Computer-aided worst-case analyses for operator splitting

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What is this presentation about?
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Computer-assisted analyses for optimization & monotone inclusions
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(Drori & Teboulle 2014), (Kim & Fessler 2016), (Lessard, Recht & Packard 2016),
(T, Hendrickx & Glineur 2017), (Lieder 2018), (Kim 2019), (Tan, Varvitsiotis & Tan 2019),
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Worst-case analyses for operator splitting (here: Douglas-Rachford)
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Worst-case analyses for operator splitting (here: Douglas-Rachford)

(Douglas & Rachford 1956), (Lions & Mercier 1979), (Giselsson & Boyd 2017), (Giselsson
2017), (Moursi & Vandenberghe 2018), and many others.
Take-home messages

Worst-cases are solutions to optimization problems.
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Sometimes, those optimization problems are tractable.
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Worst-cases are solutions to optimization problems.

Sometimes, those optimization problems are tractable.

Often tractable for first-order methods in optimization and monotone inclusions!
Douglas-Rachford Splitting

Let $f$ and $h$ be two convex, closed, proper functions. (Overrelaxed) DRS for solving

$$\min_{x \in \mathbb{R}^d} f(x) + h(x),$$

consists in iterating:

$$x_{k+1} = \arg\min_{x \in \mathbb{R}^d} \{ \gamma h(x) + \frac{1}{2} \| x - w_k \|_2^2 \}$$

$$y_{k+1} = \arg\min_{y \in \mathbb{R}^d} \{ \gamma f(y) + \frac{1}{2} \| y - 2x_{k+1} + w_k \|_2^2 \}$$

$$w_{k+1} = w_k + \theta (y_{k+1} - x_{k+1}).$$

Let $A$ and $B$ be maximally monotone operators; and let $J_{\gamma A} := (I + \gamma A)^{-1}$ and $J_{\gamma B} := (I + \gamma B)^{-1}$ be the respective resolvents.

Monotone inclusion problem: find $x \in \mathbb{R}^d$ such that $0 \in A(x) + B(x)$.

(Overrelaxed) Douglas-Rachford for solving the monotone inclusion

$$w_{k+1} = (I - \theta J_{\gamma B} + \theta J_{\gamma A}) w_k.$$

Recover optimization setting with $A = \partial f$ and $B = \partial h$. 
Douglas-Rachford Splitting

Let $f$ and $h$ be two convex, closed, proper functions. \((\text{Overrelaxed})\) DRS for solving

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Let \( f \) and \( h \) be two convex, closed, proper functions. (Overrelaxed) DRS for solving

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\min_{x \in \mathbb{R}^d} f(x) + h(x),
\]

consists in iterating:

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\begin{align*}
x_{k+1} &= \arg\min_{x \in \mathbb{R}^d} \left\{ \gamma h(x) + \frac{1}{2} \|x - w_k\|^2 \right\} \\
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w_{k+1} &= w_k + \theta(y_{k+1} - x_{k+1}).
\end{align*}
\]

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Monotone inclusion problem:

\[
\begin{align*}
\text{find } 0 &\in A(x) + B(x), \\
\text{for } x \in \mathbb{R}^d.
\end{align*}
\]
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Let \( f \) and \( h \) be two convex, closed, proper functions. (Overrelaxed) DRS for solving

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x_{k+1} = \arg\min_{x \in \mathbb{R}^d} \left\{ \gamma h(x) + \frac{1}{2} \|x - w_k\|^2 \right\}
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w_{k+1} = w_k + \theta(y_{k+1} - x_{k+1}).
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Let \( A \), and \( B \) be maximally monotone operators; and let \( J_{\gamma A} := (I + \gamma A)^{-1} \) and \( J_{\gamma B} := (I + \gamma B)^{-1} \) be the respective resolvents.

Monotone inclusion problem:

\[
\text{find } 0 \in A(x) + B(x),
\]

(overrelaxed) Douglas-Rachford for solving the monotone inclusion

\[
w_{k+1} = (I - \theta J_{\gamma B} + \theta J_{\gamma A}(2J_{\gamma B} - I))w_k.
\]
Douglas-Rachford Splitting

Let $f$ and $h$ be two convex, closed, proper functions. (Overrelaxed) DRS for solving

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Let $A$, and $B$ be maximally monotone operators; and let $J_{\gamma A} := (I + \gamma A)^{-1}$ and $J_{\gamma B} := (I + \gamma B)^{-1}$ be the respective resolvents.

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$$w_{k+1} = (I - \theta J_{\gamma B} + \theta J_{\gamma A}(2J_{\gamma B} - I))w_k.$$

Recover optimization setting with $A = \partial f$ and $B = \partial h$. 

4
Contraction factor?
Question: When is the DRS iteration a contraction? What is the smallest $\rho$ such that

$$\| w_1 - w'_1 \| \leq \rho \| w_0 - w'_0 \|,$$

for all $w_0, w'_0 \in \mathbb{R}^d$ and $w_1, w'_1$ generated with DRS from respectively $w_0$ and $w'_0$?
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Optimization problem to find sharp contraction factor:

$$
\text{maximize} \ A, B, w_0, w'_0, w_1, w'_1 \frac{\| w_1 - w'_1 \|}{\| w_0 - w'_0 \|}
$$

subject to $w_1$ generated by DR from $w_0$,

$w'_1$ generated by DR from $w'_0$,

assumptions on $A$ and $B$.

which has operators $A$ and $B$ as variables.
Assumptions

Nontrivial rates by assuming something more on $A$ and/or $B$.

Pick assumptions among the following:

- A convex function $f$ is commonly assumed to be (for all $x, y \in \mathbb{R}^d$):
  - $\mu$-strongly convex
    \[ f(x) \geq f(y) + \langle \nabla f(y), x - y \rangle + \frac{\mu}{2} \|x - y\|^2, \]
  - $L$-smooth
    \[ f(x) \leq f(y) + \langle f'(y), x - y \rangle + \frac{L}{2} \|x - y\|^2, \]
- A max. monotone operators $B$ is commonly assumed to be (for all $x, y \in \mathbb{R}^d$):
  - $\mu$-strongly monotone
    \[ \langle B(x) - B(y), x - y \rangle \geq \mu \|x - y\|^2, \]
  - $\beta$-cocoercive
    \[ \langle B(x) - B(y), x - y \rangle \geq \beta \|B(x) - B(y)\|^2, \]
  - $L$-Lipschitz
    \[ \|B(x) - B(y)\| \leq L \|x - y\|, \]
Assumptions

Nontrivial rates by assuming something more on $A$ and/or $B$. 

⋄ A convex function $f$ is commonly assumed to be (for all $x, y \in \mathbb{R}^d$):

- $\mu$-strongly convex $f(x) \geq f(y) + \langle \partial f(y), x - y \rangle + \mu \|x - y\|^2$.

- $L$-smooth $f(x) \leq f(y) + \langle f'(y), x - y \rangle + L\|x - y\|^2$.

⋄ A max. monotone operators $B$ is commonly assumed to be (for all $x, y \in \mathbb{R}^d$):

- $\mu$-strongly monotone $\langle B(x) - B(y), x - y \rangle \geq \mu \|x - y\|^2$.

- $\beta$-cocoercive $\langle B(x) - B(y), x - y \rangle \geq \beta \|B(x) - B(y)\|^2$.

- $L$-Lipschitz $\|B(x) - B(y)\| \leq L \|x - y\|$.
Assumptions

Nontrivial rates by assuming something more on $A$ and/or $B$.

Pick assumptions among the following:
Assumptions

Nontrivial rates by assuming something more on $A$ and/or $B$.

Pick assumptions among the following:

- A convex function $f$ is commonly assumed to be (for all $x, y \in \mathbb{R}^d$):
  - $\mu$-strongly convex $f(x) \geq f(y) + \langle \partial f(y), x - y \rangle + \frac{\mu}{2} \|x - y\|^2$,
  - L-smooth $f(x) \leq f(y) + \langle f'(y), x - y \rangle + \frac{L}{2} \|x - y\|^2$.
Assumptions

Nontrivial rates by assuming something more on $A$ and/or $B$.

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  - $\beta$-cocoercive
    \[ \langle B(x) - B(y), x - y \rangle \geq \beta \| B(x) - B(y) \|^2, \]
  - L-Lipschitz
    \[ \| B(x) - B(y) \| \leq L \| x - y \|. \]
DR contraction factors

Table: Contraction factors for DR: assumptions beyond max. monotonicity.

