Abstract—In order to represent the set of transmitters simultaneously accessing a wireless network using carrier sensing based medium access protocols, one needs tractable point processes satisfying certain exclusion rules. Such exclusion rules forbid the use of Poisson point processes within this context. Matérn point processes were first advocated for representing such protocols because of their exclusion based definition; however, these point processes are known to be rather conservative and to underestimate the density of points. Random Sequential Packing models were shown to better represent these exclusion rules; however, they are known to be less tractable than e.g. Matérn type point processes. The present paper confirms that the point process induced by the Random Sequential Adsorption model, a particular instance of the Random Sequential Packing model, should be used in order to describe such point patterns. It also shows that this point process is in fact as tractable as the Matérn model. We show in particular that he generating functional of this point process is the solution of a differential equation, which is the main new mathematical result of the paper. Using this differential equation, a new heuristic method is proposed, which leads to simple bounds and estimates for several important network performance metrics of such wireless networks. These bounds and estimates are evaluated against Monte Carlo simulation.

I. INTRODUCTION

The modeling of wireless network by means of stochastic geometry has received increasing attention during the last few years, [1]–[6]. Using this method, user locations (base stations, access points, wireless devices, etc.) are modeled as points randomly scattered all over the plane. Such an approach has several advantages over the more conventional one, which models user locations as vertices of a grid. First of all, it allows one to take the irregularity of the wireless node locations into account. This irregularity comes from two main factors: 1) many wireless networks are unplanned (e.g. ad hoc networks, mesh networks, femto base stations in heterogeneous cellular networks) and hence irregular; 2) users in most wireless networks are mobile, with locations which change in an unpredictable way over time. Stochastic geometry models allow one to capture these features by giving each user a random location. Secondly, it allows one to represent aggregated interference. The pervasive nature of radio waves is well known to lead to interference between users. In the grid model, this interference is often simplified to the maximal interferer model, where one assumes that only nearest neighbor nodes interfere with each other. In fact, interference is additive and the contribution of distant users can still be large enough to disrupt one’s transmission. Stochastic geometry modeling describes the aggregated interference from all users in the network as a Shot-Noise process and uses the analytical machinery of Laplace transforms to study this process. For several special cases, [2], [7]–[10], this leads to simple closed form formulas for most network performance metrics of interest. The equivalent formulas for the grid models are considerably more complicated.

Up to now, the most popular choice of point process (p.p.) within the context of stochastic geometry has been the planar Poisson p.p. This point process is such that i) for each bounded set $A$ of the plane, the number of points in $A$ is a Poisson random variable (r.v.) and (ii) for all $n$-tuples of disjoint sets $(A_1, \cdots, A_n)$, the number of points in $A_1, \cdots, A_n$ are independent r.v.s. While these properties strongly facilitate the analysis, it is important to keep in mind that these assumptions imply the following independence property: given that there are $n$ points in $A$, these $n$ points are uniformly and independently located in $A$. However, the controls and the optimizations used in wireless networks lead to constraints on the positions of the active transmitters which are often incompatible with such an independence property. This hence precludes the use of Poisson p.p.s within this context. Consider for instance a mobile ad hoc network (MANET) with Carrier Sensing based medium access protocol (CSMA MANET). In this kind of network, whenever a user wants to transmit, it has to sense the network to guarantee that there is no ongoing transmission nearby. This Carrier Sensing rule forbids the simultaneous access of two nearby transmitters to the shared channel, as this would cause too much interference to each of them.

The Matérn hard-core model [11], which is based on simple exclusion/repulsion rules between points, has been used as an alternative to the Poisson p.p. allowing one to take MAC into account [12]–[16] within this context. However, this approach suffers of two weaknesses. First of all, the distribution of the Shot-Noise of Matérn hard-core model is not known in closed form; this led to the use of a heuristic for evaluating the law the interference created by such a p.p. based on the fact that the moment measure of order 2 of the Matérn hard-core model is known in closed form [4]. The last heuristic will be referred to as the Matérn order 2 heuristic in what follows. Secondly, it was noticed in [17] that the Matérn hard-core model underestimates the density of transmitters in the network, and consequently underestimates the aggregated interference. The Simple Sequential Inhibition (SSI) model, which is a special case of the Random Sequential Adsorption (RSA) model proposed by Rényi [18] and Palasti [19], was
proposed in [17] as a natural model for representing MAC in such networks in place of Matérn hard-core. However, the moment measures of the RSA model (and of the SSI model in particular) are not known in closed form and even the heuristic mentioned above seems intractable for this class of point processes.

In this paper, we revisit the modeling of CSMA MANETs using a RSA model (defined formally below). We study the distribution of this p.p. through its generating functional. In particular, we show that this generating functional is the solution of a differential equation, which is the main new mathematical result of the paper. This differential equation has no closed form solution. However, we show that upper and lower bounds can be derived from it for the generating functional of the RSA model. Finally, we show that the network performance metrics can be estimated in terms of the generating functional alluded to above and that the bounds lead to a new heuristic which will be referred to as the RSA heuristic below. We show that this heuristic provides an estimate for these metrics which is better than the Matérn order 2 heuristic (and any other heuristic we are aware of).

The paper is organized as follows: the stochastic model and key mathematical results are given in Section II. The specific model of the CSMA MANET is described in Section III. Using this model and the mathematical results in Section II, the performance of the CSMA MANET is analyzed in Section IV, where the bounds and the RSA heuristic are discussed. Section V gathers all our numerical and simulation result and Section VI contains our conclusion. The extension of the RSA model to the whole plane is given in Appendix A and the proof of Proposition 1 is given in Appendix B.

II. STOCHASTIC MODEL:

We start with the definition of the RSA model in Subsection II-A. The core mathematical results are gathered in Subsection II-C. For a more complete treatment of the notions introduced in this section, such as generating functionals, Palm distribution, Campbell’s formula, the reader should refer to [4], [20].

A. The RSA Model:

To avoid questions about the definition of the model, we consider it first a finite square \( A \). To further avoid boundary effect, we make this square a torus by identifying the two opposite sides. We then consider a homogeneous Poisson p.p.

\[
\Phi_\lambda = \{x_1, \ldots, x_N\},
\]

on this torus, which is augmented with independently and identically distributed (i.i.d.) time marks

\[
\Theta = \{t_1, \ldots, t_N\}.
\]

The mark \( t_i \) is associated with the point \( x_i \). These time marks are uniformly distributed in \([0, 1]\).

We now introduce the notion of contention between points: if two points contend with each other, they cannot both appear in the RSA model. This contention is represented by the Boolean random variables (r.v.s) \( C_{i,j} \), for all unordered pairs \((i, j)\) in \([1, N]^2\) (i.e. \( C_{i,j} = C_{j,i} \)). Conditionally on a realization of \( \Phi_\lambda \), the r.v.s \( C_{i,j} \) are independent. We will use the following notation: \( P(C_{i,j} = 1) = h(x_i, x_j) \) where \( h \) is a function from \( A \times A \) to \([0, 1]\). We assume that \( h \) is translation and rotation invariant, i.e. \( h(x, y) = h(x + z, y + z) \) and \( h(x, y) = h(S(x), S(y)) \) for all \( z \) and all rotations \( S \). This condition will be used in the discussion about the motion-invariance of the RSA model.

Since the torus is of finite area, \( \Phi_\lambda \) must have almost surely (a.s.) finitely many points \((N < \infty \) a.s.). We can then sort the points of \( \Phi_\lambda \) according to the (increasing order of the) time marks. In particular, we have \( \Phi_\lambda = \{x_{\sigma(1)}, \ldots, x_{\sigma(N)}\} \), where \( \sigma \) is the a.s. unique permutation of \([1, N]\) such that \( t_{\sigma(1)} < \cdots < t_{\sigma(N)} \).

The RSA p.p. \( \mathcal{R}(\Phi_\lambda) \) associated with \( \Phi_\lambda \) and the contention variables \( C \) can be formally defined as the following subset of the points of \( \Phi_\lambda \):

\[
\begin{align*}
x_{\sigma(1)} & \in \mathcal{R}(\Phi_\lambda) \\
x_{\sigma(i)} & \in \mathcal{R}(\Phi_\lambda) \text{ if and only if for all } j \in [1, i-1] \text{ either } C_{\sigma(i), \sigma(j)} = 0 \text{ or } x_{\sigma(j)} \notin \mathcal{R}(\Phi_\lambda).
\end{align*}
\]

An informal rephrasing of this definition can be given when interpreting the time mark as the time when a point arrives. Points are sequentially packed whenever they arrive (hence the Random Sequential Packing name). Upon arrival, a point checks whether it contends with any of the already packed points. If not, this point is packed (added to the set of points already packed); otherwise it is discarded.

