Libor Market Model
Calibration & Risk-Management

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Introduction

Interest rate derivatives trading

- Focus on structured products activity
- Discuss stability, speed and robustness
- How do we extract *correlation* information from market option prices?

Activity

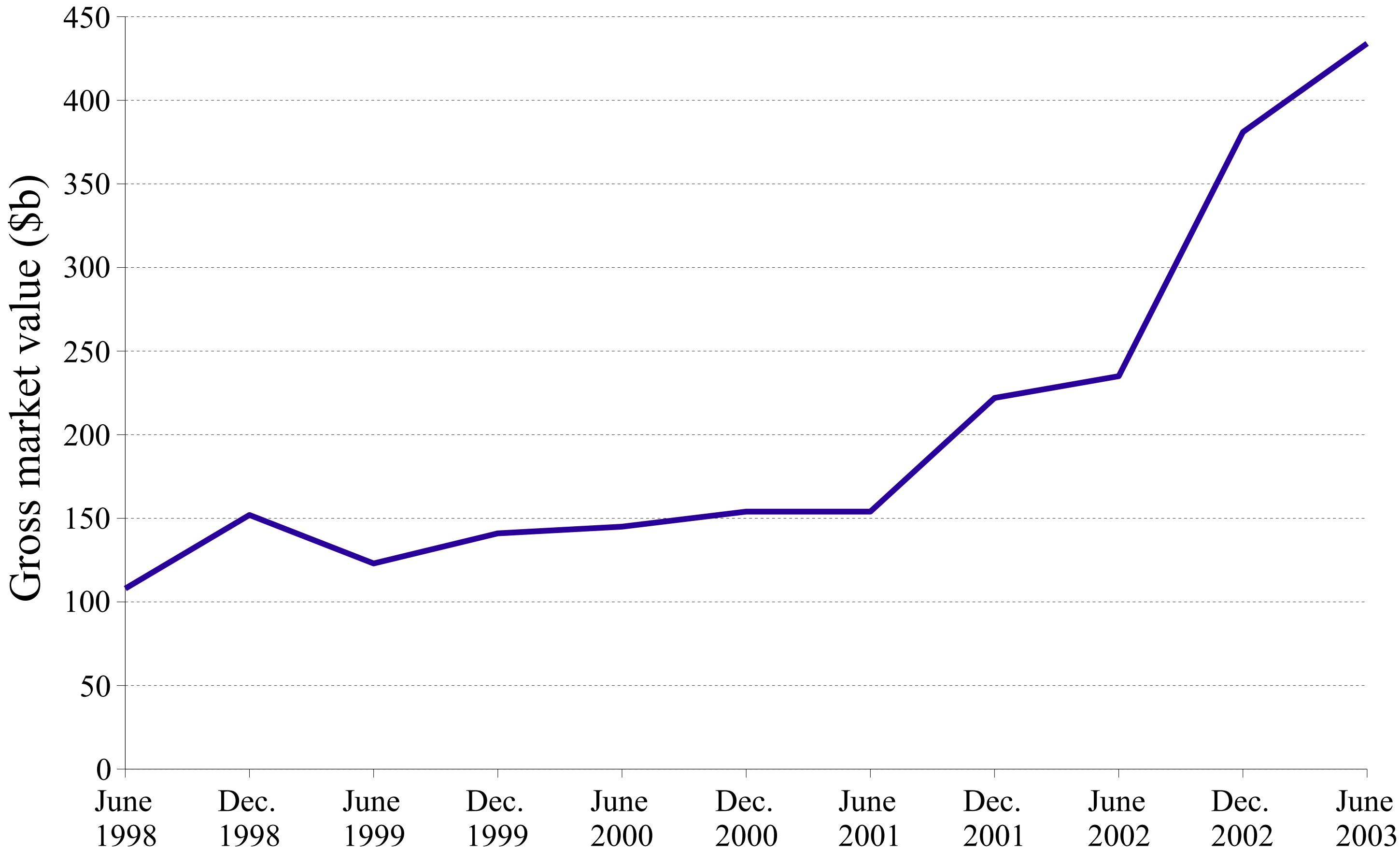


Figure 1: OTC activity in interest rate options.

Source: Bank for International Settlements.

Structured Interest Rate Products

Structured derivatives desks act as *risk brokers*

- **buy/sell** tailor made products from/to their clients
- **hedge** the resulting risk using simple options in the market
- **manage** the residual risk on the entire portfolio

Derivatives Production Cycle

market data

$\Downarrow$

model calibration

$\Downarrow$

pricing & hedging

$\Downarrow$

risk-management

Derivatives Production Cycle, *Trouble...*

market data: *illiquidity, Balkanization of the data sources*

$\Downarrow$

model calibration: *inverse problem, numerically hard*

$\Downarrow$

pricing & hedging: *American option pricing in dim.* ≥ 2

$\Downarrow$

risk-management: *all of the above. . .*

Objective

- However, *it works*
- Numerical trouble creates P&L hikes, poor risk description, . . .
- Our objective here: improve *stability*, *robustness*

- Focus first on **calibration**
- Using new cone programming techniques to calibrate models and manage portfolio risk

Outline

- **Cone programming, a brief introduction**
- IR model calibration
- Risk-management

Linear Programming

Linear program:

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax - b \succeq 0 \end{array}$$

- Simplex algorithm by Dantzig (1949): exponential worst-case complexity, very efficient in most cases
- Khachiyan (1979) used the ellispoid method to show the polynomial complexity of LP
- Karmarkar (1984) describes the first efficient polynomial time algorithm for LP, using interior point methods
- very stable and efficient numerical codes available (CPLEX, MOSEK,. . .)

Cone Programming

The standard **linear program:**

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax - b \succeq 0 \end{array}$$

becomes a **cone program:**

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax - b \in \mathcal{K} \end{array}$$

$\mathcal{K}$ is a product of symmetric cones: $\mathcal{K} = LP \times SO \times SDP$ with

$$\begin{array}{ll} \text{LP:} & \{x \in \mathbf{R}^n : x \geq 0\} \\ \text{Second order:} & \{(x, y) \in \mathbf{R}^n \times \mathbf{R} : \|x\| \leq y\} \\ \text{Semidefinite:} & \{X \in \mathbf{S}^n : X \succeq 0\} \end{array}$$

Cone Programming

In standard form:

$$\begin{array}{ll}\text{minimize} & c^T x\\ \text{subject to} & Ax - b \succeq 0\\ & \|Bx + d\| \leq d^T x + e\\ & \sum_{j=1}^n D_j x_j - D_0 \succeq 0\end{array}$$

where $D_j \in \mathbf{S}^n$ and $A \succeq B$ means here $A - B$ positive semidefinite.

- Known complexity bounds, produce proof of convergence or infeasibility, see Nesterov & Nemirovskii (1994) or Boyd & Vandenberghe (2003)
- Vast expressive power. . .
- Performance & reliability similar to linear programs
- True *black box* behavior

Example: Filling a Covariance Matrix

- Suppose that we have only partial information on the covariance between three assets (zero mean).

- We know the variance of each asset, and have some information on the sign of the correlation between others:

$$\begin{pmatrix} 1 & + & .5 \\ + & 1 & - \\ .5 & - & 1 \end{pmatrix}$$

Our objective is to find a correlation (positive semidefinite with diagonal one) matrix that *fits the data* and has the maximum possible correlation X_{12}.

Example: Filling a Covariance Matrix

The previous problem is solved by the following cone program:

$$\begin{array}{ll}
\text{maximize} & X_{12} \\
\text{subject to} & X_{11} = 1,\ X_{22} = 1,\ X_{33} = 1,\ X_{13} = .5 \\
& X_{21} \geq 0,\ X_{32} \leq 0 \\
& X = X^T \\
& X \succeq 0
\end{array}$$

and the solution is

$$\left(\begin{array}{ccc} 1 & .87 & .5 \\ .87 & 1 & 0 \\ .5 & 0 & 1 \end{array}\right)$$

According to the data, $X_{23} = .87$ is the maximum correlation possible.

Outline

- Cone programming, a brief introduction
- **IR model calibration**
- Risk-management

Swaps

The swap rate is the rate that equals the PV of a fixed and a floating leg:

$$swap(t, T_0, T_n) = \frac{B(t, T_0^{floating}) - B(t, T_{n+1}^{floating})}{level(t, T_0^{fixed}, T_n^{fixed})}$$

where

$$level(t, T_0^{fixed}, T_n^{fixed}) = \sum_{i=1}^{n+1} coverage(T_{i-1}^{fixed}, T_i^{fixed}) B(t, T_i^{fixed})$$

Swaps

The swap rate can again be written:

$$swap(t, T_0, T_n) = \sum_{i=0}^{n} w_i(t) K(t, T_i)$$

where $K(t, T_i)$ are the forward rates with maturities T_i and the weights $w_i(t)$ are given by

$$w_i(t) = \frac{coverage(T_i^{float}, T_{i+1}^{float}) B(t, T_{i+1}^{float})}{level(t, T_0^{fixed}, T_n^{fixed})}$$

In practice, these weights are very stable (see Rebonato (1998)).

Libor Market Model

- In the Libor Market Model, the zero coupon volatility is specified to make Libor rates

$$1 + \delta L(t,\theta) = exp\left(\int_{\theta}^{\theta+\delta} r(t,v)dv\right)$$

lognormal martingales under their respective measures:

$$dK(s,T_i)/K(s,T_i) = \sigma(s,T_i)dW_s^{Q^{T_i+\delta}}$$

where $\sigma(s,T_i) \in \mathbf{R}^n$ and $dW_s^{Q^{T_i+\delta}}$ is a n dimensional B.M. and

$$K(s,T_i) = L(s,T_i - s)$$

- This volatility definition and the Heath, Jarrow & Morton (1992) arbitrage conditions fully specify the model.

Pricing Swaptions

With Q_{LVL}, the swap forward martingale probability measure given by:

$$\frac{d\mathbf{Q}^{LVL}}{d\mathbf{Q}^{T}}|_t = B(t,T)\beta(T)\sum_{i=1}^{N}\frac{\delta cvg(i,b)\beta^{-1}(T_{i+1})}{Level(t,T,T_N)}$$

and following Jamshidian (1997), we can write the price of the Swaption with strike k as a that of a call on a swap rate:

$$Ps(t) = Level(t,T,T_N)E_t^{Q_{LVL}}\left[\left(\sum_{i=0}^{n}\omega_i(T)K(T,T_i) - k\right)^+\right]$$

In other words, the swaption is a *call on a basket of forwards*.

A Remark on the Gaussian HJM

We can also express the price of the swaption as that of a bond put:

$$Ps(t) = B(t,T)\mathbf{E}_t^{Q_T}\left[\left(1 - B(t,T_{N+1}) - k\delta\sum_{i=i_T}^{N} B(t,T_i)\right)^+\right]$$

In the Gaussian H.J.M. model (see El Karoui & Lacoste (1992), Musiela & Rutkowski (1997) or Duffie & Kan (1996)), this expression defines the price of a swaption as that of a *put on a basket of lognormal* zero coupon prices.

