Computer-aided worst-case analyses for operator splitting

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(Lund)

"Operator splitting performance estimation: Tight contraction factors and optimal parameter selection" (2018, arXiv:1812.00146)

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Worst-case analyses for operator splitting (here: Douglas-Rachford)

(Douglas & Rachford 1956), (Lions & Mercier 1979), (Giselsson & Boyd 2017), (Giselsson 2017), (Moursi & Vandenberghe 2018), and many others.

Take-home messages

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Often tractable for first-order methods in optimization and monotone inclusions!

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Let f and h be two convex, closed, proper functions. (Overrelaxed) DRS for solving

$$\min_{x \in \mathbb{R}^d} f(x) + h(x),$$

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Recover optimization setting with $A = \partial f$ and $B = \partial h$.

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Question: When is the DRS iteration a contraction? What is the smallest ρ such that

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for all $w_0, w_0' \in \mathbb{R}^d$ and w_1, w_1' generated with DRS from respectively w_0 and w_0'?

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- Optimization problem to find sharp contraction factor:

$$\begin{aligned} \underset{A,B,w_0,w_0',w_1,w_1'}{\text{maximize}} \quad & \frac{\|w_1 - w_1'\|}{\|w_0 - w_0'\|} \\ \text{subject to} \quad & w_1 \text{ generated by DR from } w_0, \\ & w_1' \text{ generated by DR from } w_0', \\ & \text{assumptions on } A \text{ and } B. \end{aligned}$$

  which has operators A and B as variables.

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- ◇ A convex function f is commonly assumed to be (for all $x, y \in \mathbb{R}^d$):
  - ◇ μ-strongly convex $\quad f(x) \geq f(y) + \langle \partial f(y), x - y \rangle + \frac{\mu}{2}\|x - y\|^2,$
  - ◇ L-smooth $\quad f(x) \leq f(y) + \langle f'(y), x - y \rangle + \frac{L}{2}\|x - y\|^2.$

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- A max. monotone operators B is commonly assumed to be (for all $x, y \in \mathbb{R}^d$):
  - μ-strongly monotone $\quad \langle B(x) - B(y), x - y \rangle \geq \mu\|x - y\|^2$,
  - β-cocoercive $\quad \langle B(x) - B(y), x - y \rangle \geq \beta\|B(x) - B(y)\|^2$,
  - L-Lipschitz $\quad \|B(x) - B(y)\| \leq L\|x - y\|$.

DR contraction factors

Table: Contraction factors for DR: assumptions beyond max. monotonicity.

#	Properties for A	Properties for B	Reference	Sharp	Notes
O1	∂f, f: str. cvx & smooth	∂g	[1,2]	✔	
O2	∂f, f: str. cvx	∂g, g: smooth	[3]	✖	1.
M1	str. mono. & cocoercive	-	[3]	✔	
M2	str. mono. & Lipschitz	-	[3]	✔	2.
M3	str. mono.	cocoercive	[3]	✖	
M4	str. mono.	Lipschitz	[4]	✖	3.

1. sharp rates for some parameter choices in [3]
2. Lions and Mercier [5] provided conservative rate in this setting
3. sharp rate when B is skew linear in [4]

[1] Giselsson, Boyd, Diagonal Scaling in DRS and ADMM, 2014.
[2] Giselsson, Boyd, Linear Convergence and Metric Selection in DRS and ADMM, 2017.
[3] Giselsson, Tight Global Linear Convergence Rate Bounds for DRS, 2017.
[4] Moursi, Vandenberghe. DRS for a Lipschitz continuous and a strongly monotone operator, 2018.
[5] Lions, Mercier. Splitting Algorithms for the Sum of Two Nonlinear Operators, 1979.

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- ⋄ Optimal value can be found via convex optimization! (3x3 SDP)

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- Recall DR splitting:

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- Infinite-dimensional problem: two operators as variables!

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& w_1 = w_0 + \theta(y_1 - x_1), \\
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\end{aligned}
$$

- How to remove existence constraints?