All the above can also be rewritten in the following more compact form:

\[
\mathcal{R}(\Phi_\lambda) = \{x_i \in \Phi_\lambda \text{ s.t. for all } x_j \in \Phi_\lambda \text{ with } t_j < t_i \text{ and } C_{i,j} = 1, \text{ we have } x_j \notin \mathcal{R}(\Phi_\lambda)\}.
\]

A snapshot of \( \mathcal{R}(\Phi_\lambda) \) with \( C_{i,j} = 1 \) if \(|x_i - x_j| < 2\) and \( \lambda = .1 \) is given in Fig. 1. For comparison, the same is done for the Matérn p.p. associated with the same Poisson p.p. \( \Phi_\lambda \) (see Subsection II-B for a definition) in the same figure. We can see that the RSA model clearly has more points than the Matérn model.

B. Comparison with the Matérn Model:

For the same triple of \( \Phi_\lambda, \Theta, \{C_i\} \), the Matérn hard-core model is defined as:

\[
\mathcal{M}(\Phi_\lambda) = \{x_j \in \Phi_\lambda \text{s.t. } \forall x_j \in \Phi_\lambda, C_{i,j} = 0 \text{ or } t_i < t_j\}.
\]

From this definition, we can see that the Matérn model packs less points than the RSA model. To be more concrete, consider a simple example where \( C_{i,j} = 1 \) if \(|x_i - x_j| < 2\) and consider three points 1, 2, 3 (Fig. 2). The distance from 1 to 2 and from 2 to 3 is 1.5, the time marks are: \( t_1 = .1, t_2 = .2, t_3 = .3 \). By definition, only 1 is accepted/packed by the Matérn model, while the RSA model accepts/packs both 1 and 3.

In fact, if we say that a ball of radius 1 is located at each point of the Matérn p.p., then no two balls intersect and it is known
that when \( \lambda \) goes to \( \infty \), the space covered by the Matérn model \( \bigcup_{x \in M(\Phi_\lambda)} B(x, 1) \) covers one fourth of the plane [4], which means that there is still a lot of free space to pack points in.

C. Generating Functionals

It is shown in Appendix A that the definition of RSA model can be extended to the whole Euclidean plane. In what follows, since the mathematical results have a simpler form in this case, we present our results on the RSA model on the whole plane. Before giving the mathematical results, we would like to make a remark about the motion-invariance of \( R(\Phi_\lambda) \). A p.p. is stationary if its distribution is invariant under translations; it is isotropic if its distribution is invariant under rotations, and it is motion-invariant if it is both stationary and isotropic. The same definition applies to marked-p.p.s. As the homogeneous Poisson p.p. \( \Phi_\lambda \) equipped with timer marks \( \Theta \) and the contention r.v.s \( \{C_{i,j}\} \) is motion-invariant, and \( R(\Phi_\lambda) \) is deterministically determined by \( \Phi_\lambda, \Theta, \{C_{i,j}\} \), it follows that \( R(\Phi_\lambda) \) is also motion-invariant.

Below, we concentrate on the generating functional of the RSA model, which is defined as:

\[
f(\lambda, v(.)) = \mathbb{E} \left[ \prod_{x \in R(\Phi_\lambda)} v(x) \right],
\]

where \( v(.) \) is a function from \( \mathbb{R} \) to \([0, 1]\) satisfying:

\[
\int_{\mathbb{R}^2} (1 - v(x))dx < \infty.
\]

For a more complete introduction to generating functionals of p.p.s, the readers should refer to [20], [21].

Our first result is that:

**Proposition 1:** For all functions \( v(.) \) satisfying (2), we have:

- \( f(\lambda, v) \) is continuous in \( \lambda \).
- \( f(\lambda, v) \) satisfies the following differential equation:

\[
\frac{df}{d\lambda}(\lambda, v) = - \int_{\mathbb{R}^2} (1 - v(x))f(\lambda, (1 - h(x,.))v(.))dx.
\]

**Proof:** See Appendix B.

Next, we consider the generating functional of the RSA p.p. under its Palm distribution. Loosely speaking, the Palm distribution of a p.p. with respect to (w.r.t) \( x \) is the distribution of this p.p. conditioned on the fact that it has a point at \( x \). The Palm distribution is essential in the analysis of CSMA MANETs, as one has to consider a typical user and use the Palm distribution w.r.t. the location of this user to compute its coverage probability (see Subsection IV-B). Below, we use an estimate for this Palm generating functional which will be shown below to be quite reliable.

If the RSA model has a point at \( x \), it must not contain any point that contends with \( x \). Guided by that, we construct a new p.p. as follows: we consider a Poisson p.p. \( \Phi'_\lambda \) of intensity \( \lambda \), augmented with time marks \( \Theta' \) and contention r.v.s \( \{C_{i,j}\} \) as with \( \Phi_\lambda \). We further equip this p.p. with conditionally independent Boolean marks \( \{C_i\} \) with distribution

\[
P(C_i = 1) = h(x_i, x),
\]

conditionally on \( \Phi'_\lambda \). The meaning of this mark is that \( x_i \) contends with \( x \) iff \( C_i = 1 \). Let \( \Phi'_{\lambda,x} = \{x_i \in \Phi'_\lambda \text{ s.t. } C_i = 0\} \); this is the process of points in \( \Phi'_\lambda \) that do not contend with \( x \). This point process is Poisson but non–homogeneous. We then construct an RSA model based on \( \Phi'_{\lambda,x} \), and call \( R(\Phi'_{\lambda,x}) \) the induced p.p. The Palm distribution w.r.t. \( x \) of the original RSA model is approximated by the distribution of \( R(\Phi'_{\lambda,x}) \).
Let \( P_\lambda \) be this distribution and \( E_x \) be the corresponding expectation, we put:

\[
f_x(\lambda, v) = E_x \left[ \prod_{y \in \mathcal{R}(\Phi_\lambda) \setminus \{x\} } v(x) \right],
\]

with \( v \) satisfying (2). Denote by \( o \) the center of the plane. The following result holds, with a proof similar to that of Proposition 1:

**Proposition 2:** For any \( x \in \mathbb{R}^2 \) and all functions \( v(.) \) satisfying (2), we have:

- \( f_x(\lambda, v) = f_o(\lambda, v_x) \) with \( v_x(y) = v(y-x) \).
- \( f_o(\lambda, v) \) is continuous in \( \lambda \).
- \( f_o(\lambda, v) \) satisfies the following differential equation:

\[
\frac{df_o}{d\lambda}(\lambda, v) = \int_{\mathbb{R}^2} (1 - v(x)) \, f_o(\lambda, (1 - h(x, .))v(.))(1 - h(\alpha, x)) \, dx. 
\]

(4)

The differential equations (3) and (4) are functional differential equation, the solutions of which are not easy to obtain in explicit form. One may try to evaluate \( f(\lambda, v) \) for small values of \( \lambda \) using the following Taylor expansion, see Appendix C:

\[
1 + \sum_{k=1}^{\infty} (-1)^k \frac{\lambda^k}{k!} \int_{(\mathbb{R}^2)^k} \prod_{i=1}^k \left( 1 - v(x_i) \prod_{j=1}^{i-1} (1 - h(x_j, x_i)) \right) dx_1 \cdots dx_k.
\]

However, such an approach has high computational complexity (because of the multiple nested integrals) and is hence numerically impractical.

This explains why the present paper resorts to bounds and heuristics. Let us start with the former:

**Proposition 3:** For any \( x \in \mathbb{R}^2 \) and any \( v \) satisfying (2), we have:

\[
e^{-\lambda \int_{\mathbb{R}^2} (1 - v(x)) \, dx} \leq f(\lambda, v) \leq 1 - \int_{\mathbb{R}^2} (1 - v(x)) \frac{1 - e^{-\lambda \int_{\mathbb{R}^2} (1 - h(x, y))v(y) \, dy}}{\int_{\mathbb{R}^2} (1 - h(x, y))v(y) \, dy} \, dx,
\]

\[
e^{-\lambda \int_{\mathbb{R}^2} (1 - v(x)) (1 - h(x, o)) \, dx} \leq f_o(\lambda, v) \leq 1 - \int_{\mathbb{R}^2} (1 - v(x)) \frac{1 - e^{-\lambda \int_{\mathbb{R}^2} (1 - h(x, y))v(y)(1 - h(\alpha, o)) \, dy}}{\int_{\mathbb{R}^2} (1 - h(x, y))v(y)(1 - h(\alpha, x)) \, dy} (1 - h(o, x)) \, dx.
\]

(5)

(6)

**Proof:** We only give the proof for \( f \); that for \( f_o \) is similar. As \( v(.) \) is always smaller than 1 and as \( \mathcal{R}(\Phi_\lambda) \) is a thinning of \( \Phi_\lambda \), we have:

\[
f(\lambda, v) := E \left[ \prod_{x \in \mathcal{R}(\Phi_\lambda)} v(x) \right] 
\geq E \left[ \prod_{x \in \Phi_\lambda} v(x) \right] = e^{-\lambda \int_{\mathbb{R}^2} (1 - v(x)) \, dx}.
\]