Approximations

We will make two approximations:

- We replace the weights $w_i(s)$ by their value today $w_i(t)$.
- We approximate the swap rate $\sum_{i=0}^{n} w_i(t) K(s, T_i)$ by a sum of Q^{LVL} lognormal martingales F_s^i with:

$$F_t^i = K(t, T_i)$$

and

$$dF_s^i / F_s^i = \sigma(s, T_i - s) dW_s^{LVL}$$

Error

Try to quantify the error:

- What's the contribution of the weights in the swap's volatility?
- What about the drift terms coming from the forwards under Q^{LVL}?
- How do we compute the price of a call on a basket of lognormals?

Weights contribution

The swap dynamics are given by:

$$dswap(s,T,T_N) = \sum_{i=i_T}^{N} \omega_i(s) K(s,T_i) \left(\gamma(s,T_i - s) + \eta(s,T_i)\right) dW_s^{LVL}$$

where the contribution of the weights is:

$$\eta(s,T_i) = \left(\sum_{k=i_T}^{N} \omega_i(s) \left(\sigma^B(s,T_i - s) - \sigma^B(s,T_k - s)\right) \right)$$

with $\sigma^B(t,\theta)$, the ZC volatility. In practice $\delta K(s,T_j) \simeq 1\%$ and

$$\sum_{i=i_T}^{N} \omega_i(s) K(s,T_i) \eta(s,T_i) = \sum_{i=i_T}^{N} \omega_i(s) \left(K(s,T_i) - swap(s,T,T_N)\right) \eta(s,T_i)$$

with $\sum_{i=i_T}^{N} \omega_i(s) = 1$ et $0 \leq \omega_i(s) \leq 1$. This is *zero when the curve is flat*

Change of forward measure

With

$$F_t^i = K(t, T_i) \quad \text{and} \quad dF_s^i / F_s^i = \sigma(s, T_i - s) dW_s^{LVL}$$

we approximate the swap by

$$dY_s = \sum_{i=i_T}^{N} w_i(t) F_s^i \sigma(s, T_i - s) dW_s^{LVL}, \quad Y_t = swap(t, T, T_N).$$

The error can be bounded by

$$E\left[\left(\sup_{t \leq s \leq T} (swap(s) - Y_s)\right)^2\right] \leq \ldots \left\|\tilde{\sigma}_s^i\right\|_4^2 + \left(K(t, T_{k^*}) \delta \left(N - i_T\right) \bar{\sigma}^2\right)^2$$

where $\tilde{\sigma}_s^i$ is a residual volatility:

$$\tilde{\sigma}_s^i = \sigma_s^i - \sigma_s^\omega \quad \text{with} \quad \sigma_s^w = \sum_{j=1}^{n} \widehat{w}_{i,t} \sigma_s^j$$

Multivariate Black-Scholes

With these assumptions, we have reduced the problem of pricing a swaption in the Libor Market Model, to that of *pricing a basket in a generic Black & Scholes (1973) model* with n assets F_s^i such that:

$$dF_s^i/F_s^i = \sigma_s^i dW_s$$

where $\sigma_i \in \mathbf{R}^n$ and dW_s^{LVL} is a n dimensional B.M. under a swap measure Q^{LVL}.

. . . Still, no closed form formula for basket calls

Price Approximation

The price of a basket call

$$B(t,T)\mathbf{E}\left[\left(\sum_{i=1}^{n} w_i F_T^i - K\right)^+\right]$$

is approximated by a regular call price

$$C = BS(w^T F_t, K, T, V_T) \quad \text{with} \quad V_T = \int_t^T \mathbf{Tr}\left(\Omega_t X_s\right) ds$$

where

$$\begin{aligned}\mathbf{Tr}\left(\Omega_t X_s\right) &= \textstyle\sum_{i,j=1}^{n} \Omega_{t,i,j} X_{s,i,j} \\ &= \textstyle\sum_{i,j=1}^{n} \hat{w}_{i,t}\hat{w}_{j,t}\sigma_s^{iT}\sigma_s^j\end{aligned}$$

and

$$\Omega_t = \hat{w}_t \hat{w}_t^T \quad \text{with} \quad \hat{w}_{i,t} = \frac{w_i F_t^i}{w^T F_t}$$

Price Approximation: Error Term

In fact, we have:

$$C^{\varepsilon} = C^{0} + C^{(1)}\varepsilon + o(\varepsilon)$$

here both C^0 and $C^{(1)}$ can be computed explicitly. C^0 is given by the BS formula above:

$$C^{0} = BS(w^T F_t, K, T, V_T)$$

We get $C^{(1)}$ as:

$$\begin{aligned} C^{(1)} \quad = \quad & w^T F_t \int_t^T \sum_{j=1}^{n} \widehat{w}_{j,t} \frac{\left\langle \tilde{\sigma}_s^j, \sigma_s^w \right\rangle}{V_T^{1/2}} \exp\left(2 \int_t^s \left\langle \tilde{\sigma}_u^j, \sigma_u^w \right\rangle du\right) \\ & N\left(\frac{\ln \frac{w^T F_t}{K} + \int_t^s \left\langle \tilde{\sigma}_u^j, \sigma_u^w \right\rangle du + \frac{1}{2} V_T}{V_T^{1/2}}\right) ds \end{aligned}$$

where

$$\tilde{\sigma}_s^i = \sigma_s^i - \sigma_s^{\omega} \quad \text{with} \quad \sigma_s^w = \sum_{j=1}^{n} \widehat{w}_{i,t} \sigma_s^j \quad \text{and} \quad \epsilon \sim \sum_{j=1}^{n} \widehat{w}_{i,t} \tilde{\sigma}_s^j$$

Hedging Interpretation

- Suppose we are hedging the option with the approximate vol. σ_s^ω and, as in El Karoui, Jeanblanc-Picqué & Shreve (1998), we study the hedging error:

$$e_T = \frac{1}{2}\int_t^T \left(\left\| \sum_{i=1}^n \widehat{\omega}_{i,s}\sigma_s^i \right\|^2 - \|\sigma_s^\omega\|^2 \right) (F_s^\omega)^2 \frac{\partial^2 C^0(F_s^\omega, V_{t,T})}{\partial x^2} ds$$

- At the first order in $\tilde{\sigma}_s^j$, we get:

$$e_T^{(1)} = \int_t^T \sum_{i=1}^n \left\langle \tilde{\sigma}_s^i, \sigma_s^\omega \right\rangle \widehat{\omega}_{i,s} F_s^\omega \frac{n(h(V_{s,T}, F_s^\omega))}{V_{s,T}^{1/2}} ds$$

- This is also:

$$C^{(1)} = E\left[e_T^{(1)}\right]$$

Price Approximation: Precision

We plot the difference between two distinct sets of swaption prices in the Libor Market Model.

- One is obtained by Monte-Carlo simulation using enough steps to make the 95% confidence margin of error always less than 1bp.
- The second set of prices is computed using the order zero approximation.

The plots are based on the prices obtained by calibrating a BGM model to EURO Swaption prices on November 6 2000, using all cap volatilities and the following swaptions: 2Y into 5Y, 5Y into 5Y, 5Y into 2Y, 10Y into 5Y, 7Y into 5Y, 10Y into 2Y, 10Y into 7Y, 2Y into 2Y, 1Y into 9Y (choice based on liquidity).

Price Approximation: Precision

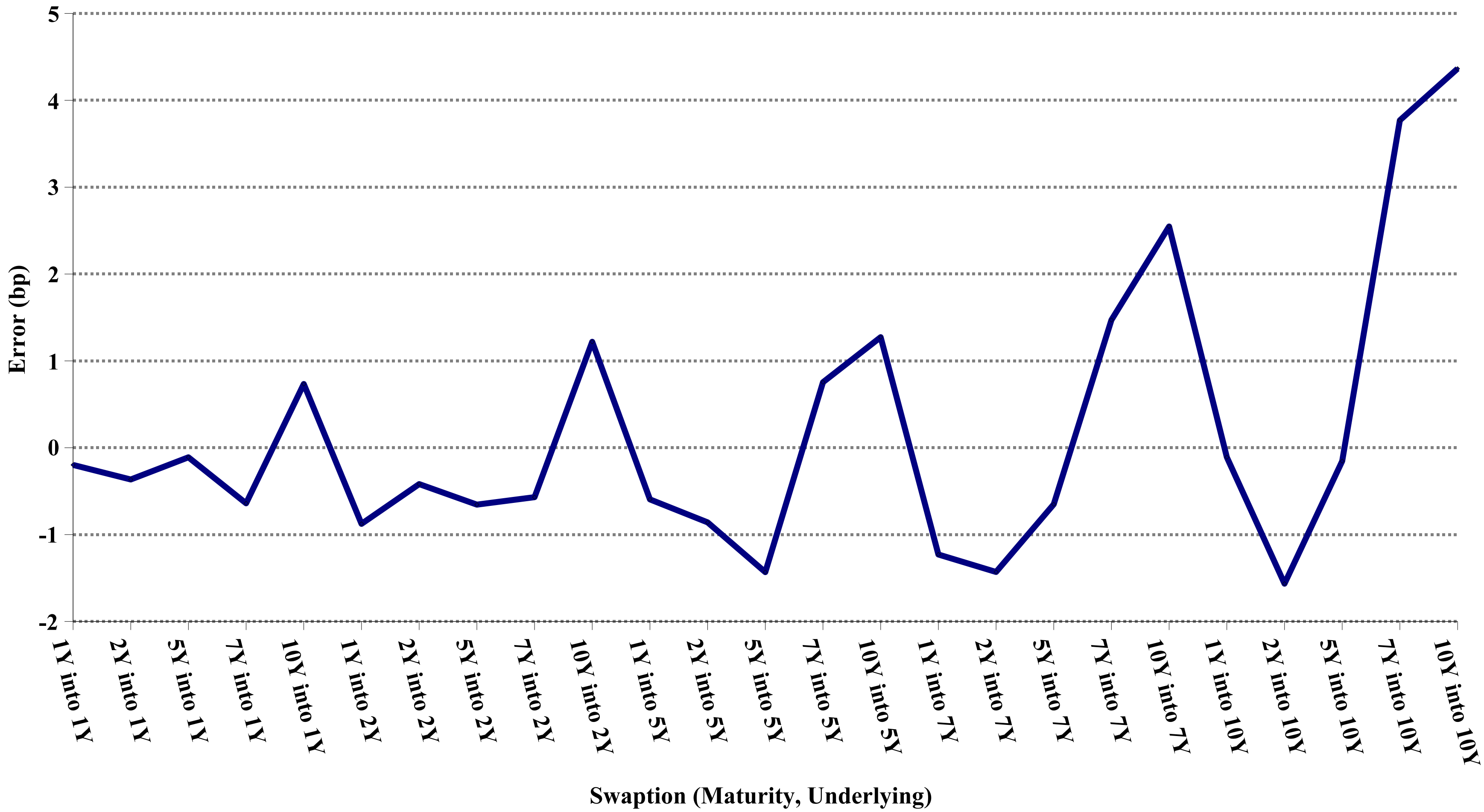


Figure 2: Error (in bp) for various ATM swaptions.