<table>
<thead>
<tr>
<th>#</th>
<th>Properties for A</th>
<th>Properties for B</th>
<th>Reference</th>
<th>Sharp</th>
<th>Notes</th>
</tr>
</thead>
<tbody>
<tr>
<td>O1</td>
<td>$\partial f, f$: str. cvx &amp; smooth</td>
<td>$\partial g$</td>
<td>[1,2]</td>
<td>✔</td>
<td></td>
</tr>
<tr>
<td>O2</td>
<td>$\partial f, f$: str. cvx</td>
<td>$\partial g, g$: smooth</td>
<td>[3]</td>
<td>✗</td>
<td>1.</td>
</tr>
<tr>
<td>M4</td>
<td>str. mono.</td>
<td>Lipschitz</td>
<td>[4]</td>
<td>✗</td>
<td></td>
</tr>
</tbody>
</table>

1. sharp rates for some parameter choices in [3]
2. Lions and Mercier [5] provided conservative rate in this setting
3. sharp rate when $B$ is skew linear in [4]

Contraction factor

Question:
When is the DRS iteration a contraction? What is the smallest $\rho$ such that
$$
\|w_1 - w'_1\| \leq \rho \|w_0 - w'_0\|,
$$
for all $w_0, w'_0 \in \mathbb{R}^d$ and $w_1, w'_1$ generated with DRS from respectively $w_0$ and $w'_0$?

⋄ Optimization problem to find sharp contraction factor:
$$
\max A, B, w_0, w'_0, w_1, w'_1 \|w_1 - w'_1\| \|w_0 - w'_0\|
$$
subject to $w_1$ generated by DR from $w_0$, $w'_1$ generated by DR from $w'_0$, $A$ is $\mu$-strongly monotone and $B$ is $\beta$-cocoercive.

which has operators $A$ and $B$ as variables.

⋄ Optimal value can be found via convex optimization! (3x3 SDP)
Contraction factor

**Question:** When is the DRS iteration a contraction? What is the smallest $\rho$ such that

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**Contraction factor**

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for all $w_0, w'_0 \in \mathbb{R}^d$ and $w_1, w'_1$ generated with DRS from respectively $w_0$ and $w'_0$?

◊ Optimization problem to find sharp contraction factor:

\[
\begin{align*}
\text{maximize} & \quad \frac{\|w_1 - w'_1\|}{\|w_0 - w'_0\|} \\
\text{subject to} & \quad w_1 \text{ generated by DR from } w_0, \\
& \quad w'_1 \text{ generated by DR from } w'_0, \\
& \quad A \text{ is } \mu\text{-strongly monotone and } B \text{ is } \beta\text{-cocoercive.}
\end{align*}
\]

which has operators $A$ and $B$ as variables.
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◊ Optimal value can be found via convex optimization! (3x3 SDP)
Problem reformulation

Recall DR splitting:

\[ x_1 = J_{\gamma B}(w_0) \]

with \( J_{\gamma B} := (I + \gamma B)^{-1} \),

\[ y_1 = J_{\gamma A}(2x_1 - w_0) \]

with \( J_{\gamma A} := (I + \gamma A)^{-1} \),

\[ w_1 = w_0 + \theta(y_1 - x_1) \]

\[ w'_1 = w'_0 + \theta(y'_1 - x'_1) \]

Require \( w_1 \) and \( w'_1 \) to be generated by DR:

\[
\begin{align*}
\text{maximize} & \quad A, B, w_0, w'_0, w_1, w'_1, x_1, x'_1, y_1, y'_1 \\
\text{subject to} & \quad x_1 = J_{\gamma B}(w_0), \\
& \quad x'_1 = J_{\gamma B}(w'_0), \\
& \quad y_1 = J_{\gamma A}(2x_1 - w_0), \\
& \quad y'_1 = J_{\gamma A}(2x'_1 - w'_0), \\
& \quad w_1 = w_0 + \theta(y_1 - x_1), \\
& \quad w'_1 = w'_0 + \theta(y'_1 - x'_1) \\
& \text{A is } \mu \text{-strongly monotone and } B \text{ is } \beta \text{-cocoercive.}
\end{align*}
\]
Problem reformulation

⋄ Recall DR splitting:

\[ x_1 = J_{\gamma B}(w_0) \quad \text{with} \quad J_{\gamma B} := (I + \gamma B)^{-1}, \]
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\text{subject to} & \quad x_1 = J_{\gamma B}(w_0), \\
& \quad x'_1 = J_{\gamma B}(w'_0), \\
& \quad y_1 = J_{\gamma A}(2x_1 - w_0), \\
& \quad y'_1 = J_{\gamma A}(2x'_1 - w'_0), \\
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A, B, w_0, w'_0, w_1, w'_1, x_1, x'_1, y_1, y'_1 & \quad \text{subject to} \quad x_1 = J_{\gamma B}(w_0),
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  w_1 = w_0 + \theta(y_1 - x_1).
  \]

- Require \( w_1 \) and \( w'_1 \) to be generated by DR:
  
  \[
  \begin{align*}
  &\underset{A, B, w_0, w'_0, w_1, w'_1}{\text{maximize}} & \frac{\|w_1 - w'_1\|}{\|w_0 - w'_0\|} \\
  &\text{subject to} & x_1 = J_{\gamma B}(w_0), \\
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& \quad x_1' = J_{\gamma B}(w_0'), \\
& \quad y_1 = J_{\gamma A}(2x_1 - w_0), \\
& \quad y_1' = J_{\gamma A}(2x_1' - w_0'), \\
& \quad w_1 = w_0 + \theta(y_1 - x_1),
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\( A \) is \( \mu \)-strongly monotone and \( B \) is \( \beta \)-cocoercive.

◊ Infinite-dimensional problem: two operators as variables!
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\]
\[
y_1 = J_{\gamma A}(2x_1 - w_0) \quad \text{with} \quad J_{\gamma A} := (I + \gamma A)^{-1},
\]
\[
w_1 = w_0 + \theta(y_1 - x_1).
\]

⋄ Require \( w_1 \) and \( w'_1 \) to be generated by DR:

\[
\begin{aligned}
\text{maximize} & \quad \frac{\|w_1 - w'_1\|}{\|w_0 - w'_0\|} \\
\text{subject to} & \quad x_1 = J_{\gamma B}(w_0), \\
& \quad x'_1 = J_{\gamma B}(w'_0), \\
& \quad y_1 = J_{\gamma A}(2x_1 - w_0), \\
& \quad y'_1 = J_{\gamma A}(2x'_1 - w'_0), \\
& \quad w_1 = w_0 + \theta(y_1 - x_1), \\
& \quad w'_1 = w'_0 + \theta(y'_1 - x'_1), \\
A \text{ is } \mu\text{-strongly monotone and } B \text{ is } \beta\text{-cocoercive.}
\end{aligned}
\]
Problem reformulation

⋄ Recall DR splitting:

\[ x_1 = J_{\gamma B}(w_0) \] with \( J_{\gamma B} := (I + \gamma B)^{-1}, \)

\[ y_1 = J_{\gamma A}(2x_1 - w_0) \] with \( J_{\gamma A} := (I + \gamma A)^{-1}, \)

\[ w_1 = w_0 + \theta(y_1 - x_1). \]

⋄ Require \( w_1 \) and \( w'_1 \) to be generated by DR:

\[
\begin{align*}
\text{maximize} & \quad \frac{\|w_1 - w'_1\|}{\|w_0 - w'_0\|} \\
\text{subject to} & \quad x_1 = J_{\gamma B}(w_0), \\
& \quad x'_1 = J_{\gamma B}(w'_0), \\
& \quad y_1 = J_{\gamma A}(2x_1 - w_0), \\
& \quad y'_1 = J_{\gamma A}(2x'_1 - w'_0), \\
& \quad w_1 = w_0 + \theta(y_1 - x_1), \\
& \quad w'_1 = w'_0 + \theta(y'_1 - x'_1), \\
& \quad A \text{ is } \mu\text{-strongly monotone and } B \text{ is } \beta\text{-cocoercive.}
\end{align*}
\]

⋄ Infinite-dimensional problem: two operators as variables!
Discrete version

- Remove $A$ and $B$ from the variables?
Discrete version

Remove $A$ and $B$ from the variables?

\[
\begin{align*}
\text{maximize} & \quad \frac{\|w_1 - w'_1\|}{\|w_0 - w'_0\|} \\
\text{subject to} & \quad \exists B\beta -\text{cocoercive such that} \\
& \quad \begin{cases}
\gamma_B(w_0) = x_1, \\
\gamma_B(w'_0) = x'_1,
\end{cases} \\
& \quad \exists A\mu -\text{-strongly monotone such that} \\
& \quad \begin{cases}
\gamma_A(2x_1 - w_0) = y_1, \\
\gamma_A(2x'_1 - w'_0) = y'_1,
\end{cases} \\
& \quad w_1 = w_0 + \theta(y_1 - x_1), \\
& \quad w'_1 = w'_0 + \theta(y'_1 - x'_1).
\end{align*}
\]
Remove $A$ and $B$ from the variables?

\[
\begin{align*}
\text{maximize} & \quad \frac{\|w_1 - w'_1\|}{\|w_0 - w'_0\|} \\
\text{subject to} & \quad \exists B \ \beta\text{-cocoercive such that} \\
& \quad \left\{ \begin{array}{l}
\ x_1 = J_{\gamma B}(w_0), \\
\ x'_1 = J_{\gamma B}(w'_0), \\
\end{array} \right.
\end{align*}
\]
Discrete version