Interpolation of operators

Interpolation of operators

- Define the duplets (x, x_+) and (y, y_+). Then

$$\langle x - y, x_+ - y_+ \rangle \geq (\gamma\mu + 1)\|x_+ - y_+\|^2$$

  iff there exists a μ-strongly monotone operator A such that
  - $x_+ = J_{\gamma A}(x)$
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$$\langle x - y, x_+ - y_+ \rangle \geq \frac{\beta}{\gamma}\|x - x_+ - (y - y_+)\|^2 + \|x_+ - y_+\|^2$$

  iff there exists a β-cocoercive operator B such that
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Replace constraints

Replace constraints

- Interpolation conditions allows to remove red constraints

$$
\begin{aligned}
\underset{\substack{w_0, w_0', w_1, w_1' \\ x_1, x_1', y_1, y_1'}}{\text{maximize}} \quad & \frac{\|w_1 - w_1'\|}{\|w_0 - w_0'\|} \\
\text{subject to} \quad & \exists B \ \beta\text{-cocoercive such that } \begin{cases} x_1 = J_{\gamma B}(w_0), \\ x_1' = J_{\gamma B}(w_0'), \end{cases} \\
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& w_1 = w_0 + \theta(y_1 - x_1), \\
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\end{aligned}
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$$\underset{\substack{w_0,w_0',w_1,w_1'\\x_1,x_1',y_1,y_1'}}{\text{maximize}} \quad \frac{\|w_1 - w_1'\|}{\|w_0 - w_0'\|}$$

$$\text{subject to} \quad \exists B \ \beta\text{-cocoercive such that } \begin{cases} x_1 = J_{\gamma B}(w_0), \\ x_1' = J_{\gamma B}(w_0'), \end{cases}$$

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$$w_1 = w_0 + \theta(y_1 - x_1),$$

$$w_1' = w_0' + \theta(y_1' - x_1').$$

◇ replacing them by:

$$\langle y_1 - y_1', 2(x_1 - x_1') - (w_0 - w_0')\rangle \geq (\gamma\mu + 1)\|y_1 - y_1'\|^2,$$

and

$$\langle w_0 - w_0', x_1 - x_1'\rangle \geq \tfrac{\beta}{\gamma}\|w_0 - w_0' - (x_1 - x_1')\|^2 + \|x_1 - x_1'\|^2.$$

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◇ Note: optimal value is the same! No relaxation.

Reformulations (cont'd)

Reformulations (cont'd)

- Equivalent problem without operator class constraints:

$$
\begin{array}{ll}
\underset{\substack{w_0, w_0', w_1, w_1' \\ x_1, x_1', y_1, y_1'}}{\text{maximize}} & \dfrac{\|w_1 - w_1'\|}{\|w_0 - w_0'\|} \\
\text{subject to} & \left\langle y_1 - y_1', 2(x_1 - x_1') - (w_0 - w_0') \right\rangle \geq (\gamma\mu + 1)\left\|y_1 - y_1'\right\|^2, \\
& \left\langle w_0 - w_0', x_1 - x_1' \right\rangle \geq \frac{\beta}{\gamma}\left\|w_0 - w_0' - (x_1 - x_1')\right\|^2 + \left\|x_1 - x_1'\right\|^2, \\
& w_1 = w_k + \theta(y_1 - x_1), \\
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Reformulations (cont'd)

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\end{aligned}
$$

◇ Yet another reformulation

$$
\begin{aligned}
\underset{\substack{w_0, w_0' \\ x_1, x_1', y_1, y_1'}}{\text{maximize}} \quad & \frac{\|w_0 + \theta(y_1 - x_1) - w_0 - \theta(y_1' - x_1')\|^2}{\|w_0 - w_0'\|^2} \\
\text{subject to} \quad & \left\langle y_1 - y_1', 2(x_1 - x_1') - (w_0 - w_0') \right\rangle \geq (\gamma\mu + 1)\left\|y_1 - y_1'\right\|^2, \\
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\end{aligned}
$$

Semidefinite lifting

Semidefinite lifting

- All parts of optimization problem are quadratic:

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\end{aligned}
$$

- They can therefore be represented with a Gram matrix. Let

$$
G = \begin{bmatrix}
\|w_0 - w_0'\|^2 & \langle w_0 - w_0', x_1 - x_1' \rangle & \langle w_0 - w_0', y_1 - y_1' \rangle \\
\langle x_1 - x_1', w_0 - w_0' \rangle & \|x_1 - x_1'\|^2 & \langle x_1 - x_1', y_1 - y_1' \rangle \\
\langle y_1 - y_1', w_0 - w_0' \rangle & \langle y_1 - y_1', x_1 - x_1' \rangle & \|y_1 - y_1'\|^2
\end{bmatrix}
$$

where $G \succeq 0$ by construction

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$$G = \begin{bmatrix} \|w_0 - w_0'\|^2 & \langle w_0 - w_0', x_1 - x_1'\rangle & \langle w_0 - w_0', y_1 - y_1'\rangle \\ \langle x_1 - x_1', w_0 - w_0'\rangle & \|x_1 - x_1'\|^2 & \langle x_1 - x_1', y_1 - y_1'\rangle \\ \langle y_1 - y_1', w_0 - w_0'\rangle & \langle y_1 - y_1', x_1 - x_1'\rangle & \|y_1 - y_1'\|^2 \end{bmatrix}$$

where $G \succeq 0$ by construction, and reformulate to:

$$\begin{array}{ll} \underset{G}{\text{maximize}} & \dfrac{\mathrm{Tr}(A_o G)}{\mathrm{Tr}(A_s G)} \\ \text{subject to} & \mathrm{Tr}(A_1 G) \geq 0 \\ & \mathrm{Tr}(A_2 G) \geq 0 \\ & G \succeq 0. \end{array}$$

with appropriate A_o, A_s, A_1, A_2 for picking correct elements in G

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  with appropriate A_o, A_s, A_1, A_2 for picking correct elements in G
- Note: assuming $w_0, w_0', x_1, x_1', y_1, y_1' \in \mathbb{R}^d$ with $d \geq 3$, same optimal cost!

Last part in convexification

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- The constraints are positively homogeneous of deg. 1 and the cost is constant under scaling of G

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& G \succeq 0.
\end{array}
$$

- Therefore an equivalent *convex* problem is

$$
\begin{array}{ll}
\underset{G}{\text{maximize}} & \mathrm{Tr}(A_o G) \\
\text{subject to} & \mathrm{Tr}(A_1 G) \geq 0 \\
& \mathrm{Tr}(A_2 G) \geq 0 \\
& \mathrm{Tr}(A_s G) = 1 \\
& G \succeq 0.
\end{array}
$$

which is a 3x3 semidefinite program.

Dual problem

Dual problem

- Introduce dual variables τ, λ_1 and λ_2

$$
\begin{array}{lll}
\underset{G}{\text{maximize}} & \mathrm{Tr}(A_o G) & \\
\text{subject to} & \mathrm{Tr}(A_1 G) \geq 0 & : \lambda_1 \\
& \mathrm{Tr}(A_2 G) \geq 0 & : \lambda_2 \\
& \mathrm{Tr}(A_s G) = 1 & : \tau \\
& G \succeq 0 &
\end{array}
$$

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$$\begin{array}{ll} \underset{\tau,\lambda_1,\lambda_2}{\text{minimize}} & \tau \\ \text{subject to} & \lambda_i \geq 0 \\ & S = A_o + \sum_{i=1}^{2} \lambda_i A_i - \tau A_s \preceq 0 \end{array}$$

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◇ In this example:

$$S = \begin{bmatrix} -\tau - \frac{\beta\lambda_2}{\gamma} + 1 & -\theta + \frac{\lambda_2}{2} + \frac{\beta\lambda_2}{\gamma} & \theta - \frac{\lambda_1}{2} \\ -\theta + \frac{\lambda_2}{2} + \frac{\beta\lambda_2}{\gamma} & \theta^2 - \lambda_2 - \frac{\beta\lambda_2}{\gamma} & \lambda_1 - \theta^2 \\ \theta - \frac{\lambda_1}{2} & \lambda_1 - \theta^2 & \theta^2 - \lambda_1 - \gamma\lambda_1\mu \end{bmatrix}$$

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- Strong duality holds (existence of a Slater point): $\mathrm{rank}(G) + \mathrm{rank}(S) \leq 3$.

A few more examples

Warning for the next few slides:

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- ◇ expressions are horrible,

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Intuitions can be developed, but this is another story ☺

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Note I: the methodology offers 3 ways to proceed:

- ◇ play with primal formulation,
- ◇ play with primal-dual saddle-point formulation,
- ◇ play with dual formulation.