Now, using Proposition 1, we get:

\[
f(\lambda, v) = 1 - \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} f(\tau, (1 - h(x, .))v(.))(1 - v(x)) \, dx \, d\tau \\
\leq 1 - \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} e^{-\tau \int_{\mathbb{R}^2} (1 - (1 - h(x,y))v(y)) \, dy}(1 - v(x)) \, dx \, d\tau \\
= 1 - \int_{\mathbb{R}^2} (1 - v(x)) \frac{1 - e^{-\lambda \int_{\mathbb{R}^2} (1 - (1 - h(x,y))v(y)) \, dy}}{\int_{\mathbb{R}^2} (1 - (1 - h(x,y))v(y)) \, dy} \, dx.
\]

(7)

III. Network Model

In this section, we use the RSA p.p. to model the locations of transmitters in a controlled wireless network. The p.p. \( \Phi_\lambda \) is the set of all potential transmitters in the network, while \( \mathcal{R}(\Phi_\lambda) \) is the set of transmitters who actually transmit. The first question is that of the definition of the contention r.v.s \( \{C_{i,j}\} \). Here we take \( C_{i,j} = 1_{\|y_i - y_j\| > \rho} \) with \( \eta_{i,j} \) the fading variable between transmitter \( i \) and \( j \); we assume these fading variables to be i.i.d. exponential r.v.s with parameter \( \mu \) (Rayleigh fading) for all unordered pairs \( (i, j) \); here \( \alpha \) is the pathloss exponent and \( \rho \) is the detection threshold. The physical meaning of this definition is that if there is an ongoing transmission with unit power from user \( x_i \), the signal sensed by user \( x_j \) has power \( \eta_{i,j} \|x_i - x_j\|^{-\alpha} \). The transmission is detected if this sensed signal power is larger than the sensing threshold \( \rho \). The same thing happens in case the ongoing transmission is at \( x_j \). For this definition of \( \{C_{i,j}\} \), the function \( h \) is

\[
h(x, y) = e^{-\mu \|x-y\|^\alpha}.
\]

One can easily verify that this \( h \) function satisfies the translation and rotation invariance properties.

The three main parameters here are \( \lambda, \alpha \) and \( \rho \), where \( \alpha \) is usually defined by the nature of the network; this leaves only \( \lambda \) and \( \rho \) as tunable parameters.

For the wireless network model, we assume that each user \( x_i \) has an intended receiver \( y_i \), whose location is uniformly distributed on a circle or radius \( r \) centered at \( x_i \). The distance \( r \) is hence the transmission range. One can also consider a more general setting where the transmission range is random. Nevertheless, such a generality adds a lot of complexity to the analysis while giving little insight into the network performance and is not pursued here. For the wireless network model, we adopt the SINR-Signal to Interference and Noise Ratio framework [4], defined as follows:

- The fading from transmitter \( x_i \) to receiver \( y_j \) is \( \kappa_{i,j} \).
  - For all \( i, j \), \( \kappa_{i,j} \) are i.i.d. exponentially distributed with parameter \( \mu \) (Rayleigh fading).
- The SINR of user \( i \) is:

\[
\text{SINR}_i = \frac{\kappa_{i,i} \|y_i - x_i\|^{-\alpha}}{W + \sum_{j \neq i} \kappa_{j,i} \|y_i - x_j\|^{-\alpha}},
\]
where $W$ is the thermal noise power, which is assumed to be independent with everything else; $\alpha$ is the path loss exponent, which is larger than 2. The sum
\[
\sum_{j \neq i} \frac{\kappa_{j,i} |y_i - x_j|^{-\alpha}}{}
\]
which is taken over all points in $\mathcal{R}(\Phi)$ is the Shot-Noise of the RSA p.p. or equivalently the aggregated interference from all other actual transmitters in the network.

- User $i$ successfully transmits to its receiver iff SINR$_i$ is larger than some threshold $T$.

**Remark:** Here we suppose that the transmitters have no information about the interference level at their receivers. Thus, the purpose of Carrier Sensing is to guarantee that other transmitters are far away compared to the transmission range $r$. The sensed signal power is used as a proxy to estimate the distance between other transmitters and one’s receiver. As well known, if extra information is available to the transmitter, Carrier Sensing can be further improved. For example, one can let the receiver sense the network and then tell the transmitter to transmit when it senses no ongoing transmission. Such an improvement can be implemented by hand shaking techniques such as the RTS-CTS, [12].

The model discussed above can also be easily modified to analyze CSMA MANETs with RTS-CTS, [22], with some extra computational details.

**IV. PERFORMANCE ANALYSIS**

**A. Performance metrics:**

For a CSMA like protocol, there are three important performance metrics, namely the medium access probability (MAP), the coverage probability (COP) and the spatial density of throughput (SDT). The MAP is defined as the probability that a typical user in the network gets access to the medium:

\[
p_{\text{MAP}}(\lambda, \rho) = \mathbb{P}_{x(0),\Phi}(x(0) \in \mathcal{R}(\Phi)),
\]

where the subscript $o, \Phi$ means that this is the Palm distribution of the p.p. $\Phi$ w.r.t $o$. Here we take the point $x(0)$ as the center $o$ of the plane.

The COP is defined as the probability that the SINR of a typical receiver is larger than the threshold $T$:

\[
p_{\text{COP}}(\lambda, \rho) = \mathbb{P}_{o}(\text{SINR}(y(0)) > T),
\]

Note that now the Palm distribution is that of $\mathcal{R}(\Phi)$. The SDT is defined as the average number of users per unit of area who (i) gain access to the medium and (ii) have their SINR larger than $T$:

\[
S(\lambda, \rho) = \mathbb{E} \left[ \sum_{x \in \mathcal{R}(\Phi) \cap D} 1_{\text{SINR}_i > T} \right],
\]

where $D$ is a subset of $\mathbb{R}^2$ with unit area. As the p.p. $\mathcal{R}(\Phi)$ is motion invariant, the choice of $D$ is not important in the above definition.

The physical meaning of the above performance metrics is as follows: The MAP measures how often a user can use the wireless medium to transmit its own messages. Generally, for each individual user, the larger the MAP, the better. However, this comes with a trade-off, as more users transmitting concurrently also implies stronger interference, and hence a lesser chance that the intended receiver decodes the message. This negative effect is captured by the COP. The SDT then measures the combined effect of the two phenomena above. It measures in average how many messages can be successfully transmitted per unit area by the network.

**B. Computing Performance metrics:**

**Proposition 4:** The MAP of a typical user in a CSMA network is:

\[
p_{\text{MAP}}(\lambda, \rho) = \frac{1}{\lambda} \int_0^\lambda f(t, 1 - e^{-\mu \rho |t|^\alpha}) dt.
\]

**Proof:** Let $m(\lambda, \rho)$ be the intensity of the (stationary) p.p. $\mathcal{R}(\Phi)$. We have

\[
m(\lambda, \rho)|D| = \mathbb{E}[\mathcal{R}(\Phi)(D)],
\]

with $|.|$ the Lebesgue measure, for all Borel subsets $D$ of $\mathbb{R}^2$. By the stationarity of both $\mathcal{R}(\Phi)$ and $\Phi$, and by Campbell’s formula, we have:

\[
m(\lambda, \rho)|D| = \lambda \mathbb{P}_{x(0),\Phi}(x(0) \in \mathcal{R}(\Phi))|D|,
\]

for all $D$. Thus:

\[
p_{\text{MAP}}(\lambda, \rho) = m(\lambda, \rho)/\lambda.
\]

Now, we compute $m(\lambda, \rho)$. For all $D$, we have:

\[
\mathbb{E}[e^{-s\mathcal{R}(\Phi)(D)}] = \mathbb{E} \left[ \prod_{x \in \mathcal{R}(\Phi)} e^{-s1_{x \in D}} \right] = f(\lambda, e^{-s1_{x \in D}}).
\]

Hence:

\[
\mathbb{E}[\mathcal{R}(\Phi)(D)] = -\frac{d}{ds} \mathbb{E}[e^{-s\mathcal{R}(\Phi)(D)}] \bigg|_{s=0} = -\frac{d}{ds} f(\lambda, e^{-s1_{x \in D}}) \bigg|_{s=0}.
\]

Applying Proposition 1 we have:

\[
f(\lambda, e^{-s1_{x \in D}}) = 1 - \int_0^\lambda \int_{\mathbb{R}^2} (1 - e^{-s1_{x \in D}}) f(t, (1 - h(x,,))e^{-s1_{x \in D}}) dx dt.
\]

Thus:

\[
\mathbb{E}[\mathcal{R}(\Phi)(D)] = \frac{d}{ds} \int_0^\lambda \int_{\mathbb{R}^2} (1 - e^{-s1_{x \in D}}) f(t, (1 - h(x,,))e^{-s1_{x \in D}}) dx dt \bigg|_{s=0}
\]

\[
= \int_0^\lambda \int_{\mathbb{R}^2} \left( \frac{d}{ds} (1 - e^{-s1_{x \in D}}) \right) f(t, (1 - h(x,,))e^{-s1_{x \in D}}) dx dt \bigg|_{s=0}.
\]
As \((1 - e^{-s1_{x \in D}})\) evaluated at \(s = 0\) is 0, we have:

\[
E[\mathcal{R}(\Phi_\lambda)(D)] = \int_0^\lambda \int_{\mathbb{R}^2} \left. \left( \frac{d}{ds} (1 - e^{-s1_{x \in D}}) \right) f(t, (1 - h(x, .))e^{-s1_{x \in D}}) dx dt \right|_{s=0} = \int_0^\lambda \int_D f(t, (1 - h(x, .))) dx dt.
\]

By the stationarity of \(\mathcal{R}(\Phi_\lambda)\), we have that \(f(t, (1 - h(x, .))) = f(t, (1 - h(0, .))) = f(t, (1 - e^{-\mu_p|x|}))\).