◊ Remove $A$ and $B$ from the variables?

\[
\begin{align*}
\text{maximize} & \quad \frac{\|w_1 - w'_1\|}{\|w_0 - w'_0\|} \\
\text{subject to} & \quad \exists B \ \beta\text{-cocoercive such that} \\
& \quad \begin{cases} \ x_1 = J_{\gamma B}(w_0) , \\ \ x'_1 = J_{\gamma B}(w'_0) , \end{cases} \\
& \quad \exists A \ \mu\text{-strongly monotone such that} \\
& \quad \begin{cases} \ y_1 = J_{\gamma A}(2x_1 - w_0) , \\ \ y'_1 = J_{\gamma A}(2x'_1 - w'_0) , \end{cases}
\end{align*}
\]
Discrete version

• Remove $A$ and $B$ from the variables?

\[
\begin{align*}
\text{maximize} & \quad \frac{\|w_1 - w_1\|}{\|w_0 - w_0\|} \\
\text{subject to} & \quad \exists B \ \beta\text{-cocoercive such that} \\
& \quad \begin{cases} 
    x_1 = J_{\gamma B}(w_0), \\
    x_1' = J_{\gamma B}(w_0'), 
\end{cases} \\
& \quad \exists A \ \mu\text{-strongly monotone such that} \\
& \quad \begin{cases} 
    y_1 = J_{\gamma A}(2x_1 - w_0), \\
    y_1' = J_{\gamma A}(2x_1' - w_0'), 
\end{cases} \\
& \quad w_1 = w_0 + \theta(y_1 - x_1), \\
& \quad w_1' = w_0' + \theta(y_1' - x_1').
\end{align*}
\]
Discrete version

- Remove $A$ and $B$ from the variables?

\[
\begin{align*}
\text{maximize} & \quad \frac{\|w_1 - w'_1\|}{\|w_0 - w'_0\|} \\
\text{subject to} & \quad \exists B \ \beta\text{-cocoercive such that} \\
& \quad \exists A \ \mu\text{-strongly monotone such that} \\
& \quad w_1 = w_0 + \theta (y_1 - x_1), \\
& \quad w'_1 = w'_0 + \theta (y'_1 - x'_1).
\end{align*}
\]

- How to remove existence constraints?
Interpolation of operators

Define the duplets \((x, x + \gamma \mu)\) and \((y, y + \gamma \mu)\). Then

\[
\langle x - y, x + \gamma \mu - y + \gamma \mu \rangle \geq (\gamma \mu + 1) \| x + \gamma \mu - y + \gamma \mu \|_2^2
\]

iff there exists a \(\mu\)-strongly monotone operator \(A\) such that

\[
-x + \gamma \mu A(x) - y + \gamma \mu A(y)
\]

Define the duplets \((x, x + \gamma \mu)\) and \((y, y + \gamma \mu)\). Then

\[
\langle x - y, x + \gamma \mu - y + \gamma \mu \rangle \geq \beta \gamma \| x - x + \gamma \mu - (y - y + \gamma \mu) \|_2^2 + \| x + \gamma \mu - y + \gamma \mu \|_2^2
\]

iff there exists a \(\beta\)-cocoercive operator \(B\) such that

\[
-x + \gamma \mu B(x) - y + \gamma \mu B(y)
\]
Interpolation of operators

Define the duplets \((x, x_+)\) and \((y, y_+)\). Then

\[
\langle x - y, x_+ - y_+ \rangle \geq (\gamma \mu + 1)\|x_+ - y_+\|^2
\]

iff there exists a \(\mu\)-strongly monotone operator \(A\) such that

- \(x_+ = J_{\gamma A}(x)\)
- \(y_+ = J_{\gamma A}(y)\)
Define the duplets \((x, x_+)\) and \((y, y_+)\). Then
\[
\langle x - y, x_+ - y_+ \rangle \geq (\gamma \mu + 1) \| x_+ - y_+ \|^2
\]
iff there exists a \(\mu\)-strongly monotone operator \(A\) such that
- \(x_+ = J_{\gamma A}(x)\)
- \(y_+ = J_{\gamma A}(y)\)

Define the duplets \((x, x_+)\) and \((y, y_+)\). Then
\[
\langle x - y, x_+ - y_+ \rangle \geq \frac{\beta}{\gamma} \| x - x_+ - (y - y_+) \|^2 + \| x_+ - y_+ \|^2
\]
iff there exists a \(\beta\)-cocoercive operator \(B\) such that
- \(x_+ = J_{\gamma B}(x)\)
- \(y_+ = J_{\gamma B}(y)\)
Replace constraints

\[ \max w_0, w'_0, w_1, w'_1 \]

\[ x_1, x'_1, y_1, y'_1 \]

\[ \|w_1 - w'_1\| \]

\[ \|w_0 - w'_0\| \]

subject to

\[ \exists B \beta \text{-cocoercive such that} \]

\[ x_1 = J_{\gamma B}(w_0), \]

\[ x'_1 = J_{\gamma B}(w'_0); \]

\[ \exists A \mu \text{-strongly monotone such that} \]

\[ y_1 = J_{\gamma A}(2x_1 - w_0), \]

\[ y'_1 = J_{\gamma A}(2x'_1 - w'_0); \]

\[ w_1 = w_0 + \theta(y_1 - x_1), \]

\[ w'_1 = w'_0 + \theta(y'_1 - x'_1). \]

Note: optimal value is the same! No relaxation.
Replace constraints

◊ Interpolation conditions allows to remove red constraints

\[
\begin{align*}
\text{maximize} & \quad \frac{\|w_1 - w'_1\|}{\|w_0 - w'_0\|} \\
\text{subject to} & \quad \exists B \text{ } \beta\text{-cocoercive such that} \\
& \exists A \text{ } \mu\text{-strongly monotone such that} \\
& w_1 = w_0 + \theta(y_1 - x_1), \\
& w'_1 = w'_0 + \theta(y'_1 - x'_1).
\end{align*}
\]
Replace constraints

- Interpolation conditions allows to remove red constraints

\[
\begin{align*}
\text{maximize} & \quad \frac{\|w_1 - w'_1\|}{\|w_0 - w'_0\|} \\
\text{subject to} & \quad \exists B \beta\text{-cocoercive such that} \quad \begin{cases} 
  x_1 = J_{\gamma B}(w_0), \\
  x'_1 = J_{\gamma B}(w'_0), 
\end{cases} \\
& \quad \exists A \mu\text{-strongly monotone such that} \quad \begin{cases} 
  y_1 = J_{\gamma A}(2x_1 - w_0), \\
  y'_1 = J_{\gamma A}(2x'_1 - w'_0), 
\end{cases} \\
& \quad w_1 = w_0 + \theta(y_1 - x_1), \\
& \quad w'_1 = w'_0 + \theta(y'_1 - x'_1).
\end{align*}
\]

- replacing them by:

\[
\langle y_1 - y'_1, 2(x_1 - x'_1) - (w_0 - w'_0) \rangle \geq (\gamma \mu + 1)\|y_1 - y'_1\|^2,
\]

and

\[
\langle w_0 - w'_0, x_1 - x'_1 \rangle \geq \frac{\beta}{\gamma} \|w_0 - w'_0 - (x_1 - x'_1)\|^2 + \|x_1 - x'_1\|^2.
\]
Replace constraints

\[ \text{maximize } \frac{\|w_1 - w'_1\|}{\|w_0 - w'_0\|} \]

subject to

\[ \exists B \beta\text{-cocoercive such that } \begin{cases} x_1 = J_{\gamma B}(w_0), \\ x'_1 = J_{\gamma B}(w'_0), \end{cases} \]

\[ \exists A \mu\text{-strongly monotone such that } \begin{cases} y_1 = J_{\gamma A}(2x_1 - w_0), \\ y'_1 = J_{\gamma A}(2x'_1 - w'_0), \end{cases} \]

\[ w_1 = w_0 + \theta(y_1 - x_1), \]

\[ w'_1 = w'_0 + \theta(y'_1 - x'_1). \]

\[ \diamond \text{ replacing them by:}\]

\[ \langle y_1 - y'_1, 2(x_1 - x'_1) - (w_0 - w'_0) \rangle \geq (\gamma \mu + 1)\|y_1 - y'_1\|^2, \]

and

\[ \langle w_0 - w'_0, x_1 - x'_1 \rangle \geq \frac{\beta}{\gamma} \|w_0 - w'_0 - (x_1 - x'_1)\|^2 + \|x_1 - x'_1\|^2. \]

\[ \diamond \text{ Note: optimal value is the same! No relaxation.} \]
Reformulations (cont’d)