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Note I: the methodology offers 3 ways to proceed:

- ◇ play with primal formulation,
- ◇ play with primal-dual saddle-point formulation,
- ◇ play with dual formulation.

Note II: that any dual feasible point can be translated into a “traditional” proof.

Douglas-Rachford Splitting

Assumptions: A μ-strongly monotone, B β-cocoercive.

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$$\rho = \begin{cases} |1 - \theta\frac{\beta}{\beta+1}| & \text{if } \mu\beta - \mu + \beta < 0, \text{ and } \theta \leq 2\frac{(\beta+1)(\mu-\beta-\mu\beta)}{\mu+\mu\beta-\beta-\beta^2-2\mu\beta^2}, \\ |1 - \theta\frac{1+\mu\beta}{(\mu+1)(\beta+1)}| & \text{if } \mu\beta - \mu - \beta > 0, \text{ and } \theta \leq 2\frac{\mu^2+\beta^2+\mu\beta+\mu+\beta-\mu^2\beta^2}{\mu^2+\beta^2+\mu^2\beta+\mu\beta^2+\mu+\beta-2\mu^2\beta^2}, \\ |1 - \theta| & \text{if } \theta \geq 2\frac{\mu\beta+\mu+\beta}{2\mu\beta+\mu+\beta}, \\ |1 - \theta\frac{\mu}{\mu+1}| & \text{if } \mu\beta + \mu - \beta < 0, \text{ and } \theta \leq 2\frac{(\mu+1)(\beta-\mu-\mu\beta)}{\beta+\mu\beta-\mu-\mu^2-2\mu^2\beta}, \\ X & \text{otherwise,} \end{cases}$$

Douglas-Rachford Splitting

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with

$$X = \frac{\sqrt{2-\theta}}{2}\sqrt{\frac{((2-\theta)\mu(\beta+1)-\theta\beta(\mu-1))\,((2-\theta)\beta(\mu+1)-\theta\mu(\beta-1))}{(2-\theta)\mu\beta(\mu+1)(\beta+1)-\theta\mu^2\beta^2}}.$$

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- ◇ The first four cases are achieved on 1-dimensional examples (primal is simpler).

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with

$$X = \frac{\sqrt{2-\theta}}{2}\sqrt{\frac{((2-\theta)\mu(\beta+1)-\theta\beta(\mu-1))\,((2-\theta)\beta(\mu+1)-\theta\mu(\beta-1))}{(2-\theta)\mu\beta(\mu+1)(\beta+1)-\theta\mu^2\beta^2}}.$$

- ◇ The first four cases are achieved on 1-dimensional examples (primal is simpler).
- ◇ Fifth case is achieved on 2-dimensional example (dual is simpler).

Douglas-Rachford Splitting

Assumptions: A μ-strongly monotone, B β-cocoercive.

Douglas-Rachford Splitting

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Examples on which those bounds are attained?

Douglas-Rachford Splitting

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Examples on which those bounds are attained?

- ◇ Case 1: (1-dimensional) $A = N_{\{0\}}$ (i.e., $J_{\lambda A} = 0$), $B = \frac{1}{\beta} I$ for $\rho = |1 - \theta \frac{\beta}{\beta+1}|$.

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- Case 2: (1-dimensional) $A = \mu I$, $B = \frac{1}{\beta} I$ for $\rho = |1 - \theta \frac{1+\mu\beta}{(\mu+1)(\beta+1)}|$.

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- Case 2: (1-dimensional) $A = \mu I$, $B = \frac{1}{\beta} I$ for $\rho = |1 - \theta \frac{1+\mu\beta}{(\mu+1)(\beta+1)}|$.
- Case 3: (1-dimensional) $A = N_{\{0\}}$, $B = 0$ for $\rho = |1 - \theta|$.

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- Case 2: (1-dimensional) $A = \mu I$, $B = \frac{1}{\beta} I$ for $\rho = |1 - \theta \frac{1+\mu\beta}{(\mu+1)(\beta+1)}|$.
- Case 3: (1-dimensional) $A = N_{\{0\}}$, $B = 0$ for $\rho = |1 - \theta|$.
- Case 4: (1-dimensional) $A = \mu I$, $B = 0$ for $\rho = |1 - \theta \frac{\mu}{\mu+1}|$.