So:

\[
E[\mathcal{R}(\Phi_\lambda)(D)] = \int_0^\lambda l(D) f(t, (1 - e^{-\mu_p|x|})) dt.
\]

The proposition then follows directly.

**Proposition 5:** The coverage probability of a typical user is:

\[
p_{\text{COP}}(\lambda, \rho) \approx \mathcal{L}_W(\mu Tr^\alpha) f_o(\lambda, \varpi),
\]

with \(\varpi(x) = \frac{|x - re|^\alpha}{|x - re|^\alpha + Tr^\alpha}\), \(e\) is a unit vector and \(\mathcal{L}\) is the Laplace transform.

**Proof:** We have

\[
p_{\text{COP}}(\lambda, \rho) = \mathbb{P}_0 \left[ \kappa_{0,0} > Tr^\alpha \left( W + \sum_{i \neq 0} \kappa_{i,0} |x_i - y(0)|^{-\alpha} \right) \right] = \mathbb{E}_0 \left[ \exp \left( -\mu Tr^\alpha \left( W + \sum_{i \neq 0} \kappa_{i,0} |x_i - y(0)|^{-\alpha} \right) \right) \right] = \mathbb{E}[\exp(-\mu Tr^\alpha W)] \mathbb{E}_0 \left[ \exp \left( -\mu Tr^\alpha \sum_{i \neq 0} \kappa_{i,0} |x_i - y(0)|^{-\alpha} \right) \right] = \mathcal{L}_W(\mu Tr^\alpha) \mathbb{E}_0 \left[ \prod_{i \neq 0} \exp \left( -\mu Tr^\alpha \kappa_{i,0} |x_i - y(0)|^{-\alpha} \right) \right].
\]

The sum here is taken over all points of \(\mathcal{R}(\Phi_\lambda)\), \(\mathbb{P}_0\) is the Palm distribution w.r.t. \(\mathcal{R}(\Phi_\lambda)\) and \(\mathbb{E}_0\) is the corresponding expectation. As the p.p. \(\mathcal{R}(\Phi_\lambda)\) is motion-invariant, under its Palm distribution, it is isotropic. Thus, without loss of generality we can assume that the receiver \(y_0\) is at position \(re\). Hence,

\[
\mathbb{E}_0 \left[ \prod_{i \neq 0} \exp \left( -\mu Tr^\alpha \kappa_{i,0} |x_i - y(0)|^{-\alpha} \right) \right] = \mathbb{E}_0 \left[ \prod_{i \neq 0} \varpi(x_i) \right] \approx f_o(\lambda, \varpi).
\]

So:

\[
p_{\text{COP}}(\lambda, \rho) \approx \mathcal{L}_W(\mu Tr^\alpha) f_o(\lambda, \varpi).
\]

**Proposition 6:** The spatial density of throughput of a CSMA network is:

\[
S(\lambda, \rho) = \lambda p_{\text{MAP}}(\lambda, \rho) p_{\text{COP}}(\lambda, \rho).
\]

**Proof:** Let \(D\) be a unit area subset of the Euclidean plane. We have by definition and Campbell’s formula:

\[
S(\lambda, \rho) = \mathbb{E} \left[ \sum_{x_i \in \mathcal{R}(\Phi_\lambda) \cap D} 1_{\text{SINR}_i > T} \right] = m(\lambda, \rho) \mathbb{P}_{\text{COP}}(\text{SINR}_0 > T) = \lambda p_{\text{MAP}}(\lambda, \rho) p_{\text{COP}}(\lambda, \rho).
\]

In order to obtain closed form representations of the performance metrics defined above, we use Proposition 3:

**Corollary 1:** The MAP and the COP of a typical user in a CSMA MANET satisfy the following bounds:

\[
1 - e^{-\frac{\lambda^N}{(\mu \rho)^{2/\alpha}}} \leq p_{\text{MAP}}(\lambda, \rho) \leq 1 - \int_0^\lambda \int_{\mathbb{R}^2} e^{-\rho|y|} \left( 1 - e^{-\alpha(\lambda x + y) \omega(x)} \right) dxd\tau \tag{7}
\]

and

\[
\mathcal{L}_W(\mu Tr^\alpha) e^{-\lambda f_o(1 - \varpi(x)) (1 - e^{-\mu_p|x|})} dx \leq p_{\text{COP}}(\lambda, \rho) \leq \mathcal{L}_W(\mu Tr^\alpha) - \mathcal{L}_W(\mu Tr^\alpha) \int_{\mathbb{R}^2} (1 - \varpi(x)) dx - \int_{\mathbb{R}^2} (1 - \varpi(y)) \left( 1 - e^{-\mu_p|y - x|} \right) dy \int_{\mathbb{R}^2} \left( 1 - e^{-\mu_p|y|} \right) dy \left( 1 - e^{-\rho|y|} \right) dy \tag{8}
\]

where

\[
N = \int_{\mathbb{R}^2} e^{-|y|} dy = \frac{2\pi \Gamma(2/\alpha)}{\alpha},
\]

\(\Gamma\) is the Gamma function defined as \(\Gamma(s) = \int_0^\infty e^{-t} t^{s-1} dt\) and:

\[
g(x) = \int_{\mathbb{R}^2} e^{-|y - x|} + |y|^\alpha dy.
\]

**C. Matern Order 2 Heuristic**

For the sake of comparison, we briefly discuss here the Matern order 2 heuristic of [4], [12], [15]. The model considered here is the Matern hard-core model. The main idea consists in approximating the Palm distribution of the Matern model by the distribution of a non-homogeneous Poisson p.p. The intensity measure of the Poisson p.p. is chosen in such a way to conserve the pair-correlation function of the induced p.p. All the results in this subsection are stated without proof. Further details on the method can be found in [4].

- Pair-correlation function: this is a function from \(\mathbb{R}^2\) to \([0,1]\). Loosely speaking, it is the probability that we can find a point at \(x\) conditioned on the fact that a point has
been found at \( o \). For a more formal setting, the reader should refer to [20].

The key ingredient for this analysis is the following result:

**Proposition 7:** Conditionally on the fact that \( \Phi_x \) contains two points \( o \) and \( x \), and that \( o \in \mathcal{M}(\Phi_x) \), the probability \( h(x, \lambda, \rho) \) that \( x \in \mathcal{M}(\Phi_x) \) is:

\[
h(x, \lambda, \rho) = \frac{2(\mu \rho)^{2/\alpha}}{N-g(x)} \left( 1 - e^{-\lambda \frac{N}{(\mu \rho)^{2/\alpha}}} - \frac{1}{\lambda} \frac{2 - e^{-\lambda_2 \frac{N}{(\mu \rho)^{2/\alpha}}}}{2 - e^{-\lambda_2 \frac{N}{(\mu \rho)^{2/\alpha}}}} \right),
\]

with \( N \) and \( g \) defined in Corollary 1.

**Remark:** As \( \lim_{x \to \infty} g(x) = 0 \), we have

\[
\lim_{x \to \infty} h(x, \lambda, \rho) = \frac{1 - e^{-\lambda \frac{N}{(\mu \rho)^{2/\alpha}}}}{\lambda \frac{N}{(\mu \rho)^{2/\alpha}}}.
\]

This corresponds to the fact that, as \( x \) moves far away from \( o \), there is less and less correlation between the two events \( o \in \mathcal{R}(\Phi_x) \) and \( x \in \mathcal{R}(\Phi_x) \).

If we approximate the Palm distribution w.r.t. \( o \) of the locations of users in a CSMA MANET by that of an inhomogeneous Poisson p.p. of intensity measure \( \lambda h(x, \lambda, \rho) \), the three performance metrics can be estimated as follows [4]:

- For the MAP

\[
p_{MAP}^{heur}(\lambda, \rho) = 1 - e^{-\lambda \frac{N}{\mu \rho^{2/\alpha}}}.
\]

**Remark:** The MAP computed here is the exact value, not approximation. We can see that it is exactly the same as the lowerbound for the MAP of the RSA model. This is quite expected since for the same system \( \Phi_x, \Theta, \{C_{i,j}\} \), the p.p. induced by Matérn model is always dominated by the p.p. induced by the RSA model. This again justifies the claim that the Matérn model underestimates the process of users concurrently accessing the shared medium.

- For the COP

\[
p_{COP}^{heur}(\lambda, \rho) = e^{-\lambda \int_{\mathbb{R}^2} (1 - \bar{w}(x)) h(x, \lambda, \rho) dx}.
\]

- For the SDT

\[
S^{heur}(\lambda, \rho) = \lambda p_{MAP}^{heur}(\lambda, \rho) p_{COP}^{heur}(\lambda, \rho).
\]

V. NUMERICAL AND SIMULATION RESULTS

In this section, we study the CSMA MANET by means of Monte Carlo simulation. Then we compare the results obtained by this simulation with those obtained by numerically evaluating the analytical bounds in Section IV.