Yet another reformulation

\[
\begin{align*}
\text{maximize} & \quad w_0, \quad w'_0, \quad w_1, \quad w'_1, \quad x_1, \quad x'_1, \quad y_1, \quad y'_1 \\
\text{subject to} & \quad \langle y_1 - y'_1, 2(x_1 - x'_1) - (w_0 - w'_0) \rangle \geq (\gamma \mu + 1) \left\| y_1 - y'_1 \right\|_2, \\
& \quad \langle w_0 - w'_0, x_1 - x'_1 \rangle \geq \beta \gamma \left\| w_0 - w'_0 - (x_1 - x'_1) \right\|_2 + \left\| x_1 - x'_1 \right\|_2.
\end{align*}
\]
Reformulations (cont’d)

◊ Equivalent problem without operator class constraints:

\[
\begin{align*}
\text{maximize} & \quad \frac{\|w_1 - w'_1\|}{\|w_0 - w'_0\|} \\
\text{subject to} & \quad \langle y_1 - y'_1, 2(x_1 - x'_1) - (w_0 - w'_0) \rangle \geq (\gamma \mu + 1)\|y_1 - y'_1\|^2, \\
& \quad \langle w_0 - w'_0, x_1 - x'_1 \rangle \geq \frac{\beta}{\gamma} \|w_0 - w'_0 - (x_1 - x'_1)\|^2 + \|x_1 - x'_1\|^2, \\
& \quad w_1 = w_k + \theta(y_1 - x_1), \\
& \quad w'_1 = w_k + \theta(y'_1 - x'_1).
\end{align*}
\]
Reformulations (cont’d)

◊ Equivalent problem without operator class constraints:

\[
\begin{align*}
\text{maximize} & \quad \frac{\|w_1 - w'_1\|}{\|w_0 - w'_0\|} \\
\text{subject to} & \quad \langle y_1 - y'_1, 2(x_1 - x'_1) - (w_0 - w'_0) \rangle \geq (\gamma \mu + 1) \|y_1 - y'_1\|^2, \\
& \quad \langle w_0 - w'_0, x_1 - x'_1 \rangle \geq \frac{\beta}{\gamma} \|w_0 - w'_0 - (x_1 - x'_1)\|^2 + \|x_1 - x'_1\|^2, \\
& \quad w_1 = w_k + \theta(y_1 - x_1), \\
& \quad w'_1 = w_k + \theta(y'_1 - x'_1).
\end{align*}
\]

◊ Yet another reformulation

\[
\begin{align*}
\text{maximize} & \quad \frac{\|w_0 + \theta(y_1 - x_1) - w_0 - \theta(y'_1 - x'_1)\|^2}{\|w_0 - w'_0\|^2} \\
\text{subject to} & \quad \langle y_1 - y'_1, 2(x_1 - x'_1) - (w_0 - w'_0) \rangle \geq (\gamma \mu + 1) \|y_1 - y'_1\|^2, \\
& \quad \langle w_0 - w'_0, x_1 - x'_1 \rangle \geq \frac{\beta}{\gamma} \|w_0 - w'_0 - (x_1 - x'_1)\|^2 + \|x_1 - x'_1\|^2.
\end{align*}
\]
Semidefinite lifting

All parts of optimization problem are quadratic:

\[
\text{maximize } w_0, \ w_0' \ 
\text{subject to } \
\langle y_1 - y_1', 2(x_1 - x_1') \rangle \geq (\gamma \mu + 1) \parallel y_1 - y_1' \parallel_2,
\langle w_0 - w_0', x_1 - x_1' \rangle \geq \beta \gamma \parallel w_0 - w_0' - (x_1 - x_1') \parallel_2 + \parallel x_1 - x_1' \parallel_2.
\]

They can therefore be represented with a Gram matrix. Let

\[
G = \begin{bmatrix}
\parallel w_0 - w_0' \parallel_2 \\
\langle w_0 - w_0', x_1 - x_1' \rangle \\
\langle x_1 - x_1', w_0 - w_0' \rangle \\
\langle y_1 - y_1', 2(x_1 - x_1') \rangle \\
\langle y_1 - y_1', w_0 - w_0' \rangle \\
\langle y_1 - y_1', x_1 - x_1' \rangle \\
\parallel x_1 - x_1' \parallel_2 \\
\parallel y_1 - y_1' \parallel_2 
\end{bmatrix}
\]

where

\[G \succeq 0\]

by construction,

and reformulate to:

\[
\text{maximize } G \text{Tr}(A_0 G) - \text{Tr}(A_s G)
\text{subject to } \text{Tr}(A_1 G) \geq 0, \ \text{Tr}(A_2 G) \geq 0, \ G \succeq 0.
\]

with appropriate

\[A_0, A_s, A_1, A_2\]

for picking correct elements in

\[G\]

Note: assuming

\[w_0, w_0', x_1, x_1', y_1, y_1' \in \mathbb{R}^d \text{ with } d \geq 3, \text{ same optimal cost!}\]
Semidefinite lifting

- All parts of optimization problem are quadratic:

\[
\begin{align*}
\text{maximize} & \quad \frac{\|w_0 + \theta(y_1 - x_1) - w_0 - \theta(y'_1 - x'_1)\|^2}{\|w_0 - w'_0\|^2} \\
\text{subject to} & \quad \langle y_1 - y'_1, 2(x_1 - x'_1) - (w_0 - w'_0) \rangle \geq (\gamma \mu + 1) \|y_1 - y'_1\|^2, \\
& \quad \langle w_0 - w'_0, x_1 - x'_1 \rangle \geq \frac{\beta}{\gamma} \|w_0 - w'_0 - (x_1 - x'_1)\|^2 + \|x_1 - x'_1\|^2.
\end{align*}
\]
Semantic lifting

- All parts of optimization problem are quadratic:

\[
\begin{align*}
&\text{maximize} \quad \frac{\|w_0 + \theta(y_1 - x_1) - w_0 - \theta(y'_1 - x'_1)\|^2}{\|w_0 - w'_0\|^2} \\
&\text{subject to} \quad \langle y_1 - y'_1, 2(x_1 - x'_1) - (w_0 - w'_0) \rangle \geq (\gamma \mu + 1)\|y_1 - y'_1\|^2, \\
&\quad \langle w_0 - w'_0, x_1 - x'_1 \rangle \geq \frac{\beta}{\gamma}\|w_0 - w'_0 - (x_1 - x'_1)\|^2 + \|x_1 - x'_1\|^2.
\end{align*}
\]

- They can therefore be represented with a Gram matrix. Let

\[
G = \begin{bmatrix}
\|w_0 - w'_0\|^2 & \langle w_0 - w'_0, x_1 - x'_1 \rangle & \langle w_0 - w'_0, y_1 - y'_1 \rangle \\
\langle x_1 - x'_1, w_0 - w'_0 \rangle & \|x_1 - x'_1\|^2 & \langle x_1 - x'_1, y_1 - y'_1 \rangle \\
\langle y_1 - y'_1, w_0 - w'_0 \rangle & \langle y_1 - y'_1, x_1 - x'_1 \rangle & \|y_1 - y'_1\|^2
\end{bmatrix}
\]

where \( G \succeq 0 \) by construction.
Semidefinite lifting

- All parts of optimization problem are quadratic:

\[
\begin{align*}
\text{maximize} & \quad \frac{\|w_0 + \theta(y_1 - x_1) - w_0 - \theta(y'_1 - x'_1)\|^2}{\|w_0 - w'_0\|^2} \\
\text{subject to} & \quad \langle y_1 - y'_1, 2(x_1 - x'_1) - (w_0 - w'_0) \rangle \geq (\gamma \mu + 1)\|y_1 - y'_1\|^2, \\
& \quad \langle w_0 - w'_0, x_1 - x'_1 \rangle \geq \frac{\beta}{\gamma} \|w_0 - w'_0 - (x_1 - x'_1)\|^2 + \|x_1 - x'_1\|^2.
\end{align*}
\]

- They can therefore be represented with a Gram matrix. Let

\[
G = \begin{bmatrix}
\|w_0 - w'_0\|^2 & \langle w_0 - w'_0, x_1 - x'_1 \rangle & \langle w_0 - w'_0, y_1 - y'_1 \rangle \\
\langle x_1 - x'_1, w_0 - w'_0 \rangle & \|x_1 - x'_1\|^2 & \langle x_1 - x'_1, y_1 - y'_1 \rangle \\
\langle y_1 - y'_1, w_0 - w'_0 \rangle & \langle y_1 - y'_1, x_1 - x'_1 \rangle & \|y_1 - y'_1\|^2
\end{bmatrix}
\]

where \(G \succeq 0\) by construction, and reformulate to:

\[
\begin{align*}
\text{maximize} & \quad \frac{\text{Tr}(A_o G)}{\text{Tr}(A_s G)} \\
\text{subject to} & \quad \text{Tr}(A_1 G) \geq 0 \\
& \quad \text{Tr}(A_2 G) \geq 0 \\
& \quad G \succeq 0.
\end{align*}
\]

with appropriate \(A_o, A_s, A_1, A_2\) for picking correct elements in \(G\)
Semidefinite lifting