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Examples on which those bounds are attained?

- ◇ Case 1: (1-dimensional) $A = N_{\{0\}}$ (i.e., $J_{\lambda A} = 0$), $B = \frac{1}{\beta} I$ for $\rho = |1 - \theta \frac{\beta}{\beta+1}|$.
- ◇ Case 2: (1-dimensional) $A = \mu I$, $B = \frac{1}{\beta} I$ for $\rho = |1 - \theta \frac{1+\mu\beta}{(\mu+1)(\beta+1)}|$.
- ◇ Case 3: (1-dimensional) $A = N_{\{0\}}$, $B = 0$ for $\rho = |1 - \theta|$.
- ◇ Case 4: (1-dimensional) $A = \mu I$, $B = 0$ for $\rho = |1 - \theta \frac{\mu}{\mu+1}|$.
- ◇ Case 5: (2-dimensional) for appropriate (complicated) values of a and K:

$$A = \begin{pmatrix} \mu & -a \\ a & \mu \end{pmatrix}, \qquad B = \begin{pmatrix} \beta K & -\sqrt{K - K^2\beta^2} \\ \sqrt{K - K^2\beta^2} & \beta K \end{pmatrix},$$

for $\rho = \frac{\sqrt{2-\theta}}{2} \sqrt{\frac{((2-\theta)\mu(\beta+1) - \theta\beta(\mu-1))\,((2-\theta)\beta(\mu+1) - \theta\mu(\beta-1))}{(2-\theta)\mu\beta(\mu+1)(\beta+1) - \theta\mu^2\beta^2}}$.

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Assumptions: A μ-strongly monotone, B L-Lipschitz and monotone.

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We have $\|Tx - Ty\| \leq \rho \|x - y\|$ for all $x, y \in \mathcal{H}$ with:

$$\rho = \begin{cases} \dfrac{\theta + \sqrt{\frac{(2(\theta-1)\mu+\theta-2)^2 + L^2(\theta-2(\mu+1))^2}{L^2+1}}}{2(\mu+1)} & \text{if } (a), \\[2ex] \left|1 - \theta \frac{L+\mu}{(\mu+1)(L+1)}\right| & \text{if } (b), \\[2ex] \sqrt{\frac{(2-\theta)}{4\mu(L^2+1)} \frac{\left(\theta(L^2+1) - 2\mu(\theta+L^2-1)\right)\left(\theta\left(1+2\mu+L^2\right) - 2(\mu+1)\left(L^2+1\right)\right)}{2\mu(\theta+L^2-1) - (2-\theta)(1-L^2)}} & \text{otherwise}, \end{cases}$$

with

(a) $\mu \frac{-(2(\theta-1)\mu+\theta-2)+L^2(\theta-2(1+\mu))}{\sqrt{(2(\theta-1)\mu+\theta-2)^2+L^2(\theta-2(\mu+1))^2}} \leq \sqrt{L^2+1}$,

(b) $L < 1$, $\mu > \frac{L^2+1}{(L-1)^2}$, and $\theta \leq \frac{2(\mu+1)(L+1)(\mu+\mu L^2-L^2-2\mu L-1)}{2\mu^2-\mu+\mu L^3-L^3-3\mu L^2-L^2-2\mu^2 L-\mu L-L-1}$.

Douglas-Rachford Splitting

Assumptions: A μ-strongly monotone, B L-Lipschitz and monotone.

We have $\|Tx - Ty\| \leq \rho \|x - y\|$ for all $x, y \in \mathcal{H}$ with:

$$\rho = \begin{cases} \dfrac{\theta + \sqrt{\frac{(2(\theta-1)\mu+\theta-2)^2 + L^2(\theta-2(\mu+1))^2}{L^2+1}}}{2(\mu+1)} & \text{if } (a), \\ \left|1 - \theta \frac{L+\mu}{(\mu+1)(L+1)}\right| & \text{if } (b), \\ \sqrt{\frac{(2-\theta)}{4\mu(L^2+1)} \frac{\left(\theta(L^2+1) - 2\mu(\theta+L^2-1)\right)\left(\theta\left(1+2\mu+L^2\right) - 2(\mu+1)\left(L^2+1\right)\right)}{2\mu(\theta+L^2-1) - (2-\theta)(1-L^2)}} & \text{otherwise,} \end{cases}$$

with

(a) $\mu \frac{-(2(\theta-1)\mu+\theta-2) + L^2(\theta-2(1+\mu))}{\sqrt{(2(\theta-1)\mu+\theta-2)^2 + L^2(\theta-2(\mu+1))^2}} \leq \sqrt{L^2+1}$,