Price Approximation: Precision

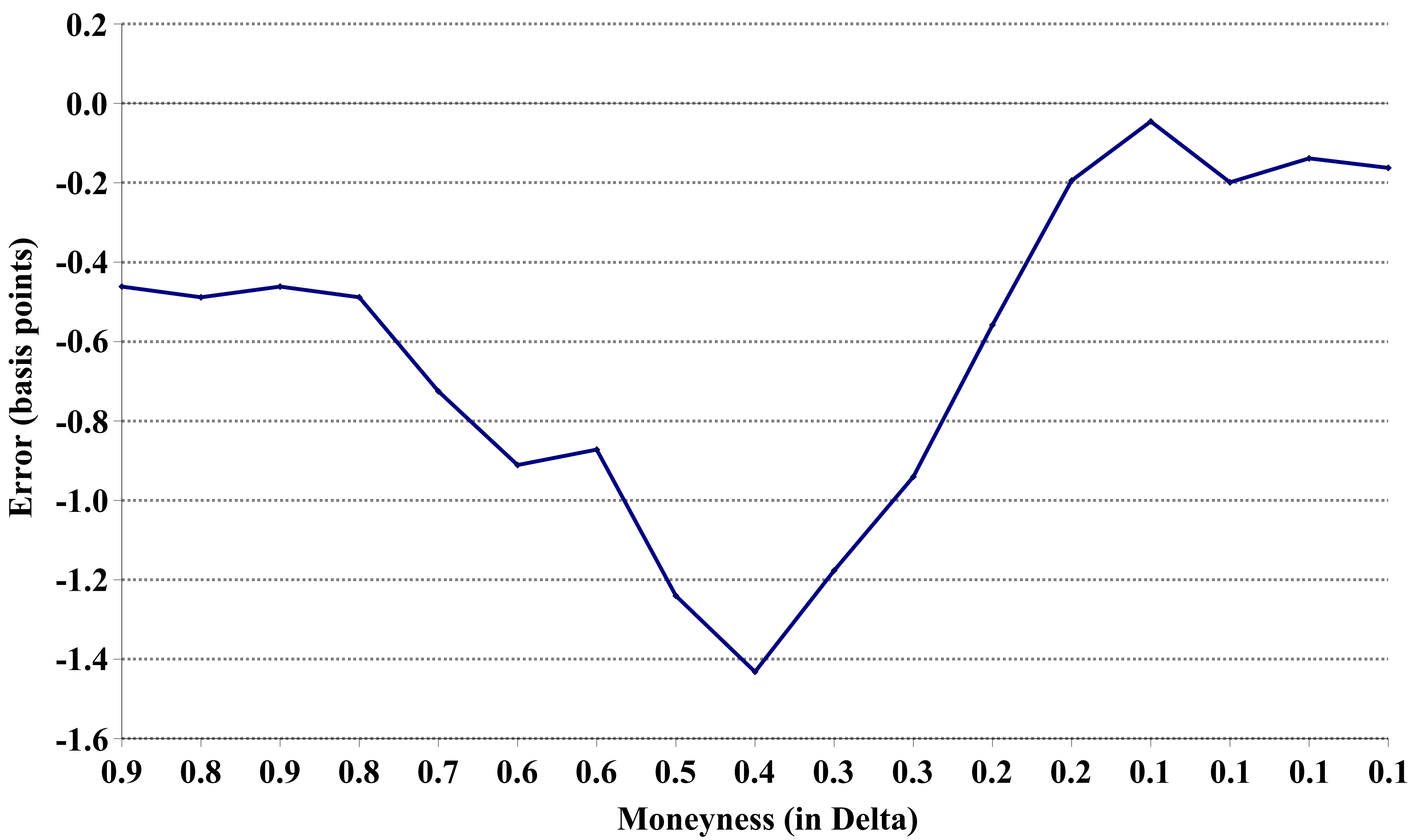


Figure 3: Error (in bp) vs. moneyness, on the 5Y into 5Y.

Price Approximation: Precision

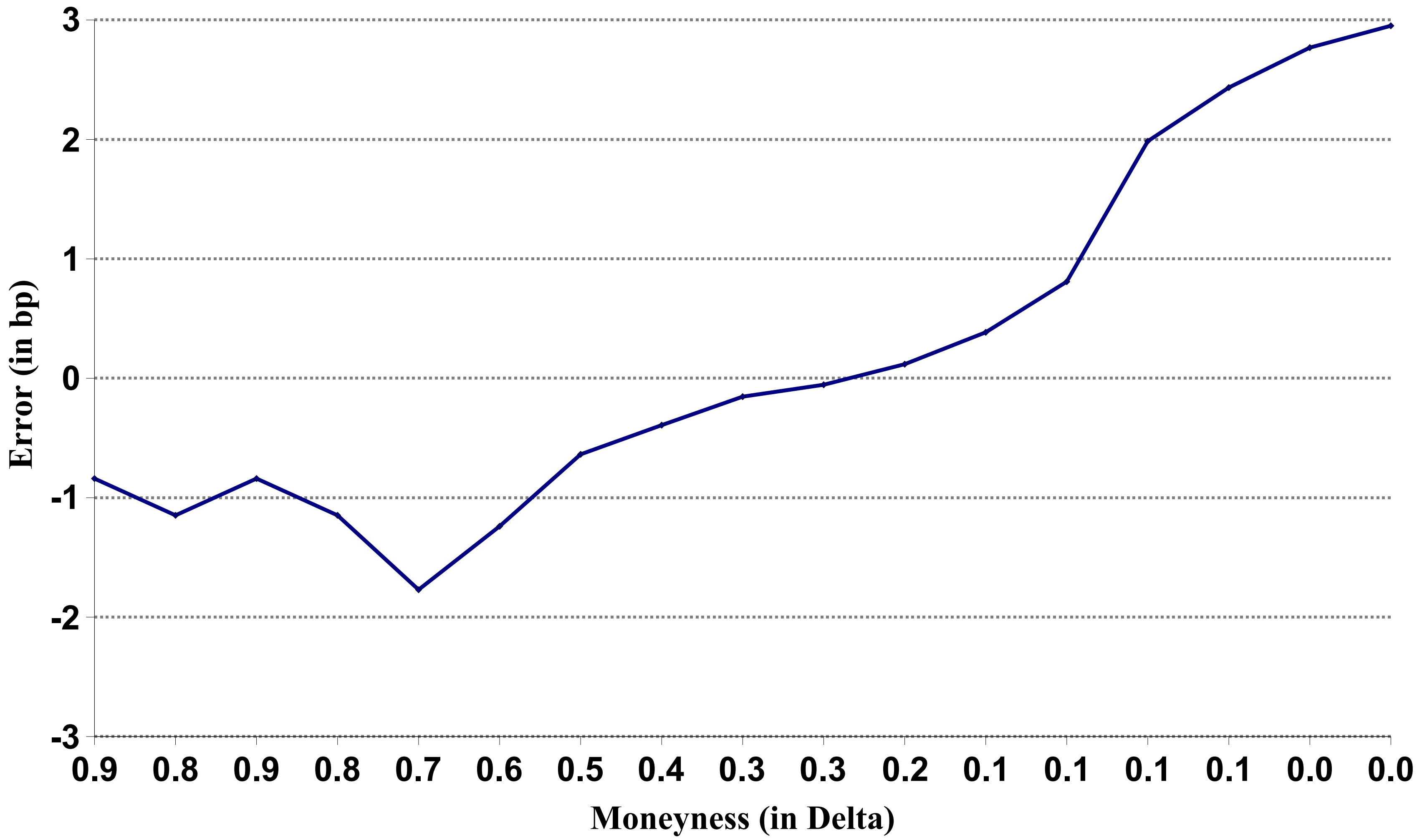


Figure 4: Error (in bp) vs. moneyness on the 5Y into 10Y.

Price Approximation: Precision

- We compare again with Monte-Carlo. The model parameters are

$$\begin{array}{rcl} F_0^i & = & \{0.07, 0.05, 0.04, 0.04, 0.04\} \\ w_i & = & \{0.2, 0.2, 0.2, 0.2, 0.2\} \end{array}$$

  $T = 5$ years, the covariance matrix is:

$$\frac{11}{100} \begin{pmatrix} 0.64 & 0.59 & 0.32 & 0.12 & 0.06 \\ 0.59 & 1 & 0.67 & 0.28 & 0.13 \\ 0.32 & 0.67 & 0.64 & 0.29 & 0.14 \\ 0.12 & 0.28 & 0.29 & 0.36 & 0.11 \\ 0.06 & 0.13 & 0.14 & 0.11 & 0.16 \end{pmatrix}$$

- These values correspond to a $5Y$ into $5Y$ swaption.

- Our goal is to measure only the error coming from the pricing formula and not from the change of measure/martingale approximation

Price Approximation: Precision

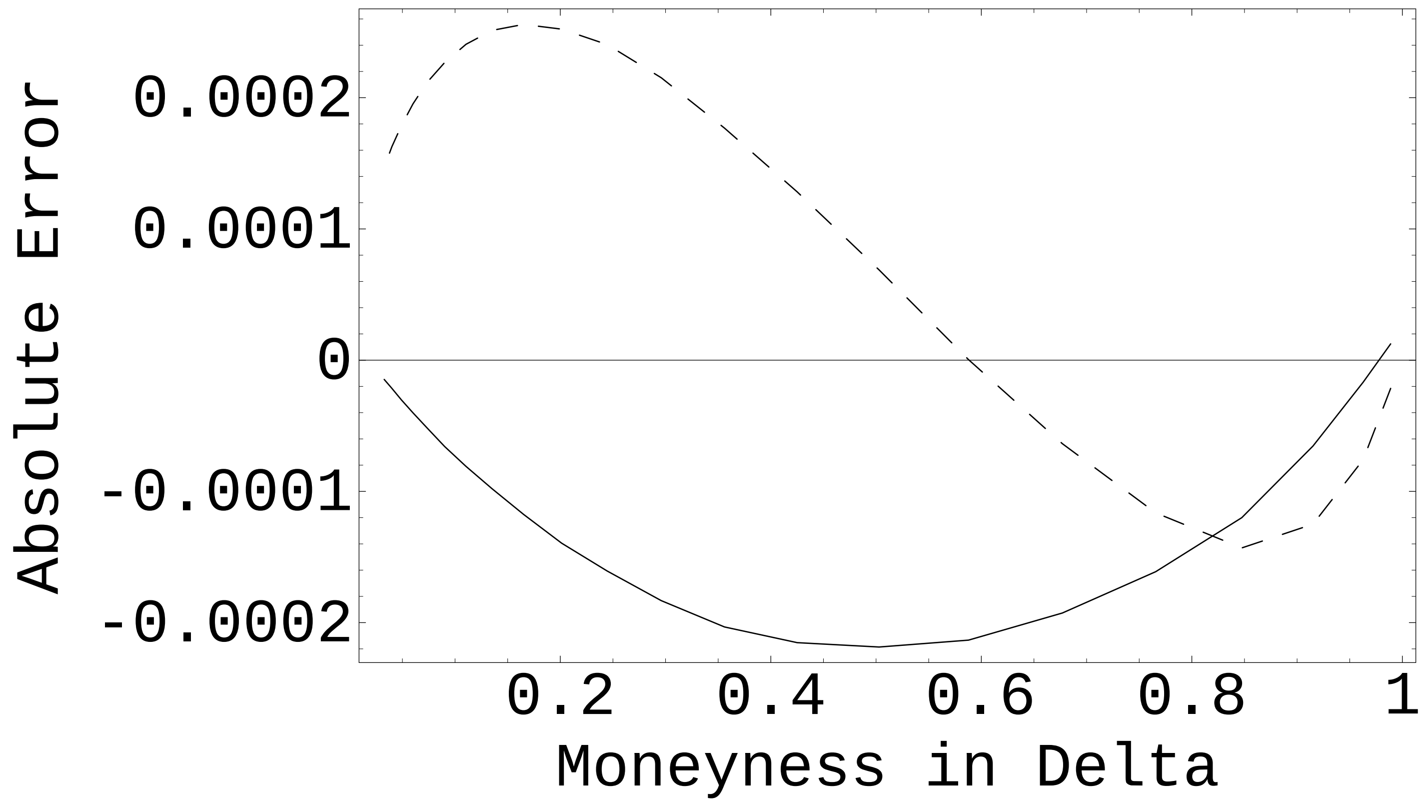


Figure 5: Order zero (dashed) and order one absolute pricing error (plain), in basis points.

