Static Arbitrage Bounds on Basket Option Prices

Alexandre d'Aspremont

MS&E Dept. & I.S.L.
Stanford University

Joint work with **Laurent El Ghaoui, UC Berkeley**

Introduction

- classic Black & Scholes (1973) option pricing based on:
  - a *dynamic hedging* argument
  - *model* for the asset dynamics (geometric BM)
- sensitive to liquidity, transaction costs, model risk ...
- what can we say about option prices with a minimal set of assumptions?

Arbitrage pricing

Fundamental theorem of asset pricing states that:

$$\text{Absence of Arbitrage} \quad \Leftrightarrow \quad \text{Price} = \mathbf{E}_{\pi}[\text{Payoff}]$$

- here π is a probability measure
- the exact meaning of arbitrage opportunity will be specified later on...

Black-Scholes

The classic Black & Scholes (1973) model:

- Lognormal asset dynamics:

$$dS/S = rdt + \sigma dW_t$$

- Pricing is based on self-financing *perfect replication* of the option payoff by *trading continuously* in stock and cash until maturity.

In particular, the distribution π of S at maturity is lognormal...

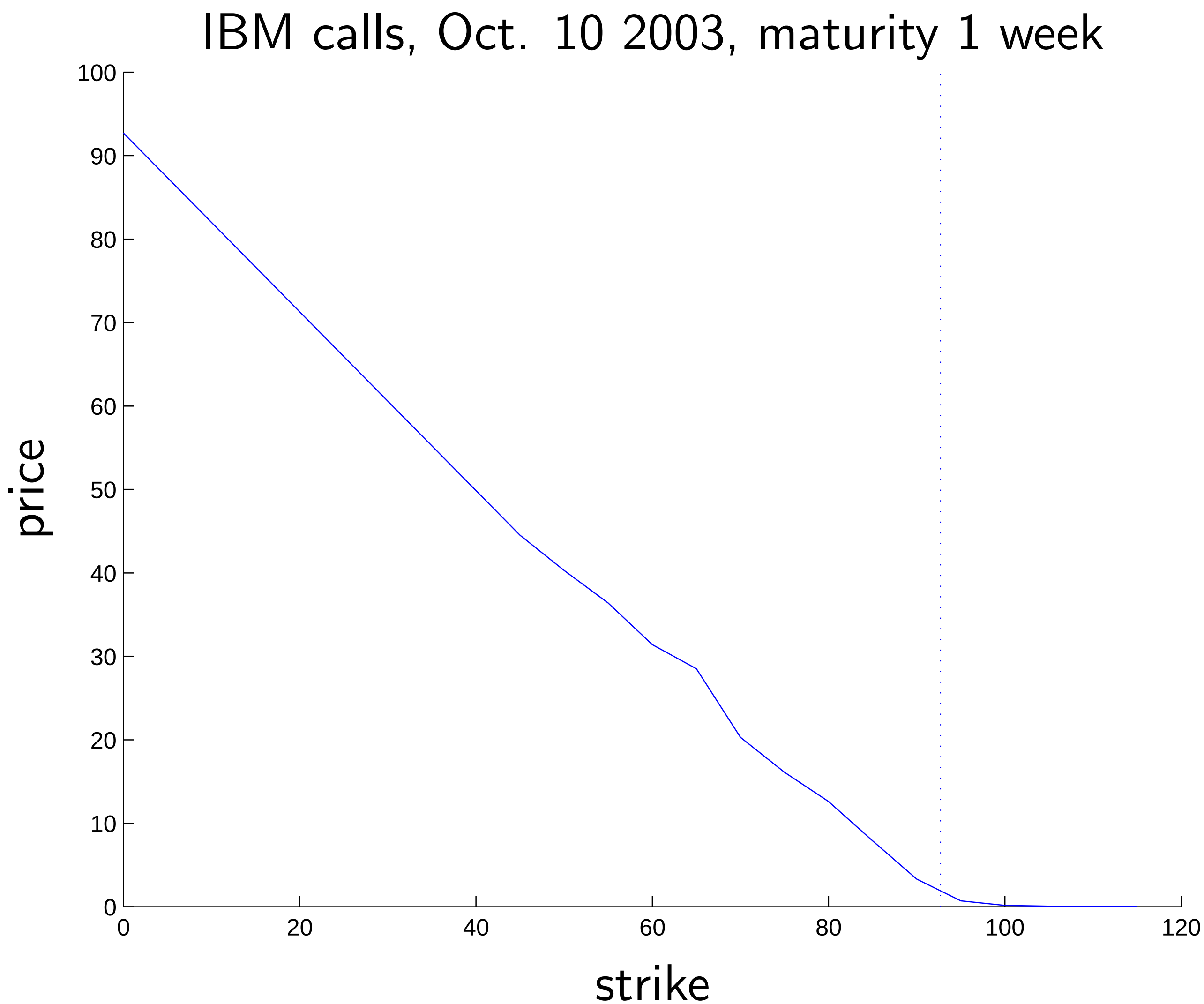
Static Arbitrage

Here instead, we rely on a minimal set of assumptions:

- *no assumption* on the asset distribution π
- *one period* model

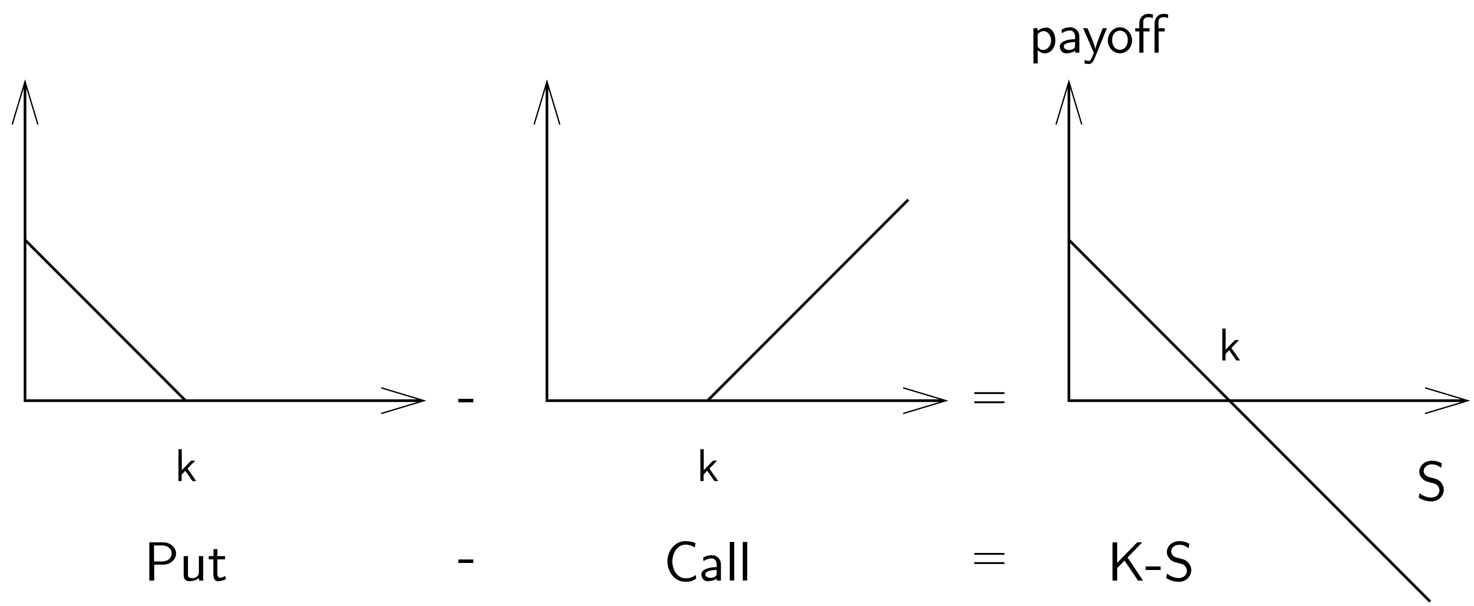
Arbitrage in this simple setting:

