time to absorption for the chain $X$. so $P_x(\tau_\delta = n) = P_{x+1}(\tau = n)$ and

$$P_x(\tau_\delta > 2n) = \sum_{v=n+1}^{\infty} f_{x+1,0}^{2v-1}. \tag{5}$$

$$\begin{aligned} & P_x(\tau > 2n) \\ \sim \quad & \sum_{v=n+1}^{\infty} \frac{(x+1)\cdot 2^{2v}}{(2\pi)^{1/2}(2v-1)^{3/2}} b^{\frac{1}{2}(2v-1-(x+1))} a^{\frac{1}{2}(2v-1+(x+1))} \\ \sim \quad & \frac{(x+1)}{(2\pi)^{1/2}} \left(\sqrt{\frac{a}{b}}\right)^x \frac{(4ab)^n}{(2n)^{3/2}} \frac{4a}{1-4ab}. \end{aligned}$$

# Checking the periodic Yaglom limit V

- Hence, for $x, y > 0$,

$$\frac{K^{2n}(x,y)}{P_x(\tau > 2n)} = \frac{1}{2}\frac{K^{2n}(x,|y|)) + {}_{\{0\}}K^{2n}(x,y)}{P_x(\tau > 2n)}$$

$$\sim \frac{\frac{1}{2}(x+1)\left(\sqrt{\frac{a}{b}}\right)^x (y+1)\left(\sqrt{\frac{b}{a}}\right)^y \sqrt{\frac{1}{\pi}\frac{(4ab)^n}{n^{3/2}}}}{\frac{(x+1)}{(2\pi)^{1/2}}\left(\sqrt{\frac{a}{b}}\right)^x \frac{(4ab)^n}{(2n)^{3/2}}\frac{4a}{1-4ab}}$$

$$+ \frac{\frac{1}{2}\frac{xy}{\sqrt{\pi}n^{3/2}}\left(\sqrt{ab}\right)^{2n}\left(\frac{a}{b}\right)^{x/2}\left(\frac{b}{a}\right)^{y/2}}{\frac{(x+1)}{(2\pi)^{1/2}}\left(\sqrt{\frac{a}{b}}\right)^x \frac{(4ab)^n}{(2n)^{3/2}}\frac{4a}{1-4ab}}$$

$$\sim \frac{1-4ab}{a}\left(\frac{1+|y|+\xi y}{2}\right)\left(\sqrt{\frac{b}{a}}\right)^y = (1+\rho)\sigma_{\xi(x)}(y).$$

# Checking the periodic Yaglom limit VI

$$\begin{aligned}
&\frac{K^{2n}(x,-y)}{P_x(\tau > 2n)} \\
&= \quad \frac{1}{2}\frac{(K^{2n}(x,|y|) - {}_{\{0\}}K^{2n}(x,y))}{P_x(\tau > 2n)} \\
&\sim \quad (y+1)\left(\sqrt{\frac{b}{a}}\right)^y \frac{1-4ab}{2a} - \frac{xy}{x+1}\left(\sqrt{\frac{b}{a}}\right)^y \frac{1-4ab}{2a} \\
&= \quad \frac{1-4ab}{a}\left(\frac{1+|y|-\xi y}{2}\right)\left(\sqrt{\frac{b}{a}}\right)^y = (1+\rho)\sigma_{\xi(x)}(-y).
\end{aligned}$$

Finally, for $y = 0$, $K^{2n}(x,0) = P^{2n}_{x+1,1}$ so

$$\begin{aligned}
\frac{K^{2n}(x,0)}{P_x(\tau > 2n)} &= \frac{P^{2n}_{x+1,1}}{P_x(\tau > 2n)} = \frac{(x+1)\left(\sqrt{\frac{a}{b}}\right)^x \sqrt{\frac{1}{\pi}}\frac{(4ab)^n}{n^{3/2}}}{\frac{(x+1)}{(2\pi)^{1/2}}\left(\sqrt{\frac{a}{b}}\right)^x \frac{(4ab)^n}{(2n)^{3/2}}\frac{4a}{1-4ab}} \\
&= \frac{1-4ab}{a} = (1+\rho)\sigma_{\xi(x)}(0).
\end{aligned}$$

# Checking the periodic Yaglom limit VII

- Therefore starting from $x$ even we have a periodic Yaglom limit with density $(1+2\sqrt{ab})\sigma_{\xi}(\cdot)$ on $S_0 = 2\mathbb{Z}$ with $\xi = x/(|x|+1) \in [0,1]$.
- Similarly, for $x, y > 0$ even, $K^{2n}(-x,y) = K^{2n}(x,-y)$ and $K^{2n}(-x,-y) = K^{2n}(x,y)$; hence, starting from $-x$ even we get a Yaglom limit $(1+2\sqrt{ab})\sigma_{\xi}(\cdot)$ on $2\mathbb{Z}$ with $\xi = x/(|x|+1)$ so $\xi \in [-1,0]$.