Price Approximation: Precision

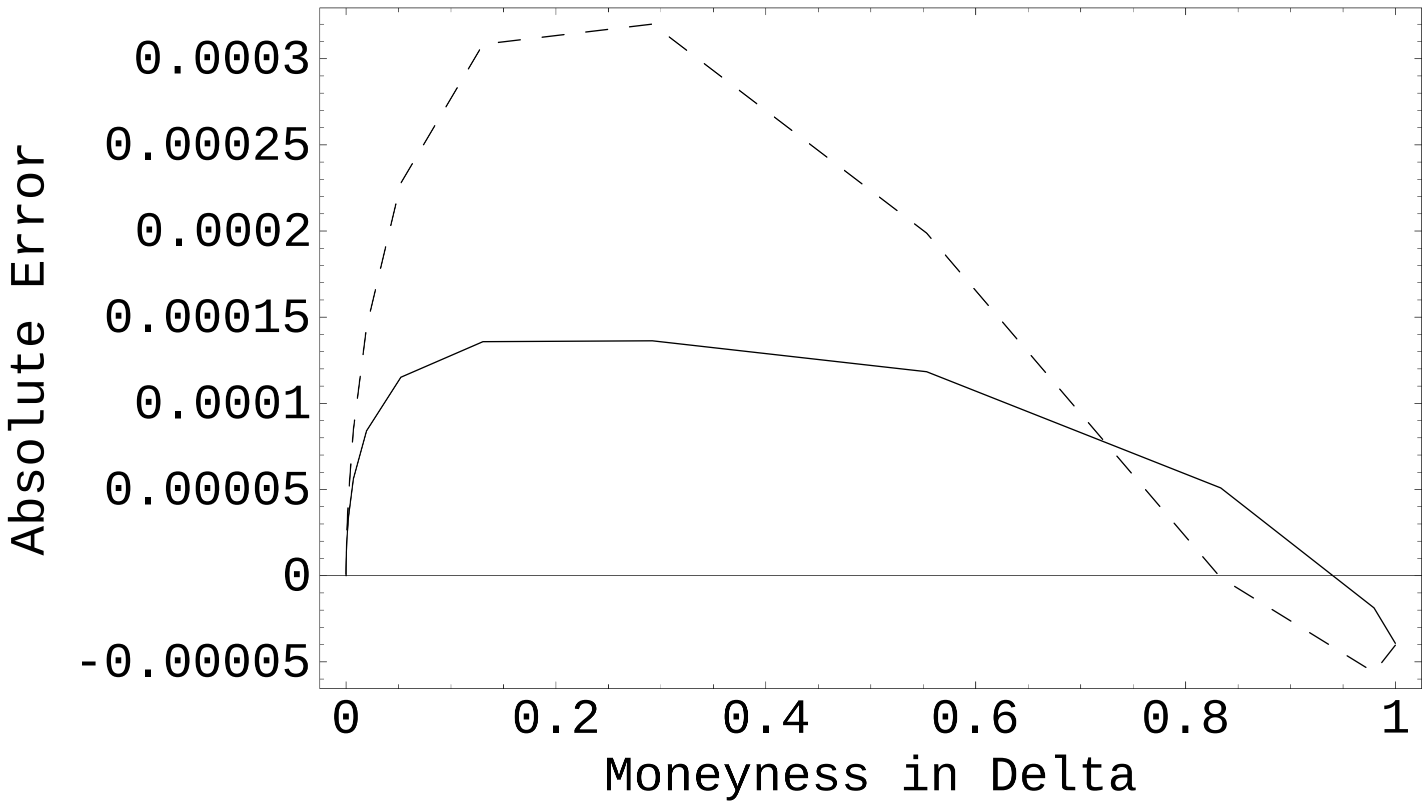


Figure 6: Order zero (dashed) and order one absolute pricing error (plain), in basis points, *zero correlation*.

Price Approximation: Precision

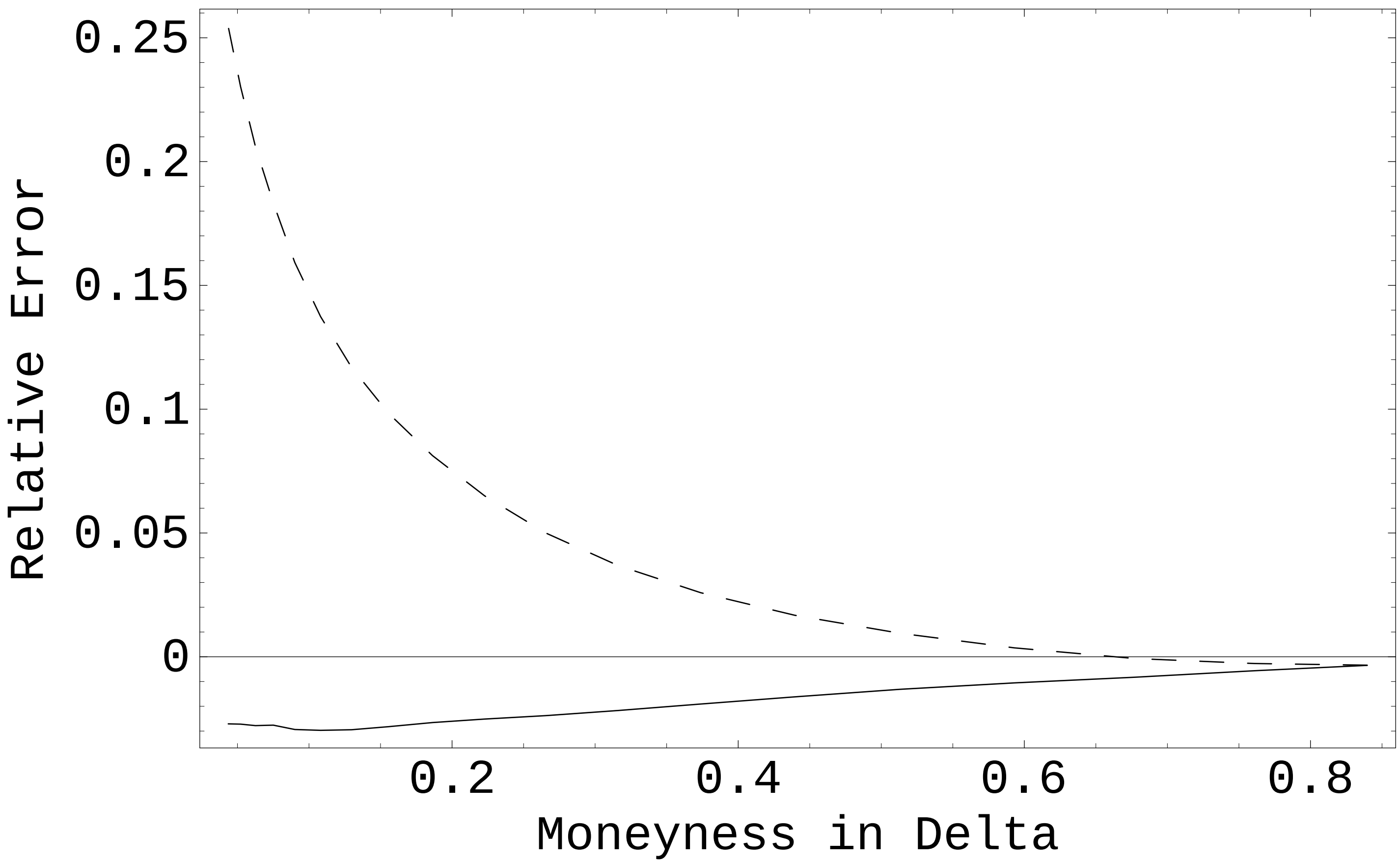


Figure 7: Order zero (dashed) and order one relative pricing error (plain), *equity case*.

Calibration

The price of a swaption was approximated by a regular call price with a well-chosen variance

$$C = BS(w^T F_t, K, T, V_T) \quad \text{with} \quad V_T = \int_t^T \mathbf{Tr}\left(\Omega_t X_s\right) ds$$

- The fundamental model parameter is the *covariance matrix* $X_s = \sigma_s^T \sigma_s$, how can we calibrate it?
- For simplicity, we discretize the covariance in time and write

$$V_T = \int_t^T \mathbf{Tr}\left(\Omega_t X_s\right) ds = \sum_{t=1}^{T} \mathbf{Tr}\left(\Omega_t X_s\right) = \mathbf{Tr}\left(\Omega X\right)$$

Calibration: Complexity

market data

$\Downarrow$

easy on calls... **model calibration** *...hard on swaptions*

$\Downarrow$

pricing & hedging

$\Downarrow$

risk-management

Calibration

- Given market prices p_i of swaptions, finding a *market calibrated* covariance matrix is equivalent to solving

$$\begin{array}{ll} \text{find} & X \\ \text{such that} & \mathbf{Tr}(\Omega_{i,t}X) = V_{T,i}, \quad i = 1, \ldots, m \\ & X \succeq 0 \end{array}$$

  with $\sigma_s^i = \sigma^i$ for simplicity and $V_{T,i}$ is computed from market prices:

$$BS(T, w_i^T F_t, V_{i,T}) = p_i, \quad i = 1, \ldots, m$$

  with p_i the prices of swaptions with weights w_i and $\Omega_{i,t} = \hat{w}_{i,t}\hat{w}_{i,t}^T$.

- This is a semidefinite feasibility problem and can be solved *very efficiently*.

Calibration

- The previous problem describes the entire set of market calibrated matrices.
- How do we pick the "best" matrix among these?

One simple choice is to try to maximize or minimize the price of another option:

$$
\begin{array}{ll}
\text{max./min.} & \mathbf{Tr}(\Omega_{0,t}X) \\
\text{subject to} & \mathbf{Tr}(\Omega_{i,t}X) = V_{T,i}, \quad i = 1, \ldots, m \\
& X \succeq 0
\end{array}
$$

to get *bounds on the model price* of this option. . .