(b) $L < 1$, $\mu > \frac{L^2+1}{(L-1)^2}$, and $\theta \leq \frac{2(\mu+1)(L+1)(\mu+\mu L^2 - L^2 - 2\mu L - 1)}{2\mu^2 - \mu + \mu L^3 - L^3 - 3\mu L^2 - L^2 - 2\mu^2 L - \mu L - L - 1}$.

- ◇ First and third cases are achieved on 2-dimensional examples (dual is simpler),

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with

(a) $\mu\,\frac{-(2(\theta-1)\mu+\theta-2)+L^2(\theta-2(1+\mu))}{\sqrt{(2(\theta-1)\mu+\theta-2)^2+L^2(\theta-2(\mu+1))^2}} \leq \sqrt{L^2+1}$,

(b) $L < 1$, $\mu > \frac{L^2+1}{(L-1)^2}$, and $\theta \leq \frac{2(\mu+1)(L+1)(\mu+\mu L^2-L^2-2\mu L-1)}{2\mu^2-\mu+\mu L^3-L^3-3\mu L^2-L^2-2\mu^2 L-\mu L-L-1}$.

- First and third cases are achieved on 2-dimensional examples (dual is simpler),
- Second case is achieved on 1-dimensional example (primal is simpler).

Douglas-Rachford Splitting

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Examples on which those bounds are attained?

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Examples on which those bounds are attained?

- Case 1: (2-dimensional) We choose (see also Moursi & Vandenberghe 2018)

$$A = \mu I + N_{\{0\}\times\mathbb{R}}, \quad B = L\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

for $\rho = \frac{\theta+\sqrt{\frac{(2(\theta-1)\mu+\theta-2)^2+L^2(\theta-2(\mu+1))^2}{L^2+1}}}{2(\mu+1)}$

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- Case 2: (1-dimensional) $A = \mu I$, $B = LI$ for $\rho = |1 - \theta\frac{L+\mu}{(\mu+1)(L+1)}|$

Douglas-Rachford Splitting

Assumptions: A μ-strongly monotone, B L-Lipschitz and monotone.

Examples on which those bounds are attained?

- Case 1: (2-dimensional) We choose (see also Moursi & Vandenberghe 2018)

$$A = \mu\, I + N_{\{0\}\times\mathbb{R}}, \quad B = L \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

for $\rho = \frac{\theta + \sqrt{\frac{(2(\theta-1)\mu+\theta-2)^2 + L^2(\theta-2(\mu+1))^2}{L^2+1}}}{2(\mu+1)}$

- Case 2: (1-dimensional) $A = \mu I$, $B = LI$ for $\rho = |1 - \theta \frac{L+\mu}{(\mu+1)(L+1)}|$
- Case 3: (2-dimensional) For appropriately chosen (complicated) K:

$$A = \mu\, I + N_{\mathbb{R}\times\{0\}}, \quad B = L \begin{pmatrix} K & -\sqrt{1-K^2} \\ \sqrt{1-K^2} & K \end{pmatrix},$$

for $\rho = \sqrt{\frac{(2-\theta)}{4\mu(L^2+1)} \frac{\left(\theta(L^2+1) - 2\mu(\theta+L^2-1)\right)\left(\theta\left(1+2\mu+L^2\right) - 2(\mu+1)\left(L^2+1\right)\right)}{2\mu(\theta+L^2-1) - (2-\theta)(1-L^2)}}.$

Avoiding semidefinite programming modeling steps?

Avoiding semidefinite programming modeling steps?