All parts of optimization problem are quadratic:

\[
\max_{w_0, w'_0, x_1, x'_1, y_1, y'_1} \frac{\|w_0 + \theta(y_1 - x_1) - w_0 - \theta(y'_1 - x'_1)\|^2}{\|w_0 - w'_0\|^2}
\]

subject to

\[
\langle y_1 - y'_1, 2(x_1 - x'_1) - (w_0 - w'_0) \rangle \geq (\gamma \mu + 1)\|y_1 - y'_1\|^2,
\]

\[
\langle w_0 - w'_0, x_1 - x'_1 \rangle \geq \frac{\beta}{\gamma} \|w_0 - w'_0 - (x_1 - x'_1)\|^2 + \|x_1 - x'_1\|^2.
\]

They can therefore be represented with a Gram matrix. Let

\[
G = \begin{bmatrix}
\|w_0 - w'_0\|^2 & \langle w_0 - w'_0, x_1 - x'_1 \rangle & \langle w_0 - w'_0, y_1 - y'_1 \rangle \\
\langle x_1 - x'_1, w_0 - w'_0 \rangle & \|x_1 - x'_1\|^2 & \langle x_1 - x'_1, y_1 - y'_1 \rangle \\
\langle y_1 - y'_1, w_0 - w'_0 \rangle & \langle y_1 - y'_1, x_1 - x'_1 \rangle & \|y_1 - y'_1\|^2
\end{bmatrix}
\]

where \(G \succeq 0\) by construction, and reformulate to:

\[
\max_{G} \frac{\text{Tr}(A_o G)}{\text{Tr}(A_s G)}
\]

subject to \(\text{Tr}(A_1 G) \geq 0\)

\(\text{Tr}(A_2 G) \geq 0\)

\(G \succeq 0\).

with appropriate \(A_o, A_s, A_1, A_2\) for picking correct elements in \(G\)

Note: assuming \(w_0, w'_0, x_1, x'_1, y_1, y'_1 \in \mathbb{R}^d\) with \(d \geq 3\), same optimal cost!
Last part in convexification
The constraints are positively homogeneous of deg. 1 and the cost is constant under scaling of $G$

\[
\begin{align*}
\text{maximize} & \quad \frac{\text{Tr}(A_o G)}{\text{Tr}(A_s G)} \\
\text{subject to} & \quad \text{Tr}(A_1 G) \geq 0 \\
& \quad \text{Tr}(A_2 G) \geq 0 \\
& \quad G \succeq 0.
\end{align*}
\]
The constraints are positively homogeneous of deg. 1 and the cost is constant under scaling of $G$

\[
\begin{align*}
\text{maximize} \quad & \frac{\text{Tr}(A_o G)}{\text{Tr}(A_s G)} \\
\text{subject to} \quad & \text{Tr}(A_1 G) \geq 0 \\
& \text{Tr}(A_2 G) \geq 0 \\
& G \succeq 0.
\end{align*}
\]

Therefore an equivalent convex problem is

\[
\begin{align*}
\text{maximize} \quad & \text{Tr}(A_o G) \\
\text{subject to} \quad & \text{Tr}(A_1 G) \geq 0 \\
& \text{Tr}(A_2 G) \geq 0 \\
& \text{Tr}(A_s G) = 1 \\
& G \succeq 0.
\end{align*}
\]

which is a 3x3 semidefinite program.
Dual problem

- Introduce dual variables \( \tau, \lambda_1, \lambda_2 \)

\[
\begin{align*}
\text{maximize} & \quad \text{Tr} (A_0 G) \\
\text{subject to} & \quad \text{Tr} (A_1 G) \geq 0 : \lambda_1 \\
& \quad \text{Tr} (A_2 G) \geq 0 : \lambda_2 \\
& \quad \text{Tr} (A_s G) = 1 : \tau \\
& \quad G \succeq 0
\end{align*}
\]

- Dual problem becomes

\[
\begin{align*}
\text{minimize} & \quad \tau, \lambda_1, \lambda_2 \\
\text{subject to} & \quad \lambda_i \geq 0 \\
& \quad S = A_0 + \sum_{i=1}^{2} \lambda_i A_i - \tau A_s \preceq 0
\end{align*}
\]

- In this example:

\[
S = \begin{bmatrix}
-\tau & -\beta \lambda_2 & \gamma + 1 - \theta + \lambda_2^2 + \beta \lambda_2^2 \\
-\theta + \lambda_1^2 - \theta & -\beta \lambda_1^2 - \gamma \lambda_1^2 - \lambda_2^2 + \theta^2 & 2 \beta \lambda_2^2 \\
\end{bmatrix}
\]

- Strong duality holds (existence of a Slater point): rank \((G) + \text{rank} (S) \leq 3\).
Dual problem

◊ Introduce dual variables $\tau$, $\lambda_1$ and $\lambda_2$

\[
\begin{align*}
\text{maximize} & \quad \text{Tr}(A_o G) \\
\text{subject to} & \quad \text{Tr}(A_1 G) \geq 0 : \lambda_1 \\
& \quad \text{Tr}(A_2 G) \geq 0 : \lambda_2 \\
& \quad \text{Tr}(A_s G) = 1 : \tau \\
& \quad G \succeq 0
\end{align*}
\]
Dual problem

- Introduce dual variables $\tau$, $\lambda_1$ and $\lambda_2$

\[
\begin{align*}
\text{maximize} \quad & \quad \text{Tr}(A_o G) \\
\text{subject to} \quad & \quad \text{Tr}(A_1 G) \geq 0 : \lambda_1 \\
& \quad \text{Tr}(A_2 G) \geq 0 : \lambda_2 \\
& \quad \text{Tr}(A_s G) = 1 : \tau \\
& \quad G \succeq 0
\end{align*}
\]

- Dual problem becomes

\[
\begin{align*}
\text{minimize} \quad & \quad \tau \\
\text{subject to} \quad & \quad \lambda_i \geq 0 \\
& \quad S = A_o + \sum_{i=1}^{2} \lambda_i A_i - \tau A_s \leq 0
\end{align*}
\]
Dual problem

◊ Introduce dual variables $\tau$, $\lambda_1$ and $\lambda_2$

$$\begin{align*}
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\end{align*}$$

◊ Dual problem becomes

$$\begin{align*}
\text{minimize } & \quad \tau, \lambda_1, \lambda_2 \\
\text{subject to } & \quad \lambda_i \geq 0 \\
& \quad S = A_o + \sum_{i=1}^2 \lambda_i A_i - \tau A_s \preceq 0
\end{align*}$$

◊ In this example:

$$S = \begin{bmatrix}
-\tau - \frac{\beta \lambda_2}{\gamma} + 1 \\
-\theta + \frac{\lambda_2}{2} + \frac{\beta \lambda_2}{\gamma} \\
\theta - \frac{\lambda_1}{2}
\end{bmatrix} \begin{bmatrix}
-\theta + \frac{\lambda_2}{2} + \frac{\beta \lambda_2}{\gamma} \\
\theta^2 - \lambda_2 - \frac{\beta \lambda_2}{\gamma} \\
\lambda_1 - \theta^2
\end{bmatrix} \begin{bmatrix}
\theta - \frac{\lambda_1}{2} \\
\lambda_1 - \theta^2 \\
\theta^2 - \lambda_1 - \gamma \lambda_1 \mu
\end{bmatrix}$$
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\[
S = \begin{bmatrix}
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-\theta + \frac{\lambda_2}{2} + \frac{\beta \lambda_2}{\gamma} & \theta^2 - \lambda_2 - \frac{\beta \lambda_2}{\gamma} & \lambda_1 - \theta^2 \\
\theta - \frac{\lambda_1}{2} & \lambda_1 - \theta^2 & \theta^2 - \lambda_1 - \gamma \lambda_1 \mu
\end{bmatrix}
\]

◊ Strong duality holds (existence of a Slater point): \( \text{rank}(G) + \text{rank}(S) \leq 3 \).
A few more examples

Warning for the next few slides:
A few more examples

Warning for the next few slides:

◇ expressions are horrible,
A few more examples

Warning for the next few slides:
  ◦ expressions are horrible,
  ◦ barely obtainable by hand,
A few more examples

Warning for the next few slides:

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◊ computer-generated (Mathematica),
A few more examples

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- verifiable by hand (long algebraic proofs).

Note I: the methodology offers 3 ways to proceed:

- play with primal formulation,
- play with primal-dual saddle-point formulation,
- play with dual formulation.

Note II: that any dual feasible point can be translated into a “traditional” proof.
A few more examples

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Intuitions can be developed, but this is another story 😊
A few more examples

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Douglas-Rachford Splitting

Assumptions: \( A \) \( \mu \)-strongly monotone, \( B \) \( \beta \)-cocoercive.
Douglas-Rachford Splitting

Assumptions: $A \mu$-strongly monotone, $B \beta$-cocoercive.