Swaptions: Price Bounds

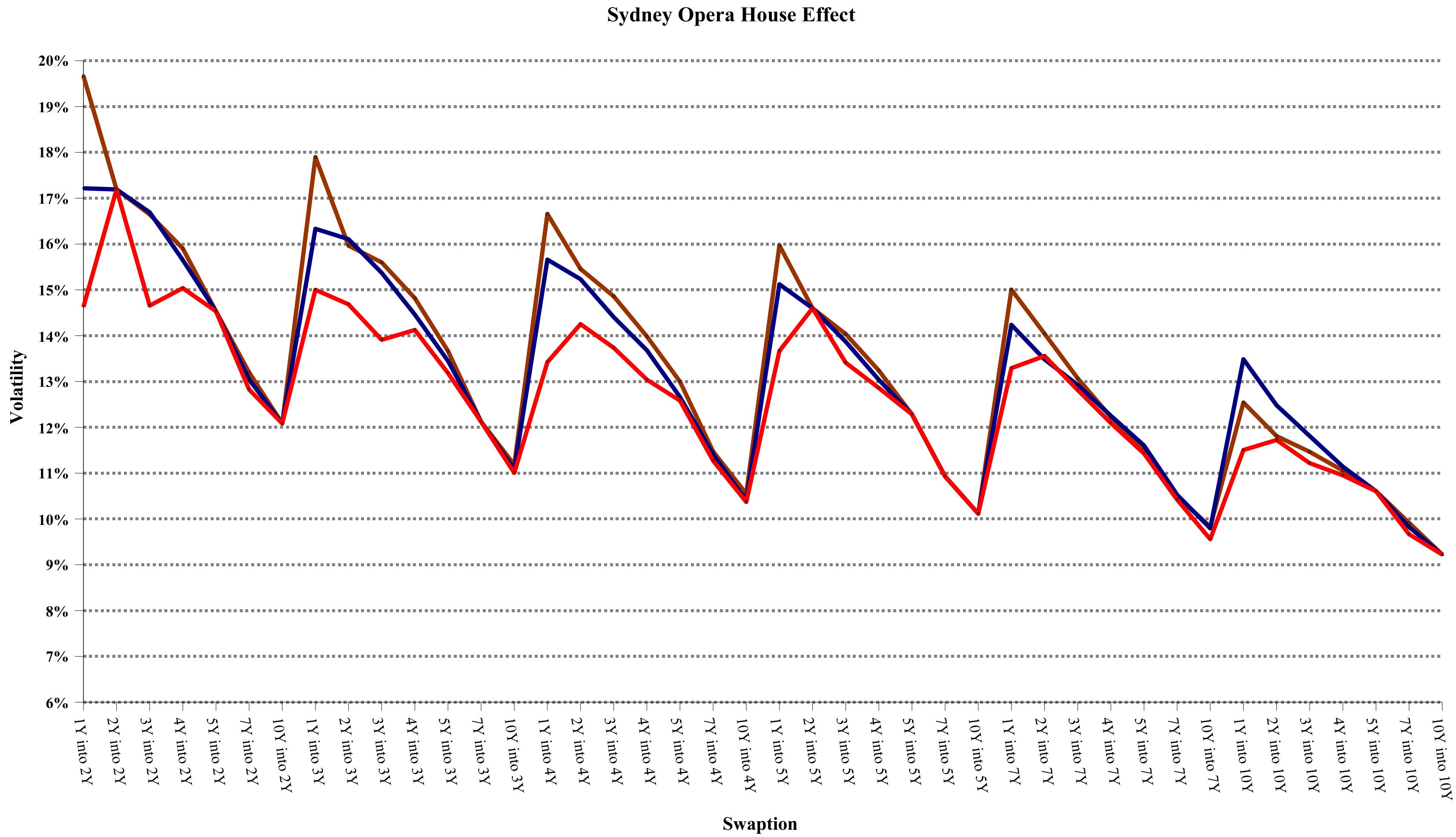


Figure 8: Upper and lower bounds on swaption prices.
(Market in blue, upper bound in brown, lower in red.)

Calibration: Objective

The last program tends to produce extreme matrices. Other possible choices include. . .

- *Norm* (see Cont (2001) on volatility surfaces regularization):

$$\begin{array}{ll} \text{minimize} & \|X\| \\ \text{subject to} & \mathbf{Tr}(\Omega_{i,t}X) = V_{T,i}, \quad i = 1,\ldots,m \\ & X \succeq 0 \end{array}$$

- *Smoothness*:

$$\begin{array}{ll} \text{minimize} & \|\Delta X_{ij}\| \\ \text{subject to} & \mathbf{Tr}(\Omega_{i,t}X) = V_{T,i}, \quad i = 1,\ldots,m \\ & X \succeq 0 \end{array}$$

- *Distance* to a given matrix C:

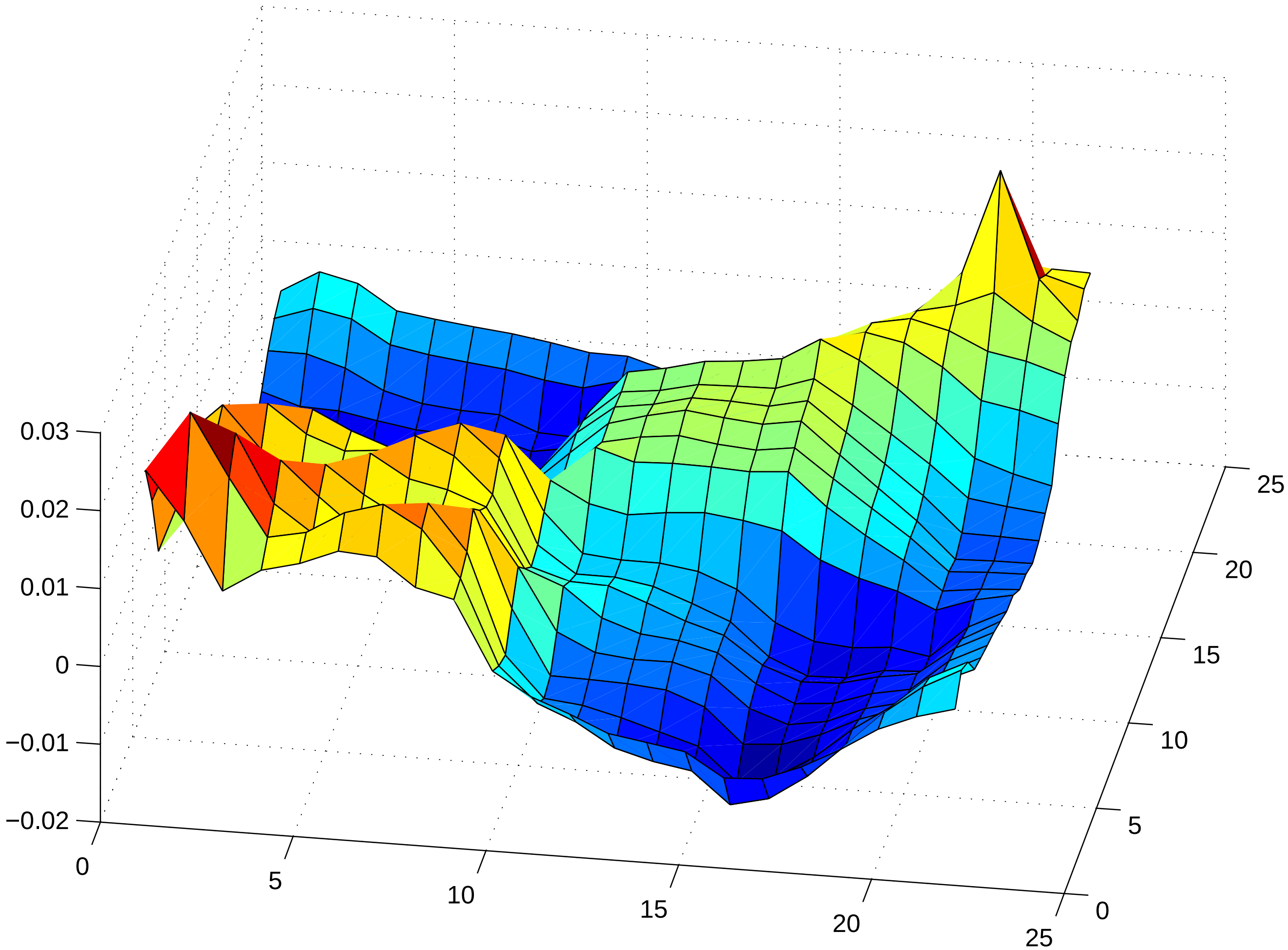
$$\begin{array}{ll} \text{minimize} & \|X - C\| \\ \text{subject to} & \mathbf{Tr}(\Omega_{i,t}X) = V_{T,i}, \quad i = 1, \ldots, m \\ & X \succeq 0 \end{array}$$

- *Robustness*, solution centering:

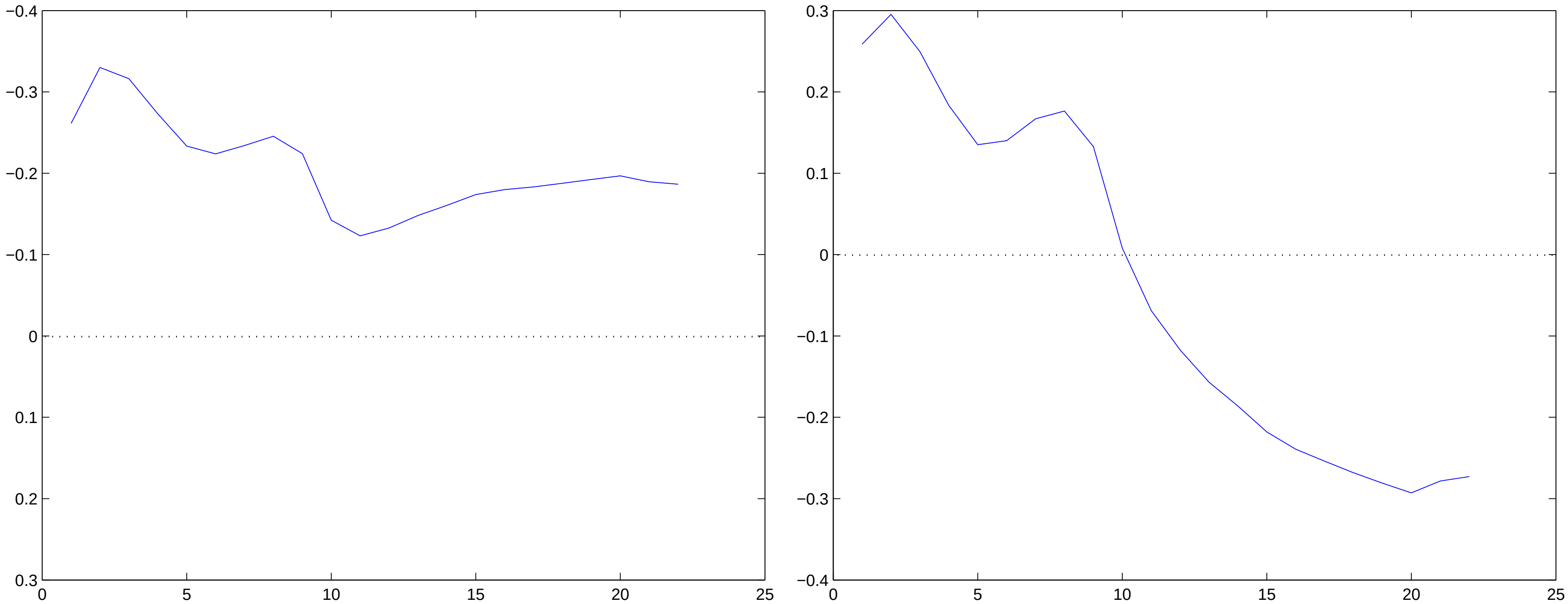
$$\begin{array}{ll} \text{maximize} & t \\ \text{subject to} & V_{T,i}^{\text{Bid}} + t \leq \mathbf{Tr}(\Omega_{i,t}X) \leq V_{T,i}^{\text{Ask}} - t, \quad i = 1, \ldots, m \\ & X \succeq 0 \end{array}$$

- Caveat: $\mathbf{Rank}(X)$. The Minimum rank problem is NP-Complete, but excellent heuristics exist (see Boyd, Fazel & Hindi (2000)).