François Glineur
(UCLouvain)

Julien Hendrickx
(UCLouvain)

"Performance Estimation Toolbox (PESTO): automated worst-case analysis of first-order optimization methods" (CDC 2017)

PESTO example: contraction factors for DRS

```
% (0) Initialize an empty PEP
P=pep();

N = 1;
% (1) Set up the class of monotone inclusions
paramA.L  =  1; paramA.mu = 0; % A is 1-Lipschitz and 0-strongly monotone
paramB.mu = .1;                % B is .1-strongly monotone

A = P.DeclareFunction('LipschitzStronglyMonotone',paramA);
B = P.DeclareFunction('StronglyMonotone',paramB);

w  = cell(N+1,1);   wp = cell(N+1,1);
x  = cell(N,1);     xp = cell(N,1);
y  = cell(N,1);     yp = cell(N,1);

% (2) Set up the starting points
w{1}    = P.StartingPoint(); wp{1}   = P.StartingPoint();
P.InitialCondition((w{1}-wp{1})^2<=1);

% (3) Algorithm
lambda = 1.3;       % step size (in the resolvents)
theta  = .9;         % overrelaxation

for k = 1 : N
    x{k}    = proximal_step(w{k},B,lambda);
    y{k}    = proximal_step(2*x{k}-w{k},A,lambda);
    w{k+1}  = w{k}-theta*(x{k}-y{k});

    xp{k}   = proximal_step(wp{k},B,lambda);
    yp{k}   = proximal_step(2*xp{k}-wp{k},A,lambda);
    wp{k+1}   = wp{k}-theta*(xp{k}-yp{k});
end

% (4) Set up the performance measure: ||z0-z1||^2
P.PerformanceMetric((w{k+1}-wp{k+1})^2);

% (5) Solve the PEP
P.solve()

% (6) Evaluate the output
double((w{k+1}-wp{k+1})^2)   % worst-case contraction factor
```

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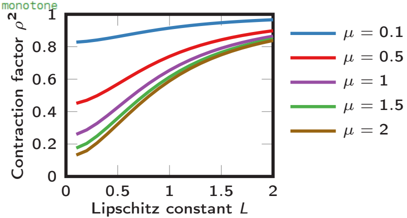
```



PESTO example: contraction factors for DRS

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    xp{k}    = proximal_step(wp{k},B,lambda);
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    wp{k+1}    = wp{k}-theta*(xp{k}-yp{k});
end

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P.solve()

% (6) Evaluate the output
double((w{k+1}-wp{k+1})^2)   % worst-case contraction factor
```

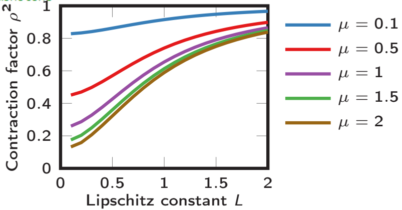

✔ fast prototyping ($\sim$ 20 effective lines)
✔ quick analyses ($\sim$ 10 minutes)
✔ computer-aided proofs (multipliers)

Current library of examples within PESTO

Includes... but not limited to

- ◇ subgradient, gradient, heavy-ball, fast gradient, optimized gradient methods,
- ◇ proximal point algorithm,
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Among others, see works by Drori, Teboulle, Kim, Fessler, Lieder, Lessard, Recht, Packard, Van Scoy, etc.

Classes of functions/operators within PESTO

Functional class	Guaranteed tight PEP
Convex functions	✔
Convex functions (poss. bounded subdifferentials)	✔
Convex indicator functions (poss. bounded domain)	✔
Convex support functions (poss. bounded subdifferentials)	✔
Smooth strongly convex functions	✔
Smooth (possibly nonconvex) functions	✔
Smooth convex functions (poss. bounded subdifferentials)	✔
Strongly convex functions (poss. bounded domain)	✔
Operator class	
Monotone (maximally)	✔
Strongly monotone (maximally)	✔
Cocoercive	✔
Lipschitz	✔
Cocoercive and strongly monotone*	✖
Lipschitz and strongly monotone*	✖

*: for some cases (e.g., DRS/TOS's contraction factors), still tight.

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- T., Hendrickx, Glineur. "Performance Estimation Toolbox (PESTO): automated worst-case analysis of first-order optimization methods" (CDC 2017)
  [In the paper: presentation of the toolbox for first-order optimization methods]

Thanks! Questions?

www.di.ens.fr/$\sim$ataylor/

AdrienTaylor/Performance-Estimation-Toolbox on Github