We have $\|Tx - Ty\| \leq \rho \|x - y\|$ for all $x, y \in \mathcal{H}$ with:

$$\rho = \begin{cases} \left|1 - \theta \beta \right| & \text{if } \mu \beta - \mu - \beta < 0, \\ \left|1 - \theta \right| & \text{if } \mu \beta - \mu - \beta > 0, \\ \left|1 - \theta \mu \right| & \text{if } \theta \geq \frac{2}{\mu \beta + \mu + \beta}, \\ \left|1 - \theta \beta \right| & \text{if } \theta \leq \frac{2 \mu \beta - \mu - \beta - \mu \beta^2}{\mu^2 + \beta^2 + \mu \beta + \mu + \beta^2 - 2 \mu \beta^2 - 2 \mu^2 \beta^2}, \\ 1 & \text{otherwise}, \\ \end{cases}$$

⋄ The first four cases are achieved on 1-dimensional examples (primal is simpler).

⋄ Fifth case is achieved on 2-dimensional example (dual is simpler).
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$$\rho = \begin{cases} 
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\frac{1}{1 - \theta\mu} & \text{if } \mu \beta - \mu - \beta > 0, \text{ and } \theta \leq 2 \mu^2 + \beta^2 + \mu \beta + \mu + \beta - 2 \mu^2 \beta^2 \\
\frac{1}{1 - \theta} & \text{if } \theta \geq 2 \mu \beta + \mu + \beta^2 \\
\frac{1}{1 - \theta(\mu + 1)(\beta + 1)} & \text{if } \mu \beta + \mu - \beta < 0, \text{ and } \theta \leq 2(\mu + 1)(\beta - \mu - \mu \beta) 
\end{cases}$$

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\end{cases}$$

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$$
\rho = \begin{cases} 
|1 - \theta \frac{\beta}{\beta+1}| & \text{if } \mu \beta - \mu + \beta < 0, \text{ and } \theta \leq \frac{(\beta+1)(\mu - \beta - \mu \beta)}{\mu + \mu \beta - \beta - 2 \mu \beta^2}, \\
|1 - \theta \frac{1+\mu \beta}{(\mu+1)(\beta+1)}| & \text{if } \mu \beta - \mu - \beta > 0, \text{ and } \theta \leq \frac{\mu^2 + \beta^2 + \mu \beta + \mu + \beta - \mu^2 \beta^2}{\mu^2 + \beta^2 + \mu \beta^2 + \mu + \beta - 2 \mu^2 \beta^2}, \\
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|1 - \theta \frac{1 + \mu \beta}{(\mu + 1)(\beta + 1)}| & \text{if } \mu \beta - \mu - \beta > 0, \text{ and } \theta \leq 2 \frac{\mu^2 + \mu \beta^2 + \mu + \beta - \mu \beta^2}{\mu^2 + \beta^2 + \mu \beta + \mu + \beta - 2 \mu \beta^2} , \\
|1 - \theta| & \text{if } \theta \geq 2 \frac{\mu \beta + \mu + \beta}{2 \mu \beta + \mu + \beta} , \\
\end{cases}$$

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We have \( \| Tx - Ty \| \leq \rho \| x - y \| \) for all \( x, y \in H \) with:

\[
\rho = \begin{cases} 
|1 - \theta \frac{\beta}{\beta+1}| & \text{if } \mu \beta - \mu + \beta < 0, \text{ and } \theta \leq 2 \frac{(\beta+1)(\mu-\beta-\mu \beta)}{\mu + \mu \beta - \beta - \beta^2 - 2 \mu \beta^2}, \\
|1 - \theta \frac{1+\mu \beta}{(\mu+1)(\beta+1)}| & \text{if } \mu \beta - \mu - \beta > 0, \text{ and } \theta \leq 2 \frac{\mu^2 + \beta^2 + \mu \beta + \mu + \beta - \mu^2 \beta^2}{\mu^2 + \beta^2 + \mu \beta^2 + \mu + \beta - 2 \mu^2 \beta^2}, \\
|1 - \theta| & \text{if } \theta \geq 2 \frac{\mu \beta + \mu + \beta}{2 \mu \beta + \mu + \beta}, \\
|1 - \theta \frac{\mu}{\mu+1}| & \text{if } \mu \beta + \mu - \beta < 0, \text{ and } \theta \leq 2 \frac{(\mu+1)(\beta-\mu-\mu \beta)}{\beta + \mu \beta - \mu - \mu^2 - 2 \mu^2 \beta}, \\
|1 - \theta| & \text{otherwise,}
\end{cases}
\]
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\[
\rho = \begin{cases}
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|1 - \theta \frac{\mu+\beta}{(\mu+1)(\beta+1)}| & \text{if } \mu \beta - \mu - \beta > 0, \text{ and } \theta \leq 2 \frac{\mu^2+\beta^2+\mu+\beta-\mu^2\beta^2}{\mu^2+\beta^2+\mu^2\beta^2+\mu+\beta-2\mu^2\beta} , \\
|1 - \theta \frac{1+\mu \beta}{(\mu+1)(\beta+1)}| & \text{if } \theta \geq 2 \frac{\mu \beta+\mu+\beta}{2\mu \beta+\mu+\beta}, \\
|1 - \theta \frac{\mu}{\mu+1}| & \text{if } \mu \beta + \mu - \beta < 0, \text{ and } \theta \leq 2 \frac{\mu+1-\beta-\mu \beta}{\beta+\mu \beta-\mu^2-2\mu^2\beta}, \\
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|1 - \theta| & \text{if } \theta \geq 2 \frac{\mu \beta + \mu + \beta}{2 \mu \beta + \mu + \beta}, \\
|1 - \theta \frac{\mu}{\mu + 1}| & \text{if } \mu \beta + \mu - \beta < 0, \text{ and } \theta \leq 2 \frac{(\mu + 1)(\beta - \mu - \mu \beta)}{\beta + \mu \beta - \mu^2 - 2 \mu \beta^2}, \\
x & \text{otherwise},
\end{cases}
\]

with

\[
x = \frac{\sqrt{2 - \theta}}{2} \sqrt{\frac{(2 - \theta)\mu(\beta + 1) - \theta \beta(\mu - 1))((2 - \theta)\beta(\mu + 1) - \theta \mu(\beta - 1))}{(2 - \theta)\mu \beta(\mu + 1)(\beta + 1) - \theta \mu^2 \beta^2}}.
\]
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|1 - \theta| & \text{if } \theta \geq 2 \frac{\mu \beta + \mu + \beta}{2 \mu \beta + \mu + \beta}, \\
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X & \text{otherwise,}
\end{cases}$$

with

$$X = \frac{\sqrt{2 - \theta}}{2} \sqrt{\frac{(2 - \theta)\mu(\beta + 1) - \theta \beta(\mu - 1))((2 - \theta)\beta(\mu + 1) - \theta \mu(\beta - 1))}{(2 - \theta)\mu \beta(\mu + 1)(\beta + 1) - \theta \mu^2 \beta^2}}.$$
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|1 - \theta| & \text{if } \theta \geq 2 \frac{\mu \beta + \mu + \beta}{2 \mu \beta + \mu + \beta}, \\
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$$X = \frac{\sqrt{2 - \theta}}{2} \sqrt{\frac{(2 - \theta)(\mu(\beta + 1) - \theta \beta(\mu - 1))((2 - \theta)\beta(\mu + 1) - \theta \mu(\beta - 1))}{(2 - \theta)\mu \beta(\mu + 1)(\beta + 1) - \theta \mu^2 \beta^2}}.$$ 

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Douglas-Rachford Splitting

Assumptions: $A \mu$-strongly monotone, $B \beta$-cocoercive.

Examples on which those bounds are attained?

\[ A = N \{0\} \text{ (i.e., } J_{\lambda A} = 0), \quad B = 1 \beta I \text{ for } \rho = |1 - \theta\beta\beta + 1|. \]

\[ A = \mu I, \quad B = 1 \beta I \text{ for } \rho = |1 - \theta\beta + \mu\beta(\mu + 1)(\beta + 1)|. \]

\[ A = N \{0\}, \quad B = 0 \text{ for } \rho = |1 - \theta|. \]

\[ A = \mu I, \quad B = 0 \text{ for } \rho = |1 - \theta\mu + \mu| \text{.} \]

\[ \text{Case 5: (2-dimensional) for appropriate (complicated) values of } a \text{ and } K: \]

\[ A = (\mu - a\mu), \quad B = (\beta K - \sqrt{K - K^2\beta^2}) \text{, for } \rho = \sqrt{2 - \theta^2 \sqrt{(2 - \theta)\mu(\beta + 1) - \theta\beta(\mu - 1)(\beta + 1)(\mu + 1) - \theta\mu^2\beta^2}}. \]
Douglas-Rachford Splitting

Assumptions: $A$ $\mu$-strongly monotone, $B$ $\beta$-cocoercive.