Smooth calibrated matrix



Factors



(a) Level (b) Spread

Infeasibility

- If the program is not feasible, we get a Farkas type certificate:

$$\lambda \in \mathbf{R}^m : \quad 0 \preceq \sum_{i=1}^{m} \lambda_i \Omega_{i,t} \text{ and } \lambda^T V_T < 0$$

- This detects an arbitrage: the options with variance $V_{T,i}$ cannot constitute a viable price system within the model.

- Detecting the smallest set of products that admits an arbitrage is NP-complete (MINCARD), but same heuristics apply (see Boyd et al. (2000)).

Existing techniques

The first fully implicit calibration technique is due to Rebonato (1999).

- Parameterize the factors from vectors on an hypersphere

$$C = BB^T$$

with

$$\begin{array}{l} b_{ij} = \cos(\theta_{ij}) \prod_{k=1}^{j-1} \sin(\theta_{ij}) \\ b_{in} = \prod_{k=1}^{n-1} \sin(\theta_{ij}) \end{array}$$

- Calibrate the variance separately.

This provides solutions with a given (low) number of factors:

- Nonconvex, NP-Hard, Convergence not guaranteed. . .
- Stability?

Existing techniques

Other techniques include the recent results by Schoenmakers (2002):

- The correlation now only depends of two parameters:

$$\rho_{ij} = \exp\left(-\frac{|j-i|}{m-1}\left(-\ln\rho_\infty + \eta C_{ij}\right)\right)$$

$$C_{ij} = \frac{i^2+j^2+ij-3mi-3mj+3i+3j+2m^2-m-4}{(m-2)(m-3)}$$

- Covariance again calibrated separately.

Full rank solutions

- Again nonconvex, NP-Hard, Convergence not guaranteed. . .
- But less parameters, stability included in the RMS objective

Existing techniques

To summarize:

	Rank	Complexity	Stability	Fit
Rebonato	$choice$	$NP\ Hard$	N/A	$best$
Schoenmakers	$full$	$NP\ Hard$	RMS	$best$
SDP	$full$	$Polynomial$	$choice$	$exact$

Rank versus Stability

- The final tradeoff is between getting a low rank solution and stabilizing the calibration
- The heuristical methods (see Boyd et al. (2000)) provide a low rank solution using semidefinite programming.
- They are however conflicting with stability objectives. . .

Stability versus Rank

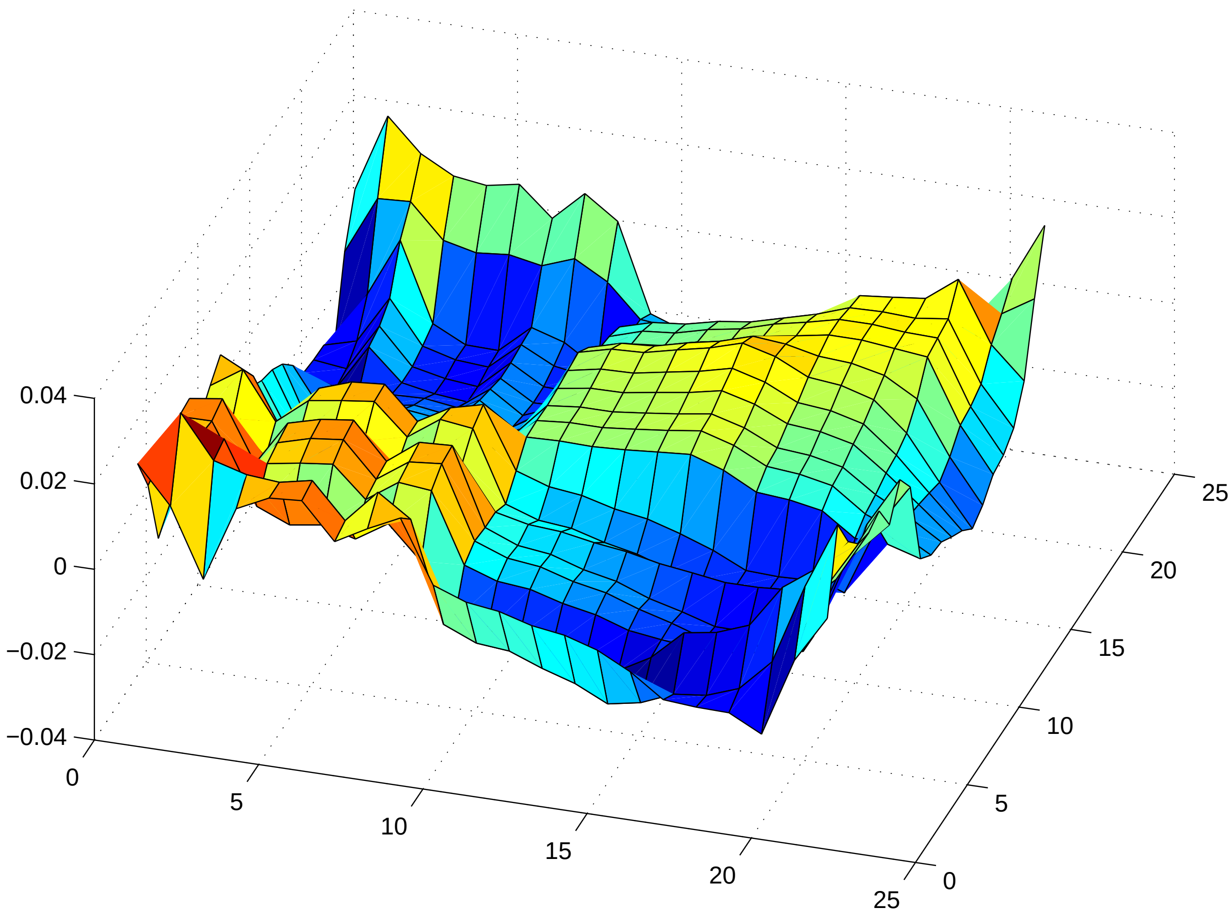


Figure 9: Low rank solution

Stability versus Rank

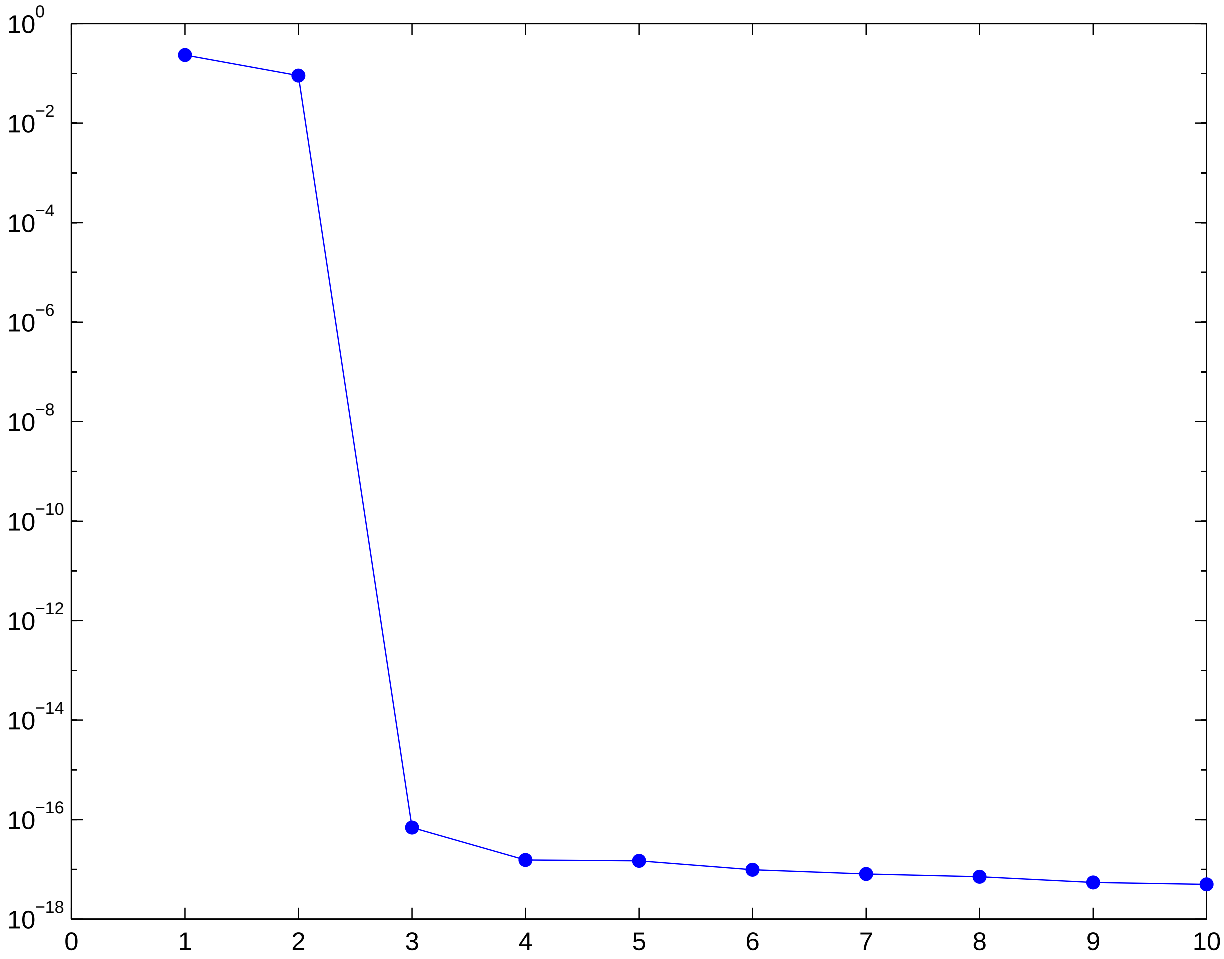


Figure 10: Low rank solution: eigenvalues (semilog).