- form a portfolio at no cost today with a strictly positive payoff at maturity
- no trading involved between today and the option's maturity



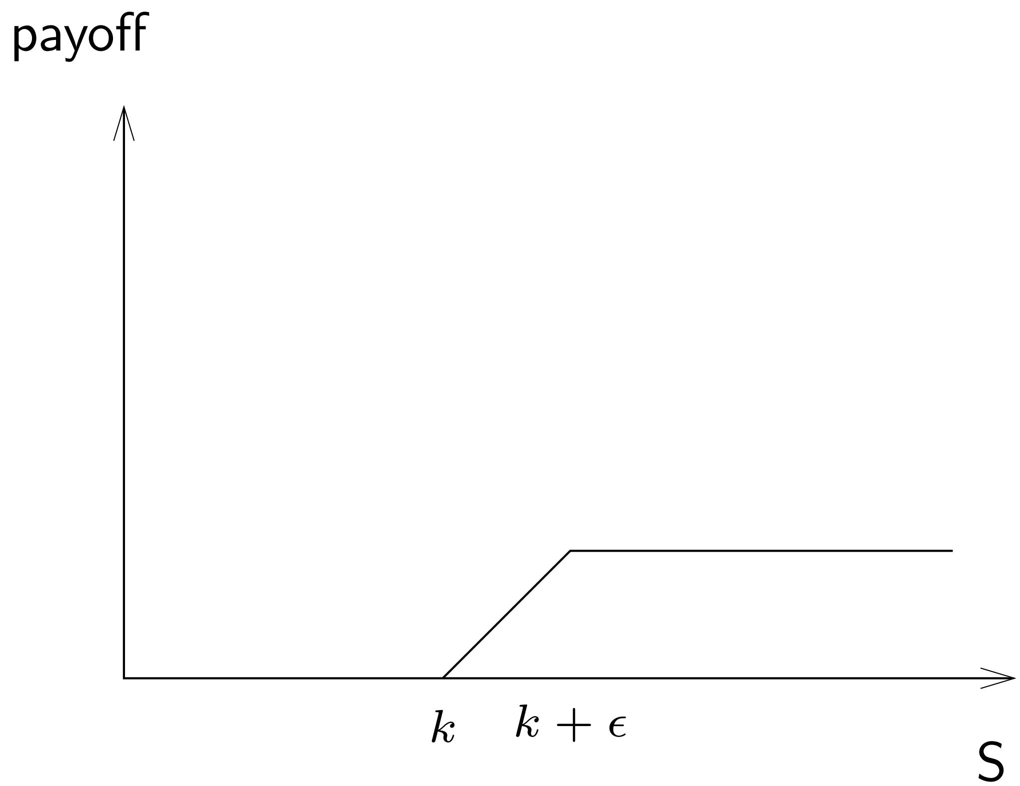
We note $C(K)$ the price of the call with payoff $(S - K)^+$

Simplest of all: put call parity



If we know the forward prices (price of the asset S at maturity T), then we can deduce call prices from puts, ...