Examples on which those bounds are attained?

\[\begin{array}{l}
\diamondsuit \text{ Case 1: (1-dimensional) } A = N_{\{0\}} \text{ (i.e., } J_{\lambda A} = 0), \quad B = \frac{1}{\beta} I \text{ for } \rho = |1 - \theta \frac{\beta}{\beta + 1}|. \\
\end{array}\]
Douglas-Rachford Splitting

Assumptions: $A$ $\mu$-strongly monotone, $B$ $\beta$-cocoercive.

Examples on which those bounds are attained?

- Case 1: (1-dimensional) $A = N\{0\}$ (i.e., $J_{\lambda A} = 0$), $B = \frac{1}{\beta} I$ for $\rho = |1 - \theta \frac{\beta}{\beta+1}|$.
- Case 2: (1-dimensional) $A = \mu I$, $B = \frac{1}{\beta} I$ for $\rho = |1 - \theta \frac{1+\mu\beta}{(\mu+1)(\beta+1)}|$. 
Douglas-Rachford Splitting

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- Case 3: (1-dimensional) $A = N_{\{0\}}$, $B = 0$ for $\rho = |1 - \theta|$.
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- Case 3: (1-dimensional) $A = N_{\{0\}}$, $B = 0$ for $\rho = |1 - \theta|$.
- Case 4: (1-dimensional) $A = \mu I$, $B = 0$ for $\rho = |1 - \theta \frac{\mu}{\mu+1}|$. 

- Case 5: (2-dimensional) for appropriate (complicated) values of $a$ and $K$:

  $$A = (\mu - a \alpha) I, B = (\beta K - \sqrt{K} - K^2 \beta^2 \sqrt{K} - K^2 \beta) I,$$

  for $\rho = \sqrt{2} - \theta \sqrt{((2 - \theta)^\mu (\beta + 1) - \theta \beta (\mu - 1)) (2 - \theta)(\mu + 1)(\beta + 1) - \theta \mu^2 \beta^2}$. 


Douglas-Rachford Splitting

Assumptions: $A \mu$-strongly monotone, $B \beta$-cocoercive.

Examples on which those bounds are attained?

- Case 1: (1-dimensional) $A = N\{0\}$ (i.e., $J_{\lambda A} = 0$), $B = \frac{1}{\beta} I$ for $\rho = |1 - \theta \frac{\beta}{\beta+1}|$.
- Case 2: (1-dimensional) $A = \mu I$, $B = \frac{1}{\beta} I$ for $\rho = |1 - \theta \frac{1+\mu\beta}{(\mu+1)(\beta+1)}|$.
- Case 3: (1-dimensional) $A = N\{0\}$, $B = 0$ for $\rho = |1 - \theta|$.
- Case 4: (1-dimensional) $A = \mu I$, $B = 0$ for $\rho = |1 - \theta \frac{\mu}{\mu+1}|$.
- Case 5: (2-dimensional) for appropriate (complicated) values of $a$ and $K$:

$$A = \begin{pmatrix} \mu & -a \\ a & \mu \end{pmatrix}, \quad B = \begin{pmatrix} \beta K & -\sqrt{K - K^2\beta^2} \\ \sqrt{K - K^2\beta^2} & \beta K \end{pmatrix},$$

for $\rho = \frac{\sqrt{2 - \theta}}{2} \sqrt{\frac{(2 - \theta)\mu(\beta+1) - \theta \beta(\mu - 1))((2 - \theta)\beta(\mu+1) - \theta \mu(\beta - 1))}{(2 - \theta)\mu \beta(\mu+1)(\beta+1) - \theta \mu^2 \beta^2}}$. 


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Assumptions: $A$ $\mu$-strongly monotone, $B$ $L$-Lipschitz and monotone.
Douglas-Rachford Splitting

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We have $\|Tx - Ty\| \leq \rho\|x - y\|$ for all $x, y \in \mathcal{H}$ with:

$$\rho = \begin{cases} 
\theta + \sqrt{(2(\theta - 1)\mu + \theta - 2)^2 + L^2(\theta - 2(\mu + 1))} & \text{if } (a), \\
\sqrt{L^2 + 1} & \text{if } (b), \\
\sqrt{2 - \theta} & \text{otherwise,}
\end{cases}$$

where

- $(a)$ $\mu - \left(2(\theta - 1)\mu + \theta - 2\right)^2 + L^2(\theta - 2(\mu + 1)) \leq \sqrt{L^2 + 1}$,
- $(b)$ $L < 1$, $\mu > L^2 + 1$, and
- $\theta \leq 2(\mu + 1)(L + 1)(\mu + \mu L - L - 1)^2$.
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We have $\|Tx - Ty\| \leq \rho \|x - y\|$ for all $x, y \in \mathcal{H}$ with:

$$
\rho = \begin{cases}
\theta + \sqrt{\frac{(2(\theta-1)\mu+\theta-2)^2 + L^2(\theta-2(\mu+1))^2}{L^2+1}} \frac{2(\mu+1)}{L^2+1} & \text{if (a),}
|1 - \theta \frac{L+\mu}{(\mu+1)(L+1)}| & \text{if (b),}
\sqrt{\frac{(2-\theta)}{4\mu(L^2+1)}} \frac{(\theta(L^2+1)-2\mu(\theta+L^2-1))(\theta(1+2\mu+L^2)-2(\mu+1)(L^2+1))}{2\mu(\theta+L^2-1)-(2-\theta)(1-L^2)} & \text{otherwise,}
\end{cases}
$$

with

(a) $\mu \frac{-(2(\theta-1)\mu+\theta-2)+L^2(\theta-2(1+\mu))}{\sqrt{(2(\theta-1)\mu+\theta-2)^2 + L^2(\theta-2(\mu+1))^2}} \leq \sqrt{L^2+1}$,

(b) $L < 1$, $\mu > \frac{L^2+1}{(L-1)^2}$, and $\theta \leq \frac{2(\mu+1)(L+1)(\mu+\mu L^2-L^2-2\mu L-1)}{2\mu^2 - \mu+\mu L^3 - L^3 - 3\mu L^2 - L^2 - 2\mu^2 L - \mu L - L - 1}$. 

⋄ First and third cases are achieved on 2-dimensional examples (dual is simpler),

⋄ Second case is achieved on 1-dimensional example (primal is simpler).
Douglas-Rachford Splitting

Assumptions: $A$ $\mu$-strongly monotone, $B$ $L$-Lipschitz and monotone.

We have $\|Tx - Ty\| \leq \rho\|x - y\|$ for all $x, y \in H$ with:

$$\rho = \begin{cases} 
\theta + \sqrt{\frac{(2(\theta - 1)\mu + \theta - 2)^2 + L^2(\theta - 2(\mu + 1))^2}{L^2 + 1}} - \frac{L + \mu}{2(\mu + 1)} & \text{if (a),} \\
|1 - \theta \frac{L + \mu}{(\mu + 1)(L + 1)}| & \text{if (b),} \\
\sqrt{\frac{2 - \theta}{4\mu(L^2 + 1)}} \left(\frac{\theta(L^2 + 1) - 2\mu(\theta + L^2 - 1)}{2\mu(\theta + L^2 - 1) - (2 - \theta)(1 - L^2)}\right) & \text{otherwise,}
\end{cases}$$

with

(a) $\mu \leq \frac{(2(\theta - 1)\mu + \theta - 2) + L^2(\theta - 2(1 + \mu))}{\sqrt{2(\theta - 1)\mu + \theta - 2}^2 + L^2(\theta - 2(\mu + 1))^2} \leq \sqrt{L^2 + 1},$

(b) $L < 1$, $\mu > \frac{L^2 + 1}{(L - 1)^2}$, and $\theta \leq \frac{2(\mu + 1)(L + 1)(\mu + \mu L^2 - L^2 - 2\mu L - 1)}{2\mu^2 - \mu + \mu L^3 - \mu L^3 - 3\mu L^2 - L^2 - 2\mu^2 L - \mu L - L - 1}.$

◊ First and third cases are achieved on 2-dimensional examples (dual is simpler).
Douglas-Rachford Splitting

Assumptions: A $\mu$-strongly monotone, $B L$-Lipschitz and monotone.