Stability versus Rank

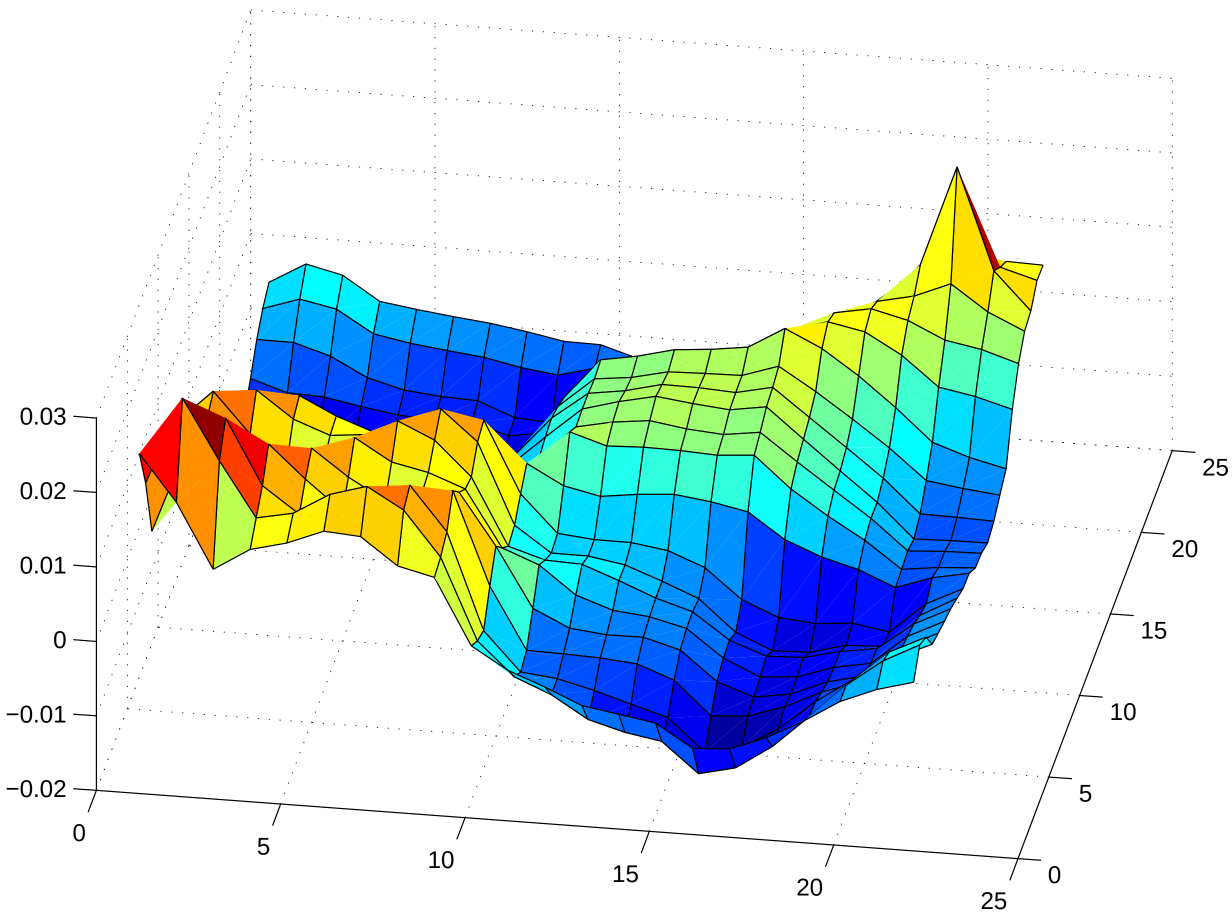


Figure 11: Smooth calibrated matrix

Stability versus Rank

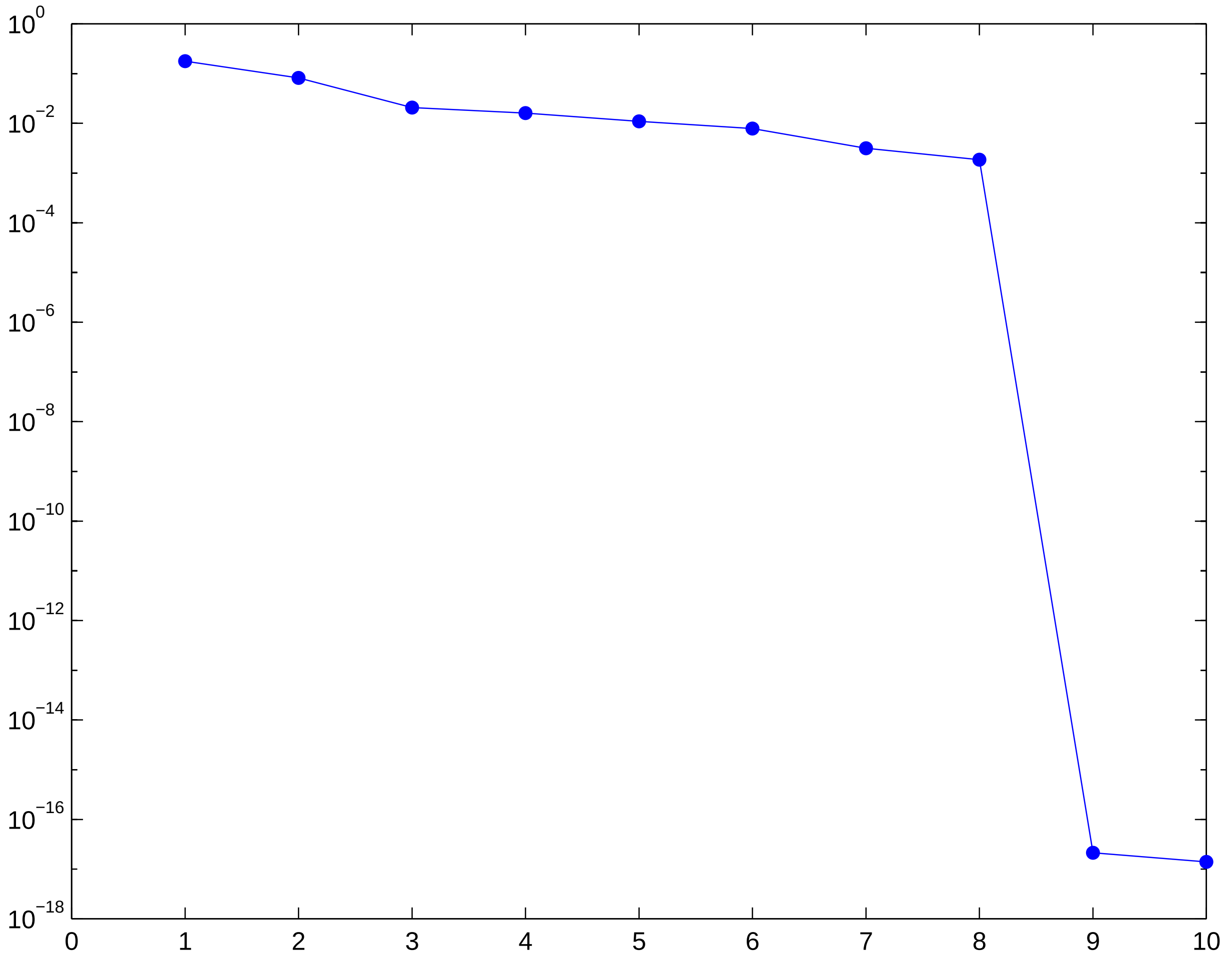


Figure 12: Smooth solution: eigenvalues (semilog).

Stability: Example

Simple numerical example:

- Simulate sample rates with a "real" covariance matrix.
- Price some basket options using this covariance.

Compare the stability of the daily P&L using various calibration objectives

- Use the basket prices to calibrate a covariance matrix.
- Update the hedge daily, or every other day.
- Evaluate daily P&L, compare P&L stability.

Stability: Example

- The real covariance is given by:

$$\begin{bmatrix} 0.045 & 0.022 & 0.022 & 0.022 & 0.022 \\ 0.022 & 0.045 & 0.022 & 0.022 & 0.022 \\ 0.022 & 0.022 & 0.045 & 0.022 & 0.022 \\ 0.022 & 0.022 & 0.022 & 0.045 & 0.022 \\ 0.022 & 0.022 & 0.022 & 0.022 & 0.045 \end{bmatrix}$$

- 5 assets with initial value $.1$.
- 10% Bid-Ask spread, 15% noise on prices.
- Maturity 1 year, 20 time steps, 200 scenarios generated.
- Price data from all single asset calls and the following baskets:

$$(1,1,0,0,0),(0,0,1,0,1),(0,0,1,1,1)$$

Stability: Example

- Model recalibrated every 3 steps.
- Price and hedge a new basket ATM call $(0,1,0.2,1,0)$.

Hedged basket relative P&L statistics:

	Min. rank	Min. norm	Real covar.
Mean:	-0.0024	-0.0017	-0.0010
Stdev:	0.0255	0.0242	0.02399

Proportional transaction costs:

	Min. rank	Min. norm	Real covar.
Mean:	2.30	2.28	2.25
Stdev:	0.68	0.65	0.64

Performed 4200 calibrations in $40'$, that's 0.6 second per cycle. . .

Stability: Example

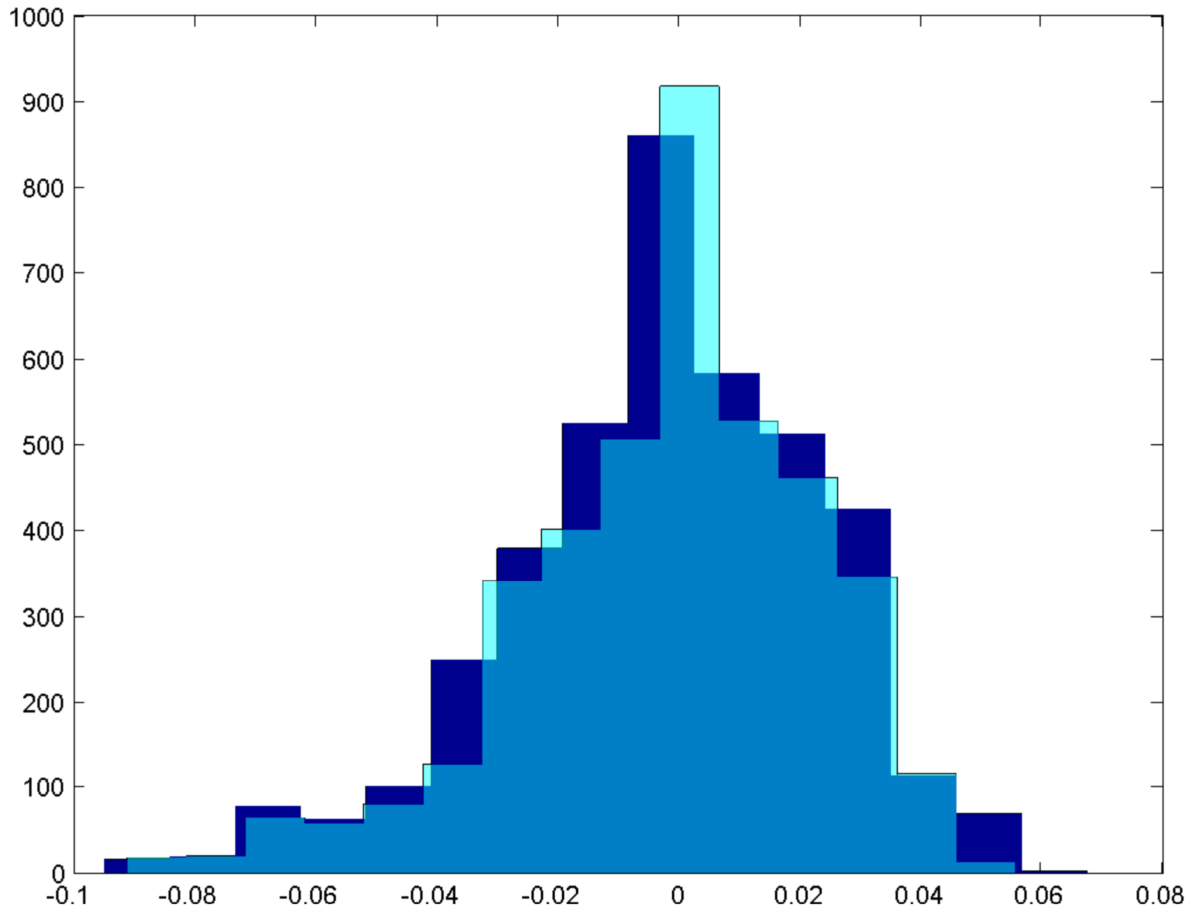


Figure 13: Relative daily P&L distribution, minimum rank (dark) and minimum norm, robust solution (light).