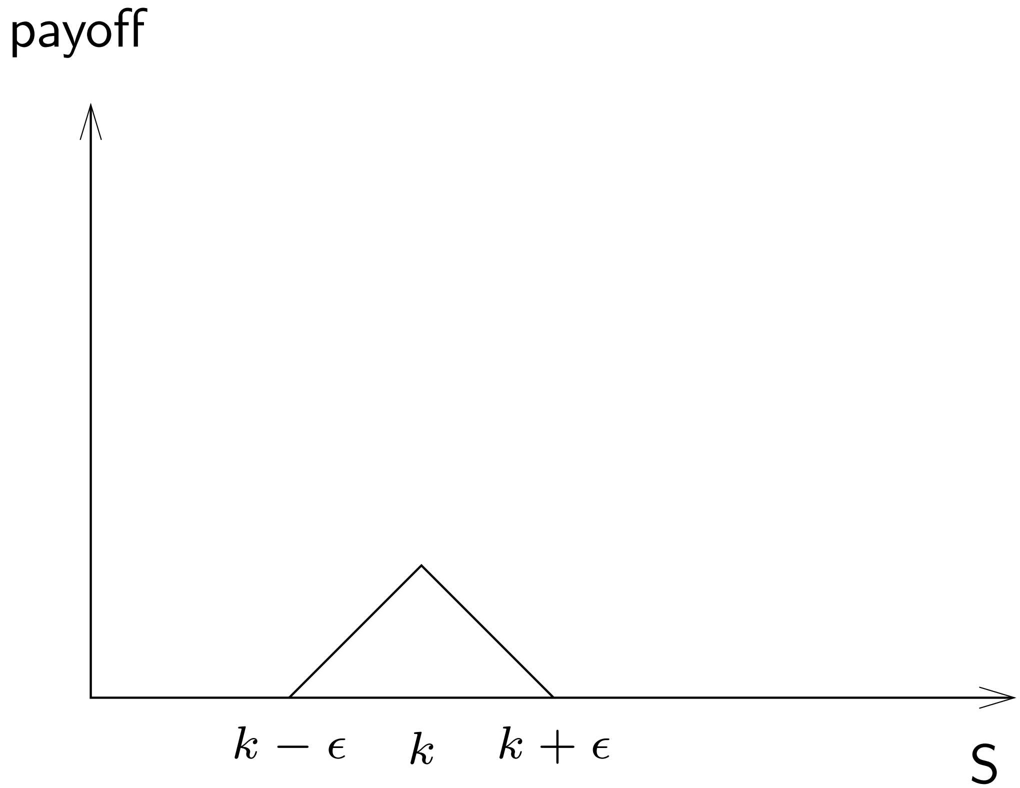
Call spread



Here, Absence of Arbitrage implies that the price of a call spread be positive, hence call prices must be *decreasing* with strike

$$C(K+\epsilon) - C(K) \leq 0$$

Butterfly spread



Absence of Arbitrage implies that the price of a butterfly spread be positive, hence call prices must be *convex* with strike

$$C(K+\epsilon)-2C(K)+C(K-\epsilon)\geq 0$$

Price constraints

Absence of Arbitrage implies that if $C(K)$ is a function giving the price of an option of strike K, then $C(K)$ must satisfy:

- $C(K)$ positive
- $C(K)$ decreasing
- $C(K)$ convex

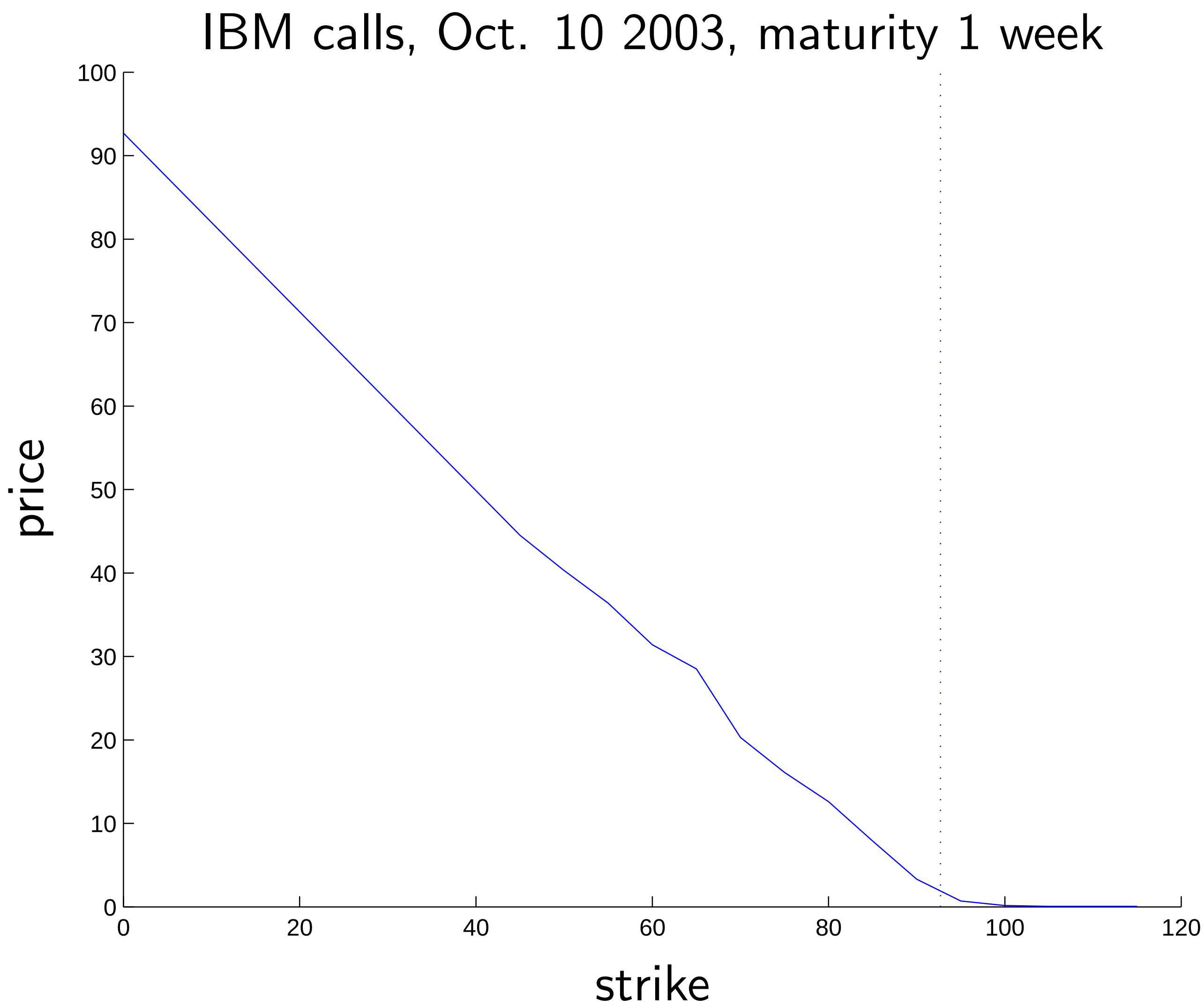
With $C(0) = S$, we have a set of *necessary* conditions for the absence of arbitrage

Sufficient conditions

In fact, these conditions are also *sufficient*, see Breeden & Litzenberger (1978), Laurent & Leisen (2000) and Bertsimas & Popescu (2002) among others

Suppose we have a set of market prices for calls $C(K_i) = p_i$, then there is no arbitrage iff there is a function C(K):

- $C(K)$ positive
- $C(K)$ decreasing
- $C(K)$ convex
- $C(K_i) = p_i$ and $C(0) = S$



Source: reuters

Why?

data quality...

- all the prices are last quotes (not simultaneous)
- low volume
- some transaction costs

Problem: this data is used to calibrate models and price other derivatives...

Dimension n: basket options

- a basket call payoff is

$$\left(\sum_{i=1}^{k} w_i S_i - K\right)_+$$

where $w_1, \ldots, w_k$ are the basket's weights and K is the option's strike price

- examples include: Index options, spread options, swaptions...

- basket option prices are used to gather information on *correlation*

We note $C(w, K)$ the price of such an option, can we get conditions to test basket price data?