In the simulation, we consider a network with the following parameters: the sensing threshold \( \rho \) is .2, the parameter \( \mu \) of fading is 1, the pathloss exponent \( \alpha \) is 3, the transmission range is \( r = 1 \) and \( \lambda \) varies from 0 to .5. The intensity of the RSA model, the COP and the SDT are computed by taking averages over 2000 samples of such a network. For simplicity we assume that there is no thermal noise. Fig. 3 plots the results obtained for the MAP.

For the COP, the same plot is proposed in Fig. 4 for \( \lambda \) varying from 0 to .5. This time, we also compare with the formulas obtained by the Matérn order 2 heuristic. Note that we did not do the same comparison for the intensity as the intensity of the p.p. induced by the Matérn model is exactly equal to the analytical lower bound for the intensity of the p.p. induced by RSA model.
We observe that the lower bound is a better estimate for the MAP while the upper bound is the better estimate for the COP. For this reason, we use the product of these two bounds as an estimate of the SDT. This is our RSA heuristic. A plot of this heuristic is given in Fig. 5. As we did for the COP, we also compare this result with that obtained by the formula of Subsection IV-C, namely the Matérn order 2 heuristic. We see that our RSA heuristic follows closely the simulation result, while the Matérn order 2 heuristic underestimates the simulation result.

In wireless networks, the parameter $\lambda$ is often given and cannot be altered. The network operator can only fine-tune its performance by playing with $\rho$. So our second task is to study how the metrics $p_{\text{COP}}$ and SDT vary with $\rho$. To this end we consider a RSA network with parameters $\lambda = .5$, transmission range $r = 1$ and pathloss exponent $\alpha = 3$. Fig. 6 plots $p_{\text{MAP}}$. Fig. 7 plots the value of the COP for $\rho$ varying from .02 to .2. We notice the well expected effect that the COP decreases as $\rho$ increases. This is because increasing $\rho$ makes Carrier Sensing less sensitive and reduces the level of protection given to each user.

Fig. 8 plots the SDT for the same range of values of $\rho$. The estimate used here is the one mentioned above, i.e. the product of the lower bound MAP and the upper bound for the COP. The optimal value of $\rho$ given by the simulation is about .12 while that value given by the RSA estimate is about .2. We can see that there is a discrepancy between the result obtained by simulation and the one obtained by estimation. Note however that the discrepancy with the Matérn order 2 heuristic is significantly larger.

Due to space constrain, we cannot go further in studying the full range of parameters. From the available result, we conclude that the RSA heuristic improves significantly on the Matérn order 2 heuristic, and in some case provides a quite good estimate.

VI. CONCLUSION:

The first achievement of this paper is to show the mathematical tractability of RSA model, a stochastic model which can be used to exactly describe the locations of users in a MANET using Carrier Sensing based medium access protocols. The generating functional of the RSA p.p. is shown to be the
solution of a differential equation. In order to cope with the computational complexity of the numerical evaluation of the solution of this differential equation, new bounds and heuristic are also proposed for the most important performance metrics in such networks. They open new avenues to fine-tune key network parameters such as the intensity \( \lambda \) of user and the pathloss exponent is \( \alpha = 3 \).

**APPENDIX A**

**RSA MODEL FOR EUCLIDEAN PLANE:**

For the extension to the whole plane of RSA model, it is useful to adopt another definition. In this definition, we assign each point \( x_i \) in \( \Phi_\lambda \) a Boolean variable \( e_i \). Then we consider the following system of equations:

\[
e_i = \prod_{j \neq i} (1 - e_j) \quad \forall \ i,
\]

with the convention that a product taken over an empty set is equal to 1.

When \( \Phi_\lambda \) has finitely many points (as in the finite window case), (9) is a finite system of equations of \( N \) equations and \( N \) variables with \( N \) is the number of points in \( \Phi_\lambda \). Recall that \( \sigma \) is the permutation such that \( t_{\sigma(1)} \leq t_{\sigma(2)} \leq \cdots \leq t_{\sigma(N)} \). We can then prove by induction on \( i \) that \( e_{\sigma(i)} = 1 \) if \( e_{\sigma(i)} \in \mathcal{R}(\Phi_\lambda) \). This is the unique solution of (9).

This motivates us to define the same system of equations for the case \( \Phi_\lambda \) is a homogeneous Poisson p.p. of intensity \( \lambda \) in the whole plane, and to define the p.p. induced by RSA model \( \mathcal{R}(\Phi_\lambda) \) by:

\[
\mathcal{R}(\Phi_\lambda) = \{ x_i \in \Phi_\lambda \ s.t. \ e_i = 1 \}.
\]

Now, the problem is that the system of equations at hand has infinitely many equations and infinitely many variables. Each equation itself involves a product of possibly infinitely many terms. This, of course, raises the question about the existence and the uniqueness of the solution. Answering the question
above is the subject of this Section.
We first introduce the notion of conflict graph which will play a central role in the forthcoming proofs. Given a p.p. \( \Phi \), equipped with time marks \( \Theta \) and conflict r.v.s \( \{C_{i,j}\} \), we define the associated conflict graph \( \mathcal{G} = (\Phi, \mathcal{E}) \), where
\[
\mathcal{E} = \{(x_i, x_j) \in \Phi^2 \text{ s.t. } t(j) < t(i), C_{i,j} = 1\}.
\]
That is, in the conflict graph, we put an edge from \( x_i \) to \( x_j \) iff \( x_i \) and \( x_j \) contend and the time mark \( t_i \) is larger than \( t_j \).
It is easy to check that \( \mathcal{G} \) is acyclic.
We now introduce the notion of conflict indicator function of a graph. To this end we consider an arbitrary acyclic directed graph \( H \) and a vertex \( o \) in it. Let \( K_H(o) \) be the positive connected component of \( o \), i.e. the set of point \( x \) such that there is a directed path in \( H \) from \( o \) to \( x \). Note that \( o \) is always contained in \( K_H(o) \) by convention. We say that \( H \) has no infinite positive percolation iff, for all \( x \) in \( \Phi \), we have \( |K_H(x)| < \infty \).
In the following, by abuse of notation, we use the same notation for a set of vertex and the subgraph induced by that set. The conflict indicator function \( e(H, o) \), which takes values in \( \{0, 1\} \), is recursively defined as:
\[
e(H, o) :=
\begin{cases}
1 & \text{if } H = (\{o\}, \emptyset) \\
\prod_{(o,x)} \text{ is an edge of } H(1 - e(K_H(x), x)) & \text{otherwise}.
\end{cases}
\]
(10)
One can easily see that when \( |K_H(o)| < \infty \), \( e(H, 0) \) is well-defined and equal to \( e(K_H(o), o) \). In what follows we will need the following lemmas in which we assume that the condition
\[
\int_{\mathbb{R}^2} h(0, x) dx = M < \infty,
\]
is satisfied:

Lemma 1: For an homogeneous Poisson p.p. \( \Phi \) of intensity \( \lambda < M^{-1} \), the conflict graph \( \mathcal{G} \) has no infinite positive percolation a.s.

Proof: For this proof we introduce the notion of infinite percolation. In \( \mathcal{G} \), we remove the direction of the edges and let \( S_i \) be the connected component of the resulting graph containing \( x_i \). We say that the graph \( \mathcal{G} \) has no infinite percolation iff for all \( i \), \( |S_i| < \infty \). Since \( K_{\mathcal{G}}(x_i) \subseteq S_i \) for all \( i \), if \( \mathcal{G} \) has no infinite percolation then it has no positive infinite percolation.