We have $\|Tx - Ty\| \leq \rho \|x - y\|$ for all $x, y \in \mathcal{H}$ with:

$$
\rho = \begin{cases} 
\theta + \sqrt{\frac{(2(\theta - 1)\mu + \theta - 2)^2 + L^2(\theta - 2(\mu + 1))^2}{L^2 + 1}} \frac{1}{2(\mu + 1)} & \text{if (a),} \\
|1 - \theta \frac{L + \mu}{(\mu + 1)(L + 1)}| & \text{if (b),} \\
\sqrt{\frac{(2 - \theta)}{4\mu(L^2 + 1)}} \frac{(\theta(L^2 + 1) - 2\mu(\theta - 2) - 2(\mu + 1))}{2\mu(\theta + L^2 - 1) - (2 - \theta)(1 - L^2)} & \text{otherwise,}
\end{cases}
$$

with

(a) $\mu \frac{- (2(\theta - 1)\mu + \theta - 2) + L^2(\theta - 2(1 + \mu))}{\sqrt{(2(\theta - 1)\mu + \theta - 2)^2 + L^2(\theta - 2(\mu + 1))^2}} \leq \sqrt{L^2 + 1},$

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◊ First and third cases are achieved on 2-dimensional examples (dual is simpler),
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Douglas-Rachford Splitting

Assumptions: $A$ $\mu$-strongly monotone, $B$ $L$-Lipschitz and monotone.
Douglas-Rachford Splitting

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Examples on which those bounds are attained?

\[ \rho = \sqrt{2 - \theta} \frac{\mu}{\theta + L^2 - 1} \left( L^2 + 1 + 2\mu \left(\theta + L^2 - 1\right) \right)^2 \mu \left(\theta + L^2 - 1\right) - (2 - \theta)(1 - L^2). \]
Douglas-Rachford Splitting

Assumptions: $A$ $\mu$-strongly monotone, $B$ $L$-Lipschitz and monotone.

Examples on which those bounds are attained?

⋄ Case 1: (2-dimensional) We choose (see also Moursi & Vandenberghe 2018)

$$A = \mu I + N_{\{0\} \times \mathbb{R}}, \quad B = L \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

for $\rho = \frac{\theta + \sqrt{2(\theta - 1)\mu + \theta + 2} + L^2(\theta - 2(\mu + 1))^2}{L^2 + 1}$

$$2(\mu + 1)$$
Douglas-Rachford Splitting

Assumptions: $A$ $\mu$-strongly monotone, $B$ $L$-Lipschitz and monotone.

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- Case 1: (2-dimensional) We choose (see also Moursi & Vandenberghe 2018)

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for $\rho = \frac{\theta + \sqrt{(2(\theta-1)\mu+\theta-2)^2 + L^2(\theta-2(\mu+1))^2}}{2(\mu+1)}$.

- Case 2: (1-dimensional) $A = \mu I$, $B = LI$ for $\rho = |1 - \theta \frac{L+\mu}{(\mu+1)(L+1)}|$
Douglas-Rachford Splitting

Assumptions: $A$ $\mu$-strongly monotone, $B$ $L$-Lipschitz and monotone.

Examples on which those bounds are attained?

- **Case 1:** (2-dimensional) We choose (see also Moursi & Vandenberghe 2018)

  $$A = \mu I + N_{\{0\} \times \mathbb{R}}, \quad B = L \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

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- **Case 2:** (1-dimensional) $A = \mu l$, $B = l I$ for $\rho = |1 - \theta \frac{L + \mu}{(\mu + 1)(L + 1)}|$

- **Case 3:** (2-dimensional) For appropriately chosen (complicated) $K$:

  $$A = \mu I + N_{\mathbb{R} \times \{0\}}, \quad B = L \begin{pmatrix} K & -\sqrt{1 - K^2} \\ \sqrt{1 - K^2} & K \end{pmatrix},$$

  for $\rho = \sqrt{\frac{(2 - \theta) L^2 + 1}{4\mu(L^2 + 1)}} \frac{(\theta L^2 + 1 - 2\mu(\theta + L^2 - 1))(\theta(1 + 2\mu + L^2) - 2(\mu + 1)(L^2 + 1))}{2\mu(\theta + L^2 - 1) - (2 - \theta)(1 - L^2)}$. 


Avoiding semidefinite programming modeling steps?
Avoiding semidefinite programming modeling steps?

François Glineur (UCLouvain)  Julien Hendrickx (UCLouvain)

“Performance Estimation Toolbox (PESTO): automated worst-case analysis of first-order optimization methods” (CDC 2017)
PESTO example: contraction factors for DRS

```matlab
% (0) Initialize an empty PEP
P=pep();

N = 1;
% (1) Set up the class of monotone inclusions
paramA.L = 1; paramA.mu = 0; % A is 1-Lipschitz and 0-strongly monotone
paramB.mu = .1; % B is .1-strongly monotone

A = P.DeclareFunction('LipschitzStronglyMonotone',paramA);
B = P.DeclareFunction('StronglyMonotone',paramB);

w = cell(N+1,1); wp = cell(N+1,1);
x = cell(N,1); xp = cell(N,1);
y = cell(N,1); yp = cell(N,1);

% (2) Set up the starting points
w{1} = P.StartingPoint(); wp{1} = P.StartingPoint();
P.InitialCondition(((w{1}-wp{1})^2<=1);

% (3) Algorithm
lambda = 1.3; % step size (in the resolvents)
theta = .9; % overrelaxation

for k = 1 : N
    x{k} = proximal_step(w{k},B,lam); % proximal step with Lipschitz constant
    y{k} = proximal_step(2*x{k} - w{k},A,lam);
    w{k+1} = w{k} - theta*(x{k} - y{k});

    xp{k} = proximal_step(wp{k},B,lam);
    yp{k} = proximal_step(2*xp{k} - wp{k},A,lam);
    wp{k+1} = wp{k} - theta*(xp{k} - yp{k});
end

% (4) Set up the performance measure: ||z0-z1||^2
P.PerformanceMetric(((w{k+1}-wp{k+1})^2);

% (5) Solve the PEP
P.solve()

% (6) Evaluate the output
double((w{k+1}-wp{k+1})^2) % worst-case contraction factor
```
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x{k} = proximal_step(w{k}, B, lambda);
y{k} = proximal_step(2*x{k} - w{k}, A, lambda);
w{k+1} = w{k} - theta*(x{k} - y{k});
xp{k} = proximal_step(wp{k}, B, lambda);
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![Graph](image_url)
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✔ fast prototyping (∼ 20 effective lines)
✔ quick analyses (∼ 10 minutes)
✔ computer-aided proofs (multipliers)
Current library of examples within PESTO

Includes... but not limited to

- subgradient, gradient, heavy-ball, fast gradient, optimized gradient methods,
- proximal point algorithm,
- projected and proximal gradient, accelerated/momentum versions,
- steepest descent, greedy/conjugate gradient methods,
- Douglas-Rachford/three operator splitting,
- Frank-Wolfe/conditional gradient,
- inexact gradient/fast gradient,
- Krasnoselskii-Mann and Halpern fixed-point iterations.

Upcoming (soon): SAG, SAGA, SGD and variants.

PESTO contains most of the recent PEP-related advances (including techniques by other groups). Clean updated references in user manual.

Among others, see works by Drori, Teboulle, Kim, Fessler, Lieder, Lessard, Recht, Packard, Van Scoy, etc.
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### Classes of functions/operators within PESTO

<table>
<thead>
<tr>
<th>Functional class</th>
<th>Guaranteed tight PEP</th>
</tr>
</thead>
<tbody>
<tr>
<td>Convex functions</td>
<td>✔</td>
</tr>
<tr>
<td>Convex functions (poss. bounded subdifferentials)</td>
<td>✔</td>
</tr>
<tr>
<td>Convex indicator functions (poss. bounded domain)</td>
<td>✔</td>
</tr>
<tr>
<td>Convex support functions (poss. bounded subdifferentials)</td>
<td>✔</td>
</tr>
<tr>
<td>Smooth strongly convex functions</td>
<td>✔</td>
</tr>
<tr>
<td>Smooth (possibly nonconvex) functions</td>
<td>✔</td>
</tr>
<tr>
<td>Smooth convex functions (poss. bounded subdifferentials)</td>
<td>✔</td>
</tr>
<tr>
<td>Strongly convex functions (poss. bounded domain)</td>
<td>✔</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Operator class</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Monotone (maximally)</td>
<td>✔</td>
</tr>
<tr>
<td>Strongly monotone (maximally)</td>
<td>✔</td>
</tr>
<tr>
<td>Cocoercive</td>
<td>✔</td>
</tr>
<tr>
<td>Lipschitz</td>
<td>✔</td>
</tr>
<tr>
<td>Cocoercive and strongly monotone*</td>
<td>✗</td>
</tr>
<tr>
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<td>✗</td>
</tr>
</tbody>
</table>

*: for some cases (e.g., DRS/TOS’s contraction factors), still tight.
Concluding remarks

Performance estimation’s philosophy
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◊ numerically allows obtaining tight bounds, rigorous baselines for proofs!
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◊ numerically allows obtaining tight bounds,
  rigorous baselines for proofs!
◊ helps designing analytical proofs (reduces to linear combinations of inequalities),
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- fast prototyping:
Concluding remarks

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  before trying to prove your crazy-algorithm works; give PEP a try!
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Concluding remarks

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Interested? Presentation mainly based on:
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◊ T., Hendrickx, Glineur. “Performance Estimation Toolbox (PESTO): automated worst-case analysis of first-order optimization methods” (CDC 2017) [In the paper: presentation of the toolbox for first-order optimization methods]
Thanks! Questions?

www.di.ens.fr/~ataylor/

AdrienTaylor/Performance-Estimation-Toolbox on Github