Sufficient conditions

Similar to dimension one...

Suppose we have a set of market prices for calls $C(w_i, K_i) = p_i$, and there is no arbitrage, then the function $C(w, K)$ satisfies:

- $C(w, K)$ positive
- $C(w, K)$ decreasing
- $C(w, K)$ jointly convex in (w, K)
- $C(w_i, K_i) = p_i$ and $C(0) = S$

Is this *tractable*?

Tractable?

The problem can be formulated as:

$$
\begin{array}{ll}
\text{find} & z \\
\text{subject to} & Az \leq b,\ Cz = d \\
& z = \left[f(x_1), \ldots, f(x_k), g_1^T, \ldots, g_k^T\right]^T \\
& g_i \text{ subgradient of } f \text{ at } x_i \quad i = 1, \ldots, k \\
& f \text{ monotone, convex}
\end{array}
$$

in the variables $f \in C(\mathbf{R}^n)$, $z \in \mathbf{R}^{(n+1)k}$, $g_1, \ldots, g_k \in \mathbf{R}^n$

- *discretize* and sample the convexity constraints to get a polynomial size LP feasibility problem

- enforce the convexity and subgradient constraints at the points $(x_i)_{i=1,\ldots,k}$ (monotonicity is a simple inequality on g) to get:

$$\begin{array}{ll} \text{find} & z \\ \text{subject to} & Cz = d,\ Az \leq b \\ & z = \left[f(x_1), \ldots, f(x_k), g_1^T, \ldots, g_k^T\right]^T \\ & \langle g_i, x_j - x_i \rangle \leq f(x_j) - f(x_i) \quad i,j = 1, \ldots, k \end{array}$$

in the variables $f(x_i)_{i=1,\ldots,k}$ and g in $\mathbf{R}^n \times \mathbf{R}^{n \times k}$

- we note $z^{\mathrm{opt}} = \left[f^{\mathrm{opt}}(x_1), \ldots, f^{\mathrm{opt}}(x_k), (g_1^{\mathrm{opt}})^T, \ldots, (g_k^{\mathrm{opt}})^T\right]^T$ a solution to this problem

- from z^{opt}, we define:

$$s(x) = \max_{i=1,\ldots,k} \left\{ f^{\mathrm{opt}}(x_i) + \left\langle g_i^{\mathrm{opt}}, x - x_i \right\rangle \right\}$$

- by construction, $s(x_i)$ solves the finite LP with:

$$s(x_i) = f^{\mathrm{opt}}(x_i), \quad i = 1, \ldots, k$$

- $s(x)$ is convex and monotone as the pointwise maximum of monotone affine functions

- so $s(x)$ is also a feasible point of the original problem

this means that the price conditions *remain tractable* on basket options...

Sufficient?

key difference with dimension one, Bertsimas & Popescu (2002) show that the exact problem is NP-Hard

- the conditions are *only necessary*...
- here however, numerical cost is minimal (small LP)
- we can show *tightness* in some particular cases
- how sharp are these conditions?

Full conditions

derived by Henkin & Shananin (1990). A function can be written

$$C(w,K)=\int_{\mathbf{R}^n_+}(w^Tx-K)_+d\pi(x)$$

with $w\in\mathbf{R}^n_+$ and $K>0$, if and only if:

- $C(w,K)$ is *convex* and *homogenous* of degree one;
- $\lim_{K\rightarrow\infty}C(w,K)=0$ and $\lim_{K\rightarrow 0^+}\frac{\partial C(w,K)}{\partial K}=-1$
- $F(w)=\int_0^\infty e^{-K}d\left(\frac{\partial C(w,K)}{\partial K}\right)$ belongs to $C_0^\infty(\mathbf{R}^n_+)$
- For some $\tilde{w}\in\mathbf{R}^n_+$ the inequalities: $(-1)^{k+1}D_{\xi_1}...D_{\xi_k}F(\lambda\tilde{w})\geq 0$, for all positive integers k and $\lambda\in\mathbf{R}_{++}$ and all $\xi_1,\ldots,\xi_k$ in $\mathbf{R}^n_+$.