Now we will show that \( \mathcal{G} \) has no infinite percolation a.s. For this, it is sufficient to prove that for each \( i \), \( |S_i| < \infty \) a.s. (note that since there are countably many points in \( \Phi \), the assertion above is equivalent to: \( |S_i| < \infty \) \forall i \) a.s.). Now, as \( \Phi \) is stationary, we can consider without loss of generality the point \( x_0 \) located at the center \( o \) of the plane:
\[
\mathbb{E}[|S_0|] \leq 1 + \sum_{n=1}^{\infty} \mathbb{E}\left[ \sum_j \mathbf{1}_{\text{there is a path of length } n \text{ from } x_0 \text{ to } x_j} \right] 
\]
Note that:
\[
\begin{align*}
\mathbb{E}\left[ \sum_j \mathbf{1}_{\text{there is a path of length } n \text{ from } x_0 \text{ to } x_j} \right] &\leq \mathbb{E}[\text{number of not self-intersecting paths of length } n \\
&\text{starting from } x_0] \\
&= \mathbb{E}\left[ \sum_{j_1, j_2, \ldots, j_n \text{ pairwise different}} \mathbf{1}_{C_{0,j_1}=1, C_{j_1,j_2}=1, \ldots, C_{j_{n-1},j_n}=1} \right].
\end{align*}
\]
Using the independence of \( C_{0,j_1}, C_{j_1,j_2}, \ldots, C_{j_{n-1},j_n} \) and the reduced Campbell formula, we have:
\[
\begin{align*}
\sum_{i=1}^{n} \mathbb{E}\left[ \sum_{j_1, j_2, \ldots, j_n \text{ pairwise different}} \mathbf{1}_{C_{0,j_1}=1, C_{j_1,j_2}=1, \ldots, C_{j_{n-1},j_n}=1} \right] &\leq \mathbb{E}\left[ \sum_{n=1}^{\infty} \int_{(\mathbb{R}^2)^n} h(o, y_1) h(y_1, y_2) \cdots h(y_{n-1}, y_n) \right] \\
&= \sum_{n=1}^{\infty} \int_{(\mathbb{R}^2)^n} h(0, y_1) h(0, y_2) \cdots h(0, y_{n-1}) \lambda^n dy_1 dy_2 \cdots dy_n \\
&= 1 + \sum_{n=1}^{\infty} \int_{(\mathbb{R}^2)^n} h(0, y_1) h(0, y_2) \cdots h(0, y_n) \lambda^n dy_1 dy_2 \cdots dy_n \\
&= 1 + \sum_{n=1}^{\infty} \lambda^n M^n = (1 - \lambda M)^{-1} < \infty.
\end{align*}
\]
Thus \( |S_0| < \infty \) a.s. and this proves our lemma.

Lemma 2: For an homogeneous Poisson p.p. \( \Phi \) of intensity \( \lambda < \infty \), the conflict graph \( \mathcal{G} \) has no infinite positive percolation.

Proof: We prove this lemma by induction on the intensity \( \lambda \). Fix a positive \( \lambda_0 < M^{-1} \). If \( \lambda < \lambda_0 \), then its conflict graph does not have infinite positive percolation thanks to Lemma 1. Now let \( n \) be a positive integer and suppose that for any homogeneous Poisson p.p. with intensity smaller than \( n\lambda_0 \), its conflict graph does not have infinite positive percolation.

Consider an homogeneous Poisson p.p. \( \Phi \) with intensity \( \lambda \) such that \( n\lambda_0 \leq \lambda < (n + 1)\lambda_0 \). We will need to prove that \( |K_{\mathcal{G}}(x_i)| < \infty \) a.s. for any \( i \).
We decompose \( \Phi \) into \( \Phi_1 = \{x_i \text{ s.t. } t_i < 1/2\} \) and \( \Phi_2 = \Phi \setminus \Phi_1 = \{x_i \text{ s.t. } t_i \geq 1/2\} \). The conflict graph \( \mathcal{G} \) is in turn decomposed into 3 parts: the subgraph \( \mathcal{G}_1 = \{\Phi_1, \mathcal{E}_1\} \) induced by \( \Phi_1 \), the subgraph \( \mathcal{G}_2 = \{\Phi_2, \mathcal{E}_2\} \) induced by \( \Phi_2 \), and the set of edges \( \mathcal{E}_3 = \mathcal{E} \setminus (\mathcal{E}_1 \cup \mathcal{E}_2) \).
For each \( x_i \) in \( \Phi \), consider first the case \( t_i < 1/2 \). Note that, when we start from a point, by following edges in \( \mathcal{E} \) we can only go to points which have smaller time marks than the starting point. Hence, we can easily see that \( K_{\mathcal{G}_1}(x_i) = \)
From these, we can conclude that \( t \) a.s. So, the union above involve only a finite number of terms. there is only a finite number of \( \lambda / G \) with \( G \) must have \( \equiv \) K for the case \( x_i \) and consequently, \( |K(x_j)| < \infty \). Thus, \( \{x_i, x_j\} \in E \) with \( x_j \in \text{K}'(x_i); \) and a path from \( x_j \) to \( x_j \) following edges in \( E_1 \). Thus, \( \text{K}'(x_i) \) can be rewritten as:

\[
K'(x_i) = \bigcup_{x_j \in K_G(x_i)} \left( \bigcup_{x_j, x_j \in E_3} K_G(x_j) \right).
\]

As \( \mathcal{G}_2 \) is the conflict graph of the p.p. \( \Phi_2 \), which is a Poisson p.p. of intensity \( \lambda / 2 < n \lambda_0 \), by induction hypothesis, \( |K_{G_2}(x_i)| < \infty \) a.s. The condition (11), implies that each vertex in \( \mathcal{G} \) has finite outdegree a.s., and hence for each \( x_j \), there is only a finite number of \( x_j \) such that \( (x_j, x_j) \in E_3 \) a.s. So, the union above involve only a finite number of terms. Each term \( K_G(x_j) \) corresponds to a point with time mark \( t_j < 1 / 2 \) and is hence a finite set. Thus, \( |K'(x_i)| < \infty \) a.s. Consequently, \( |K_G(x_i)| = |K_{G_2}(x_i)| + |K'(x_i)| < \infty \) a.s.

From these, we can conclude that \( \mathcal{G} \) has no infinite positive percolation.

The connection between the lemmas above and the well-defineness of RSA model lies in the following lemma:

**Lemma 3:** If \( \mathcal{G} \) has no infinite positive percolation, \( e_i = e(K_G(x_i), x_i) \), where \( e(.) \) is the thinning indicator function defined in (10), is the only solution of Equation (9).

**Proof:** Since \( \mathcal{G} \) has no infinite positive percolation, \( e(K_G(x_i), x_i) \) is well defined for all \( i \). Since

\[
K_G(x_i) = x_i \cup \bigcup_{x_j, x_j \in E} K_G(x_j),
\]

then by definition we have that \( e_i = e(K_G(x_i), x_i) \) satisfies Equation (9).

For uniqueness, let \( \{e_i\} \) be a solution of (9). We prove by induction on the size of \( K_G(x_i) \) that \( e_i = e(K_G(x_i), x_i) \).

For any \( x_i \) such that \( |K_G(x_i)| = 1 \), i.e. \( K_G(x_i) = \{x_i\} \) or equivalently, \( x_i \) has out degree 0 in the conflict graph, we must have \( e_i = e(K_G(x_i), x_i) = 1 \). Note that such \( x_i \) always exists by the assumption that \( \mathcal{G} \) has no infinite percolation.

Now suppose that for all \( x_i \) such that \( |K_G(x_i)| < n \), we have \( e_j = e(K_G(x_j), x_j) \) and let us consider an \( x_j \) such that \( |K_G(x_j)| = n \) (if there are any). For \( x_j \) such that \( (x_i, x_j) \in E \), we have \( |K_G(x_j)| < n \). Thus, \( e_j = e(K_G(x_j), x_j) \). Then:\

\[
e_i = \prod_{j \; s.t. \; t_j < t_i} (1 - e_j C_{i,j}) = \prod_{j \; s.t. \; t_j < t_i, C_{i,j} = 1} (1 - e_j) = \prod_{j \; s.t. \; (x_i, x_j) \in E} (1 - e_j) = e(K_G(x_i), x_i).
\]

Thus, for all \( x_i \) such that \( |K_G(x_i)| < n + 1 \) we have

\[
e_i = e(K_G(x_i), x_i).
\]

This completes the proof of uniqueness.

We conclude the three lemmas above in the following proposition:

**Proposition 8:** If the function \( h \) satisfies (11) then the RSA model of the homogeneous Poisson point process on the whole plane is well-defined in the sense that the solution \( \{e_i\} \) of (9) is well defined. It is easy to verify that \( \int_{\mathbb{R}^2} e^{-\mu p|x|^\alpha} = \frac{N}{(\mu p)^{\alpha/\alpha}} < \infty \) (\( N \) is defined in Corollary 1). Hence, it is justifiable to use the RSA model extended to the whole plane to model the location of users in a controlled wireless network.

**APPENDIX B**

**PROOF OF PROPOSITIONS 1 AND 2:**

In this Section, we provide the result that lies in the core of the proofs of Proposition 1 and 2. For this, we consider a Poisson p.p. \( \Phi \) in \( \mathbb{R}^2 \times \mathbb{R}^+ \). The third dimension here plays the role of the time marks. For this reason, the points in \( \Phi \) are called \( (x_i, t_i) \) with \( t_i \) be the position in the plane and \( t_i \) be the “time mark”. The intensity measure of \( \Phi \) is such that \( E[\Phi([0, \lambda] \times B)] = \lambda \lambda(B) \) for any positive \( \lambda \) and measurable subset \( B \) of \( \mathbb{R}^2 \), with \( \Lambda \) is a measure of \( \mathbb{R}^2 \) dominated by the Lebesgue measure \( l \). The p.p. \( \Phi \) is further equipped with the contention r.v.s \( \{C_{i,j}\} \) as in Section II.

Next, we will need the notion of time based thinning. Given any p.p. \( \Xi \) in \( \mathbb{R}^2 \times \mathbb{R}^+ \) and any time interval \( A \), the time based thinning of \( \Xi \) into \( A \) is:

\[
T_A(\Xi) = \{(x, t) \in \Xi \; s.t. \; t \in A\}.
\]

Moreover, whenever the p.p. \( \Xi \) can be considered as a p.p. in \( \mathbb{R}^2 \) equipped with time marks in \( \mathbb{R}^+ \) (this does not hold in general, see [20], [21] for more details), the notation \( x \in \Xi \) should be understood that \( " \)there exists a \( t \) such that \( (x, t) \in \Xi \)."