# Checking the periodic ratio limit

- Again taking $S_0 = 2\mathbb{Z}$,

$$\frac{K^{2n}(y,2\mathbb{Z})}{K^{2n}(x,2\mathbb{Z})} = \frac{P_y(\tau > 2n)}{P_x(\tau > 2n)} \sim \frac{(|y|+1)\left(\sqrt{a/b}\right)^{|y|}}{(|x|+1)\left(\sqrt{a/b}\right)^{|x|}} = \frac{h_0(y)}{h_0(x)}$$

- In fact $h_0$ is the unique $\rho$-harmonic function for $Q$
- in the family of $\rho$-harmonic functions for $K$

$$h_\xi(y) := [1 + |y| + \xi y]\left(\sqrt{\frac{a}{b}}\right)^{|y|} \quad \text{for } y \in \mathbb{Z}. \tag{6}$$

# Checking the periodic Yaglom limit VIII

- Applying Proposition 2, starting from $u$ odd we have a periodic Yaglom limit on the evens with density $(1+2\sqrt{ab})\sigma_{\xi(u)}(\cdot)$ on $S_0 = 2\mathbb{Z}$ with $\xi = u/(|u|+1) \in [0,1]$.
- Similarly, starting from $u$ odd we have a periodic Yaglom limit on the odds: $\frac{1+2\sqrt{ab}}{2\sqrt{ab}}\sigma_{\xi(u)}(\cdot)$

# Cone of $\rho$-invariant probabilities

- The probabilities $\sigma_\xi$ with $\xi \in [-1, 1]$ form a cone.
- The extremal elements are $\xi = -1$ and $\xi = 1$ since

$$\sigma_\xi(y) = \frac{1+\xi}{2}\sigma_1(y) + \frac{1-\xi}{2}\sigma_{-1}(y).$$

- Define the potential $G(x, y) = \sum_{n=0}^{\infty} R^n K^n(x, y)$ and
- the $\rho$-Martin kernel $M(y, x) = G(y, x)/G(y, 0)$.
- As a measure in $x$, $M(y, x) \in \mathcal{B}$ are the positive excessive measures of $R \cdot K$ normalized to be $1$ at $x = 0$; i.e. $\mu \geq R\mu K$ if $\mu \in \mathcal{B}$.
- Each point $y \in \mathbb{Z}$ is identified with the measure $M(y, \cdot) \in \mathcal{B}$, which by the Riesz decomposition theorem is extremal in $\mathcal{B}$.

# The $\rho$-Martin entrance boundary

- As $y \to +\infty$, $M(y, \cdot) \to M(+\infty, \cdot) = \sigma_1(\cdot)/\sigma_1(0)$.
- We conclude $+\infty$ is a point in the Martin boundary of $\mathbb{Z}$.
- We have therefore identified $+\infty$ in the Martin boundary with the $\rho$-invariant measure $\sigma_1(\cdot)/\sigma_1(0)$, which is identified with the point $+1$ in the topological boundary of

$$\left\{\xi = \frac{x}{1+|x|} : x \in \mathbb{Z}\right\}.$$

- By a similar argument we see $-\infty$ is also in the Martin boundary of $\mathbb{Z}$.
- As $y \to +\infty$, $M(y, \cdot) \to M(-\infty, \cdot) = \sigma_{-1}(\cdot)/\sigma_{-1}(0)$.
- Again we have identified $-\infty$ in the Martin boundary with the $\rho$-invariant measure $\sigma_{-1}(\cdot)/\sigma_{-1}(0)$ which is identified with the point $-1$ in the topological boundary of $\left\{\xi = \frac{x}{1+|x|} : x \in \mathbb{Z}\right\}$.

# Harry Kesten's example

- Kesten (1995) constructed an amazing example of a sub-Markov chain possessing most every nice property—including having a $\rho$-invariant qsd—that fails to have a Yaglom limit.
- Kesten's example has the same state space and the same structure as ours.
- The only difference is that at any state $x$ there is a probability $r_x$ of holding in state $x$ and probabilities $a(1 - r_x)$ and $b(1 - r_x)$ of moving one step closer or further from zero.
- If $\alpha = a(1 - r_0)$, then our chain is exactly Kesten's chain watched at the times his chain changes state.
- It is pretty clear Harry could have derived our example with a moment's thought, but he focused on the non-existence of Yaglom limits. His example is orders of magnitude more sophisticated and complicated than ours.