Numerical example

- two assets: x_1, x_2, we look for bounds on the price of $(x_1 + x_2 - K)^+$
- simple discrete model for the assets:

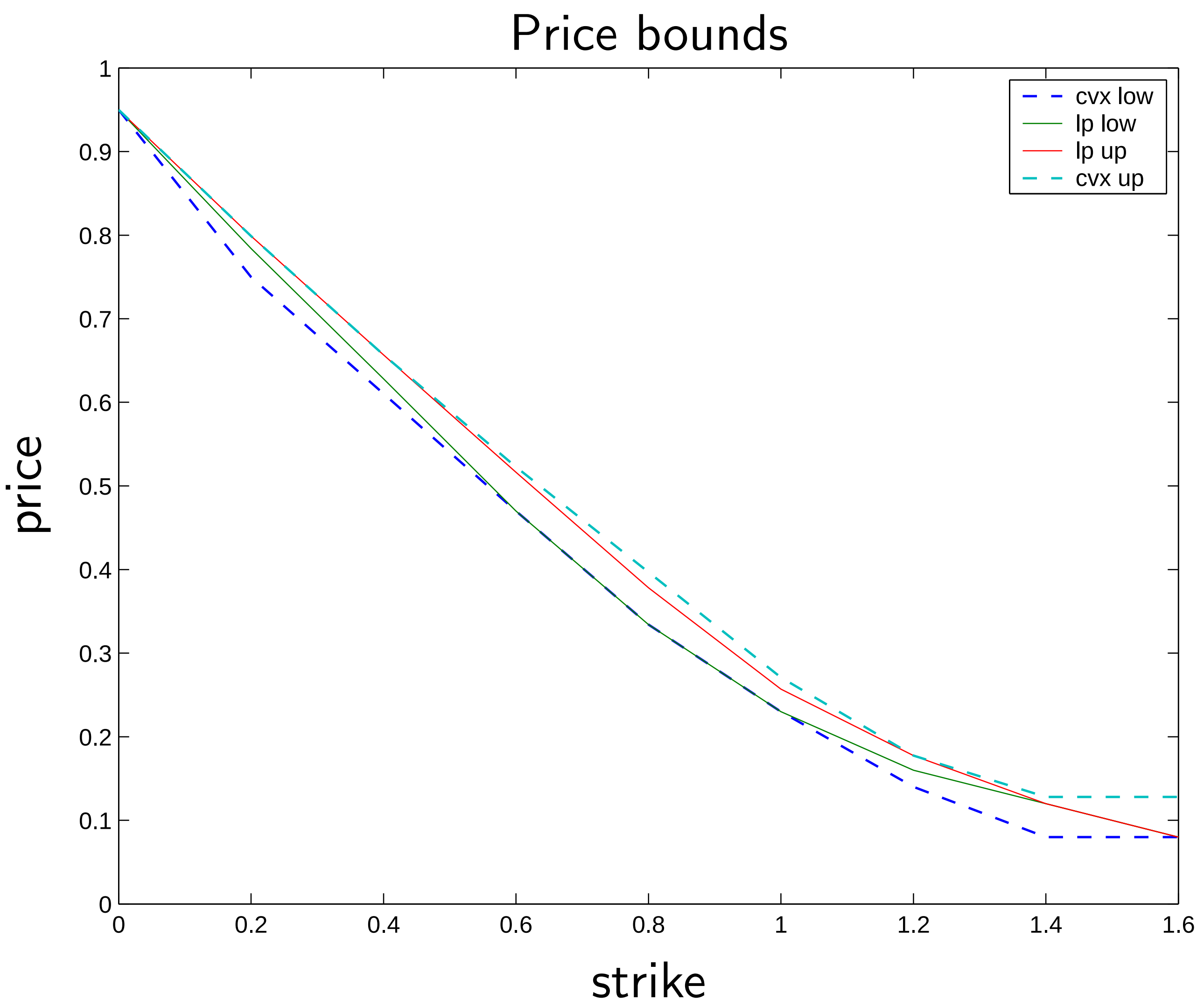
$$x = \{(0,0),(0,.8),(.8,.3),(.6,.6),(.1,.4),(1,1)\}$$