Now, come back to the p.p. \( \Phi \), given that the condition (11) is satisfied, for each \( \lambda \), we view the p.p. \( T_{[0, \lambda]}(\Phi) \) as a p.p. in \( \mathbb{R}^2 \) with uniform time marks in \([0, \lambda]\). As \( \Lambda \) is dominated by \( l \), \( T_{[0, \lambda]}(\Phi) \) is dominated by \( \Phi_\lambda \). Hence the conflict graph of \( T_{[0, \lambda]}(\Phi) \) is a subgraph of that of \( \Phi_\lambda \). For this reason, the conflict graph of \( T_{[0, \lambda]}(\Phi) \) has no infinite positive
percolation and the RSA model is well defined for this p.p. The p.p. \( \mathcal{R}(T_{[0,\lambda]}(\Phi)) \) induced by the RSA model can also be considered as a time marked p.p. by letting each point in \( \mathcal{R}(T_{[0,\lambda]}(\Phi)) \) inherit its time mark in \( \Phi \).

For each function \( v \) satisfying (2), we put:

\[
f_A(t, v) = \mathbb{E} \left[ \prod_{x \in \mathcal{R}(T_{[0,\lambda]}(\Phi))} v(x) \right].
\]

Now is the time to state the main result:

**Proposition 9:** For any function \( v \) satisfying (2), the function \( f_A(t, \cdot) \) is continuous and satisfies the following equation:

\[
\frac{df}{dt}(t,v) = -\int_{\mathbb{R}^2} (1 - v(x)) f(t,1 - h(x,\cdot)) v(\cdot) \Lambda(dx).
\]

(12)

We begin by proving that \( f_A(\cdot,v) \) is continuous

**Proof:** For all \( t_1 < t_2 \):

\[
T_{[0,t_2]}(\mathcal{R}(T_{[0,t_2]}(\Phi))) = \mathcal{R}(T_{[0,t_1]}(\Phi)).
\]

Thus, for all \( t \) positive and \( \epsilon \) positive:

\[
f_A(t + \epsilon, v) = \mathbb{E} \left[ \prod_{x \in \mathcal{R}(T_{[0,t+\epsilon]}(\Phi))} v(x) \right] = \mathbb{E} \left[ \prod_{x \in \mathcal{R}(T_{[0,t]}(\Phi))} \left( \prod_{x \in \mathcal{R}(T_{[0,t+\epsilon]}(\Phi))} v(x) \right) \right].
\]

Since \( \mathcal{R}(T_{[0,t+\epsilon]}(\Phi)) \) is dominated by \( T_{[0,t+\epsilon]}(\Phi) \), which is a Poisson p.p. of intensity \( \epsilon \Lambda \). Since \( v(x) \leq 1 \ \forall x \in \mathbb{R}^2 \), we have a.s.

\[
1 \geq \prod_{x \in \mathcal{R}(T_{[0,t+\epsilon]}(\Phi))} v(x) = \prod_{x \in T_{[0,t]}(\Phi)} v(x).
\]

Thus:

\[
f_A(t, v) = \mathbb{E} \left[ \prod_{x \in \mathcal{R}(T_{[0,t]}(\Phi))} v(x) \right] = \mathbb{E} \left[ \prod_{x \in \mathcal{R}(T_{[0,t+\epsilon]}(\Phi))} \prod_{x \in T_{[0,t]}(\Phi)} v(x) \right].
\]

Combining these inequalities we get:

\[
f_A(t - \epsilon, v) \exp \left\{ \epsilon \int_{\mathbb{R}^2} (1 - v(x)) \Lambda(dx) \right\} \geq f_A(t, v) \geq f_A(t + \epsilon, v) \exp \left\{ -\epsilon \int_{\mathbb{R}^2} (1 - v(x)) \Lambda(dx) \right\}.
\]

Letting \( \epsilon \) go to 0 completes our proof.

For the proof of the differential equation, we will need the following lemma, which shows that when we restrict to a very thin layer, the p.p. induced by RSA model behaves as a Poisson p.p.

**Lemma 4:** For any measure \( \Lambda \) dominated by the Lebesgue measure, we have:

\[
- \int_{\mathbb{R}^2} (1 - v(x)) \Lambda(dx) \leq \frac{f_A(\epsilon, v) - 1}{\epsilon} \leq \frac{1}{\epsilon} \left( \int_{\mathbb{R}^2} (1 - v(x)) \Lambda(dx) \right)^2.
\]

In particular, we have:

\[
\lim_{\epsilon \to 0} \frac{f_A(\epsilon, v) - 1}{\epsilon} = -\int_{\mathbb{R}^2} (1 - v(x)) \Lambda(dx).
\]

**Proof:** The first inequality is quite simple. We have a.s.:

\[
\prod_{x \in \mathcal{R}(T_{[0,\epsilon]}(\Phi))} v(x) \geq 1 - \sum_{x \in \mathcal{R}(T_{[0,\epsilon]}(\Phi))} (1 - v(x)).
\]

Since \( - v(x) \) is positive for all \( x \) and \( \mathcal{R}(T_{[0,\epsilon]}(\Phi)) \) is dominated by \( T_{[0,\epsilon]}(\Phi) \), then:

\[
\prod_{x \in \mathcal{R}(T_{[0,\epsilon]}(\Phi))} v(x) \geq 1 - \sum_{x \in T_{[0,\epsilon]}(\Phi)} (1 - v(x)) \text{ a.s.}
\]

Hence:

\[
\frac{f_A(\epsilon, v) - 1}{\epsilon} = \frac{\mathbb{E} \left[ \prod_{x \in \mathcal{R}(T_{[0,\epsilon]}(\Phi))} v(x) \right] - 1}{\epsilon} \geq -\mathbb{E} \left[ \sum_{x \in \mathcal{R}(T_{[0,\epsilon]}(\Phi))(1 - v(x)) \right] = -\int_{\mathbb{R}^2} (1 - v(x)) \Lambda(dx).
\]

For the second inequality, we first observe that:

\[
\prod_{x \in \mathcal{R}(T_{[0,\epsilon]}(\Phi))} v(x) \leq 1 - \sum_{x \in \mathcal{R}(T_{[0,\epsilon]}(\Phi))} (1 - v(x)) + \sum_{x \neq y \in \mathcal{R}(T_{[0,\epsilon]}(\Phi))} (1 - v(x))(1 - v(y)) \text{ a.s.}
\]

Consider the p.p.

\[
\Delta := \{ x \in T_{[0,\epsilon]}(\Phi) \text{ s.t. } \forall y \neq x \in T_{[0,\epsilon]}(\Phi), C(x, y) = 0 \}.
\]

It is easily seen that \( \mathcal{R}(T_{[0,\epsilon]}(\Phi)) \) dominates \( \Delta \). Hence:

\[
\sum_{x \in \mathcal{R}(T_{[0,\epsilon]}(\Phi))} (1 - v(x)) \geq \sum_{x \in \Delta} (1 - v(x)) \text{ a.s.}
\]
Now using the fact that $\mathcal{R}(T_{0,\epsilon})(\Phi)$ is dominated by $T_{0,\epsilon}(\Phi)$, we have:

$$
\sum_{x \not\in \mathcal{R}(T_{0,\epsilon})} (1 - v(x))(1 - v(y)) \\
\leq \sum_{x \not\in \mathcal{R}(T_{0,\epsilon})} (1 - v(x))(1 - v(y)) \text{ a.s.}
$$

Note that the intensity measure of $\Delta$ can be computed as $\epsilon e^{-\epsilon \int_{\mathbb{R}^2} h(x,y)\Lambda(dy)\Lambda(dx)}$, and

$$
\int_{\mathbb{R}^2} h(x,y)\Lambda(dy) \leq \int_{\mathbb{R}^2} h(x,y)dy = M,
$$

we have then:

$$
\mathbb{E} \left[ \sum_{x \in \mathcal{R}(T_{0,\epsilon})} (1 - v(x)) \right] \\
\geq \mathbb{E} \left[ \sum_{x \not\in \mathcal{R}(T_{0,\epsilon})} (1 - v(x)) \right] \geq \epsilon \int_{\mathbb{R}^2} (1 - v(x))e^{-\epsilon M} \Lambda(dx),
$$

$$
\mathbb{E} \left[ \sum_{x \not\in \mathcal{R}(T_{0,\epsilon})} (1 - v(x))(1 - v(y)) \right] \\
\leq \mathbb{E} \left[ \sum_{x \not\in \mathcal{R}(T_{0,\epsilon})} (1 - v(x))(1 - v(y)) \right] \\
= \epsilon^2 \left( \int_{\mathbb{R}^2} (1 - v(x))\Lambda(dx) \right)^2.
$$

Then proceeding as in the first inequality, we obtain the second inequality. The second assertion follows directly. □

Now we can prove the differential equation (12).

**Proof:** In particular, we need to prove that:

$$
\lim_{\epsilon \to 0} \frac{f_A(t + \epsilon, v) - f_A(t, v)}{\epsilon} = \lim_{\epsilon \to 0} \frac{f_A(t, v) - f_A(t - \epsilon, v)}{\epsilon} = -\int_{\mathbb{R}^2} f_A(\lambda, (1 - h(x, .))v(.))(1 - v(x))\Lambda(dx), \quad (13)
$$

for any $\lambda$ positive.

Note that for $t > s$:

$$
f_A(t, v) - f_A(s, v) = \mathbb{E} \left[ \prod_{x \in \mathcal{R}(T_{0,\epsilon})} v(x) \left( \prod_{y \in \mathcal{R}(T_{0,\epsilon})} v(y) - 1 \right) \right].
$$

In order to evaluate the last expression, we need the following conditional probability:

$$
\mathbb{E} \left[ \prod_{y \in \mathcal{R}(T_{0,\epsilon})} v(y) \middle| T_{0,\epsilon}(\Phi) \right].
$$

Let $A$ be any countable set of points such that $\prod_{x \in A}(1 - h(z, x))$ is well defined for all $x$ in $\mathbb{R}^2$. We define the $A$ constrained measure $\Lambda_A$, which is the unique measure such that $\Lambda_A(dz) = \prod_{x \in A}(1 - h(z, x))\Lambda(dz)$. This is the intensity of the process of points which do not contend with any point in $A$, belonging to a Poisson p.p. of intensity $\Lambda$. Note that $\mathcal{R}(T_{0,\epsilon})(\Phi)$ is always such that $\prod_{x \in \mathcal{R}(T_{0,\epsilon})(\Phi)}(1 - h(z, x))$ is well defined for all $x$ in $\mathbb{R}^2$. We have:

$$
\mathbb{E} \left[ \prod_{y \in \mathcal{R}(T_{0,\epsilon})} v(y) \middle| T_{0,\epsilon}(\Phi) \right] = f_A(\mathcal{R}(T_{0,\epsilon})(\Phi))(s - t, v).
$$

Put $s = t$ and $t = t + \epsilon$. As $\Lambda_{\mathcal{R}(T_{0,\epsilon})}(\Phi)$ is dominated by Lebesgue measure, we have by using Lemma 4:

$$
\lim_{\epsilon \to 0} \frac{f_A(t + \epsilon, v) - f_A(t, v)}{\epsilon} = \lim_{\epsilon \to 0} \mathbb{E} \left[ \prod_{z \in \mathcal{R}(T_{0,\epsilon})} v(z) \frac{f_A(\mathcal{R}(T_{0,\epsilon})(\Phi))}{\epsilon} \right] \quad (14)
$$

$$
= \mathbb{E} \left[ \prod_{z \in \mathcal{R}(T_{0,\epsilon})} v(z) \lim_{\epsilon \to 0} \frac{f_A(\mathcal{R}(T_{0,\epsilon})(\Phi))}{\epsilon} \right] = -\mathbb{E} \left[ \prod_{z \in \mathcal{R}(T_{0,\epsilon})} v(z) \int_{\mathbb{R}^2} (1 - v(x)) \right] (1 - h(x, z))\Lambda(dx)
$$

$$
= -\int_{\mathbb{R}^2} f_A(t - h(x, .))v(.) (1 - v(x))\Lambda(dx).
$$

Now we put $s = t - \epsilon$ and $t = t$. Using the first inequality in Lemma 4 we have:

$$
f_A(t, v) - f_A(t - \epsilon, v) = \mathbb{E} \left[ \prod_{z \in \mathcal{R}(T_{0,\epsilon})} v(z) \frac{f_A(\mathcal{R}(T_{0,\epsilon})(\Phi))}{\epsilon} \right] \quad (15)
$$

$$
\geq -\mathbb{E} \left[ \prod_{z \in \mathcal{R}(T_{0,\epsilon})} v(z) \int_{\mathbb{R}^2} (1 - v(x)) \right] (1 - h(x, z))\Lambda(dx)
$$

$$
= -\int_{\mathbb{R}^2} f_A(t, (1 - h(x, .))v(.) (1 - v(x))\Lambda(dx).
$$
Then by the second inequality:

\[
\frac{f_A(t, v) - f_A(t - \epsilon, v)}{\epsilon} = -\mathbb{E} \left[ \prod_{z \in \mathcal{R}(T[0, t-\epsilon]}(f(z)) (1 - v(x)) \right] \\
\leq -\mathbb{E} \left[ \prod_{z \in \mathcal{R}(T[0, t-\epsilon]}(f(z)) \int_{\mathbb{R}^2} (1 - v(x)) \right] \\
+ \epsilon \mathbb{E} \left[ \prod_{z \in \mathcal{R}(T[0, t-\epsilon]}(f(z)) \left( \int_{\mathbb{R}^2} (1 - v(x)) dx \right)^2 \right] \\
= -\int f_A(t, (1 - h(x, .))v(\cdot))(1 - v(x)) \Lambda(dx) + O(\epsilon).
\]

Now, we can use the continuity of \( f_A(\cdot, (1 - h(x, .))v(\cdot)) \) and let \( \epsilon \) go to 0 to get:

\[
\lim_{\epsilon \to 0} \frac{f_A(t, v) - f_A(t - \epsilon, v)}{\epsilon} = -\int f_A(t, (1 - h(x, .))v(\cdot))(1 - v(x)) \Lambda(dx).
\]

Now one can easily recognize that Propositions 1 and 2 are just applications of Proposition 9 for the Lebesgue measure and the measure \((1 - h(o, x))dx\) (this measure is dominated by the Lebesgue measure).

**APPENDIX C**

**TAYLOR EXPANSION OF \( f \) FUNCTIONAL**

In this Section, we give series representation for the functional \( f \). Recall that for a function \( g \) from \( \mathbb{R}^+ \) to \( \mathbb{R} \), which is infinitely differentiable, its Taylor expansion around 0 is:

\[
g(0) + \sum_{k=1}^{\infty} g^{(n)}(0) \frac{t^n}{n!},
\]

with \( g^{(k)} \) is the \( k^{th} \) derivative of \( g \). It is well-known that, given that this series converges, it converges exactly to \( g(t) \).

Now, using Proposition 1, we can derive the \( n^{th} \) derivative w.r.t. \( \lambda \) of \( f(\lambda, v) \) as:

\[
\frac{d^n}{d\lambda^n} f(\lambda, v) = (-1)^k \int_{\mathbb{R}^2} \prod_{i=1}^{k} \left( 1 - v(x_i) \prod_{j=1}^{i-1} (1 - h(x_j, x_i)) \right) \\
f(\lambda, \prod_{j=1}^{k} (1 - h(x_j, .))v) \prod_{j=1}^{k} dx_1 \cdots dx_k.
\]

As \( f(0, v) = 1 \) for every \( v \), the Taylor expansion of \( f(\lambda, v) \) around 0 is:

\[
1 + \sum_{k=1}^{\infty} (-1)^k \frac{\lambda^k}{k!} \int_{\mathbb{R}^2} \prod_{i=1}^{k} \left( 1 - v(x_i) \prod_{j=1}^{i-1} (1 - h(x_j, x_i)) \right) \\
dx_1 \cdots dx_k.
\]

Now, all we need is that the series above converges. In particular, we prove that it converges for \( \lambda < M^{-1} \) (\( M \) is defined in (11)) by the root criterion for series convergence. First, we have:

\[
\int_{\mathbb{R}^2} \prod_{i=1}^{k} \left( 1 - v(x_i) \prod_{j=1}^{i-1} (1 - h(x_j, x_i)) \right) dx_1 \cdots dx_k \leq \int_{\mathbb{R}^2} \prod_{i=1}^{k} \left( 1 - v(x_i) \prod_{j=1}^{i-1} (1 - h(x_j, x_i)) \right) dx_1 \cdots dx_k
\]

For any fixed \( x_1, \cdots, x_{i-1} \), by using the translation invariance of \( h \), we get:

\[
\int_{\mathbb{R}^2} \left( 1 - v(x_i) + \sum_{j=1}^{i-1} h(x_j, x_i) \right) dx_i = \int_{\mathbb{R}^2} (1 - v(x)) dx + (i - 1)M.
\]

Hence:

\[
\int_{\mathbb{R}^2} \prod_{i=1}^{k} \left( 1 - v(x_i) \prod_{j=1}^{i-1} (1 - h(x_j, x_i)) \right) dx_1 \cdots dx_k \leq \prod_{i=1}^{k} \left( \int_{\mathbb{R}^2} (1 - v(x)) dx + (i - 1)M \right).
\]

So:

\[
\frac{\lambda^k}{k!} \int_{\mathbb{R}^2} \prod_{i=1}^{k} \left( 1 - v(x_i) \prod_{j=1}^{i-1} (1 - h(x_j, x_i)) \right) dx_1 \cdots dx_k
\]

converges to \( f(\lambda, v) \) for \( \lambda < M^{-1} \).