with probability

$$p = (.2,.2,.2,.1,.1,.2)$$

- the forward prices are given, together with the following call prices:

$$(.2x_1 + x_2 - .1)^+, (.5x_1 + .8x_2 - .8)^+, (.5x_1 + .3x_2 - .4)^+,$$
$$(x_1 + .3x_2 - .5)^+, (x_1 + .5x_2 - .5)^+, (x_1 + .4x_2 - 1)^+, (x_1 + .6x_2 - 1.2)^+$$


Price bounds
cvx low
lp low
lp up
cvx up
price
strike
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
0
0.2
0.4
0.6
0.8
1
1.2
1.4
1.6

Extensions (very briefly)...

formulate as a moment problem on the payoff semigroup (see Berg, Christensen & Ressel (1984)):

$$s = (1, x_1, \ldots, x_n, |w_0^T x - K_0|, \ldots, |w_m^T x - K_m|, x_1^2, x_1 x_2, \ldots, |w_m^T x - K_m|^N)$$

this is a *semidefinite program*

$$\begin{array}{ll} \text{find} & f : s \rightarrow \mathbf{R} \\ \text{subject to} & M_N(f(s)) \succeq 0 \\ & M_N(f(s_j s)) \succeq 0, \quad \text{for } j = 1, \ldots, n, \\ & M_N\left(f((\beta - \sum_{k=0}^{n+m} s_k) s)\right) \succeq 0 \\ & f(s_j) = p_j, \quad \text{for } j = 0, \ldots, n+m \text{ and } s \in \mathbb{S} \end{array}$$

where $M_N(f(s))_{ij} = f(s_i s_j)$ and $M_N(f(s_k s))_{ij} = f(s_k s_i s_j)$

Conclusion

Simple, tractable bounds to test basket option price data...

- conditions are *only necessary*
- but... very low numerical cost
- *tightness* in some particular cases, "good" in general

Related papers...

- A. d'Aspremont, L. El Ghaoui
  "Static Arbitrage Bounds on Basket Option Prices."
  ArXiv: math.OC/0302243.

- A. d'Aspremont
  "A Harmonic Analysis Solution to the Static Basket Arbitrage Problem."
  ArXiv: math.OC/0309048.

both available on *www.stanford.edu/∽aspremon/*

References

Berg, C., Christensen, J. P. R. & Ressel, P. (1984), *Harmonic analysis on semigroups : theory of positive definite and related functions*, Vol. 100 of *Graduate texts in mathematics*, Springer-Verlag, New York.

Bertsimas, D. & Popescu, I. (2002), 'On the relation between option and stock prices: a convex optimization approach', *Operations Research* **50**(2).

Black, F. & Scholes, M. (1973), 'The pricing of options and corporate liabilities', *Journal of Political Economy* **81**, 637–659.

Breeden, D. T. & Litzenberger, R. H. (1978), 'Price of state-contingent claims implicit in option prices', *Journal of Business* **51**(4), 621–651.

Henkin, G. & Shananin, A. (1990), 'Bernstein theorems and Radon transform, application to the theory of production functions', *American*

Mathematical Society: Translation of mathematical monographs **81**, 189–223.

Laurent, J. & Leisen, D. (2000), Building a consistent pricing model from observed option prices, *in* M. Avellaneda, ed., 'Quantitative Analysis in Financial Markets', World Scientific Publishing.