Outline

- Cone programming, a brief introduction
- IR model calibration
- **Risk-management**

Risk-Management

market data

$$\Downarrow$$

model calibration

$$\Downarrow$$

pricing & hedging

$$\Downarrow$$

risk-management

Delta-Gamma

What happens next?

- individual derivative contracts and their hedges are aggregated in a portfolio (or book)
- the residual risk on these positions is then managed globally

In general, the entire book is maintained delta-neutral:

$$\frac{\partial \Pi}{\partial F_t^i} = 0$$

when rebalancing is done at time intervals Δt, the residual risk is a mix of theta (time value) and gamma (convexity) exposure:

$$\Theta = \frac{\partial \Pi}{\partial t} \quad \text{and} \quad \Gamma = \frac{\partial^2 \Pi}{\partial F^2}$$

Delta-Gamma

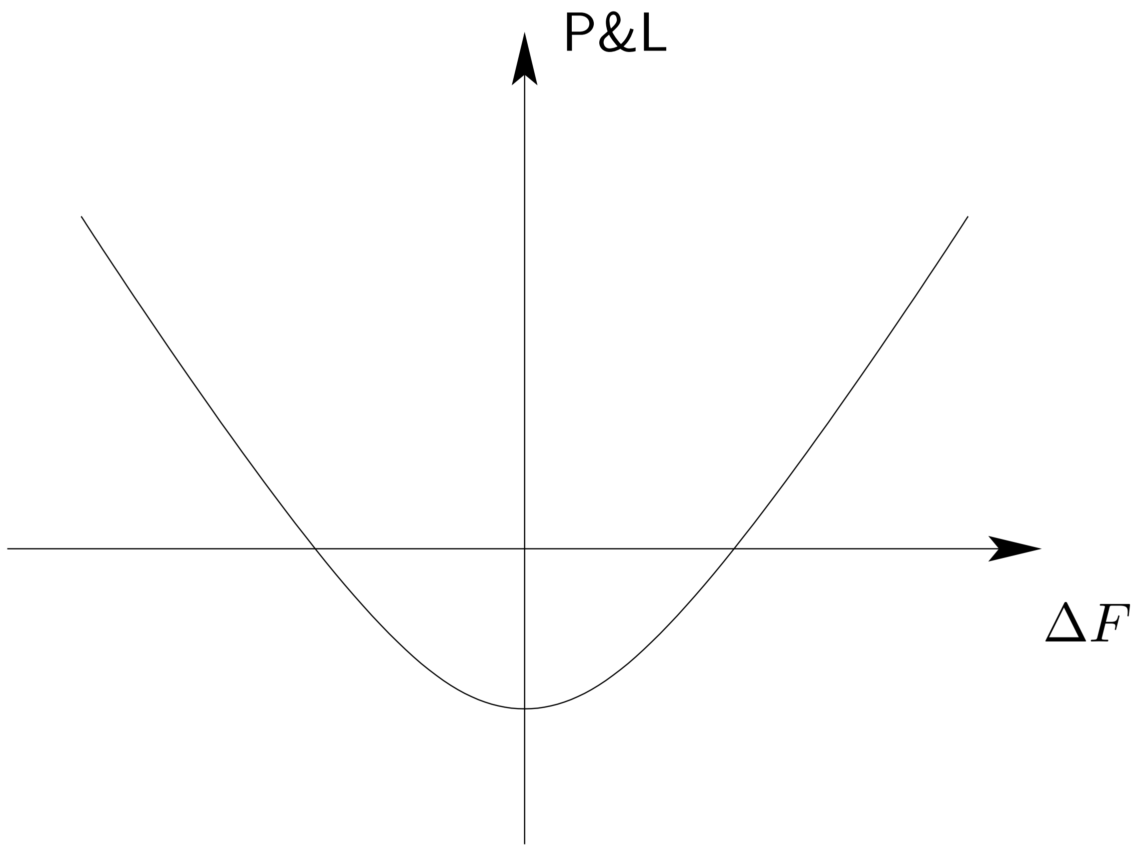


Figure 14: P&L at $t + \Delta t$ for a portfolio with positive Γ and negative Θ.

Gamma Exposure

- In dimension one, the residual exposure is then:

$$\Delta\Pi = \Theta\Delta t + \frac{1}{2}\Gamma\Delta F^2$$

- Adjusting the gamma exposure is done by buying or selling an option, the delta is then brought back to zero by trading in the stock.

- With n assets, the residual exposure is described by:

$$\Delta\Pi = \Theta\Delta t + \frac{1}{2}\,\mathbf{Tr}(\Gamma(\Delta F^i \Delta F^j))$$

- Adjusting the gamma is then non trivial, since Γ is now a matrix...

Gamma Exposure

- Following Douady (1995), suppose that we hold a delta hedged portfolio of derivatives on n assets F_t^i.
- We want to make it gamma positive, $i.e.$ $\Gamma \succeq 0$ (anticipating volatility).
- For liquidity reasons, we can only use options on each individual asset F_t^i, with gamma given by γ_i (no baskets).

The gamma positivity condition is then:

$$\Gamma + \mathbf{diag}\left(x_i \gamma_i\right) \succeq 0$$

where x_i is the number of options on asset F_t^i.

Optimal Gamma

- We suppose that the initial portfolio has zero delta and that delta neutrality is maintained by buying/selling y_i assets F_t^i.
- The price and delta of the vanilla options are given by p_i and Δ_i respectively.
- With proportional transaction costs k_i, the cheapest gamma positive portfolio is found by solving:

$$\begin{array}{ll} \text{minimize} & \sum_{i=1}^n k_i \left| x_i \right| + x_i p_i + y_i F_t^i \\ \text{subject to} & \Gamma + \mathbf{diag}\left(x_i \gamma_i \right) \succeq 0 \quad i = 1, ..., n \\ & \sum_{i=1}^n x_i \Delta_i + y_i = 0 \end{array}$$

which is a *cone program*.

To the Conclusion. . .

Cone Programs

market data

$\Downarrow$

model calibration

$\Downarrow$

pricing & hedging

$\Downarrow$

risk-management

“Hard” Price Constraints

- classic Black & Scholes (1973) option pricing based on:
  - a *dynamic hedging* argument
  - *model* for the asset dynamics (geometric BM)
- sensitive to liquidity, transaction costs, model risk ...
- what can we say about option prices with a minimal set of assumptions?

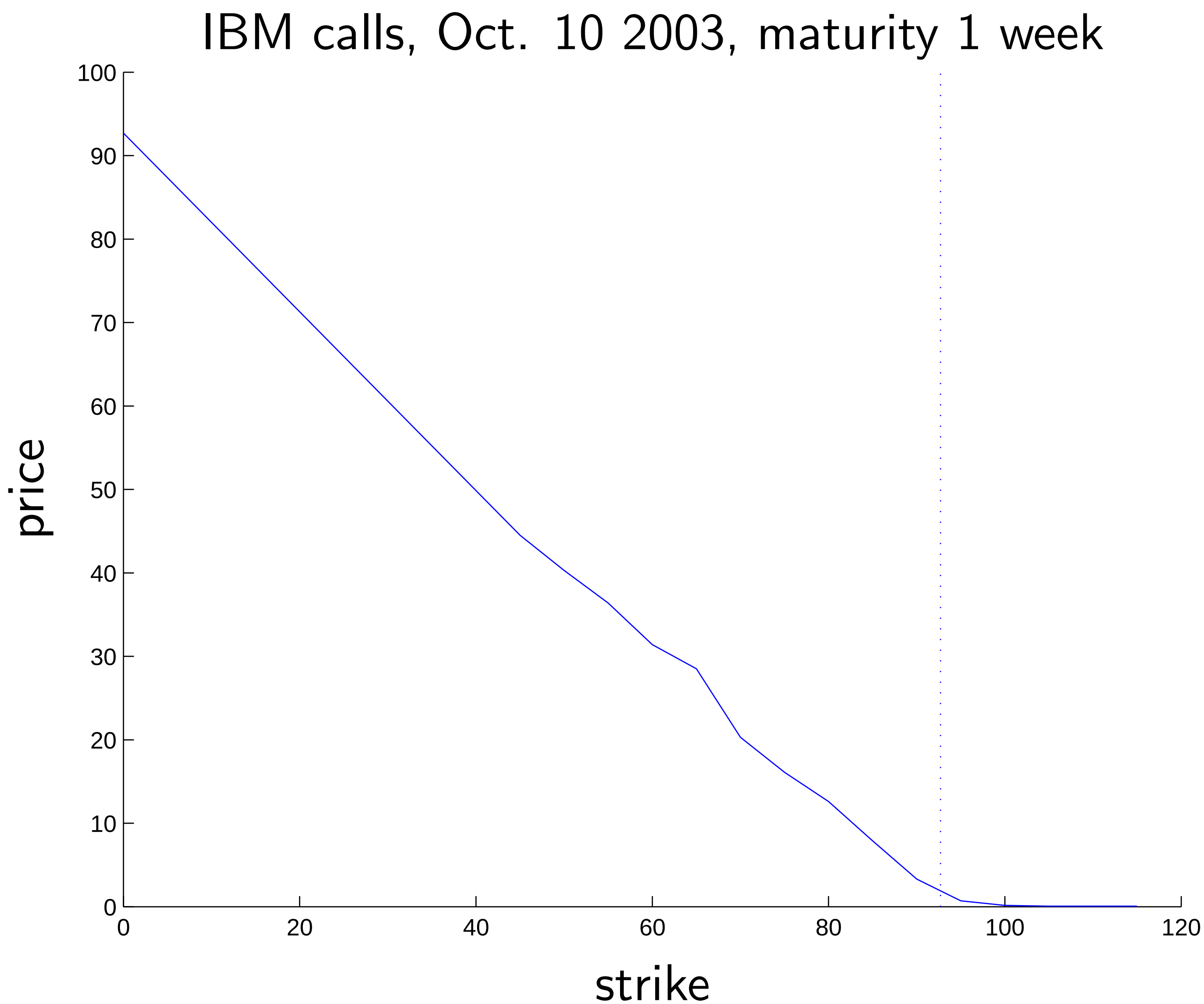
Static Arbitrage

Here, we rely on a minimal set of assumptions:

- *no assumption* on the asset distribution π
- *one period* model

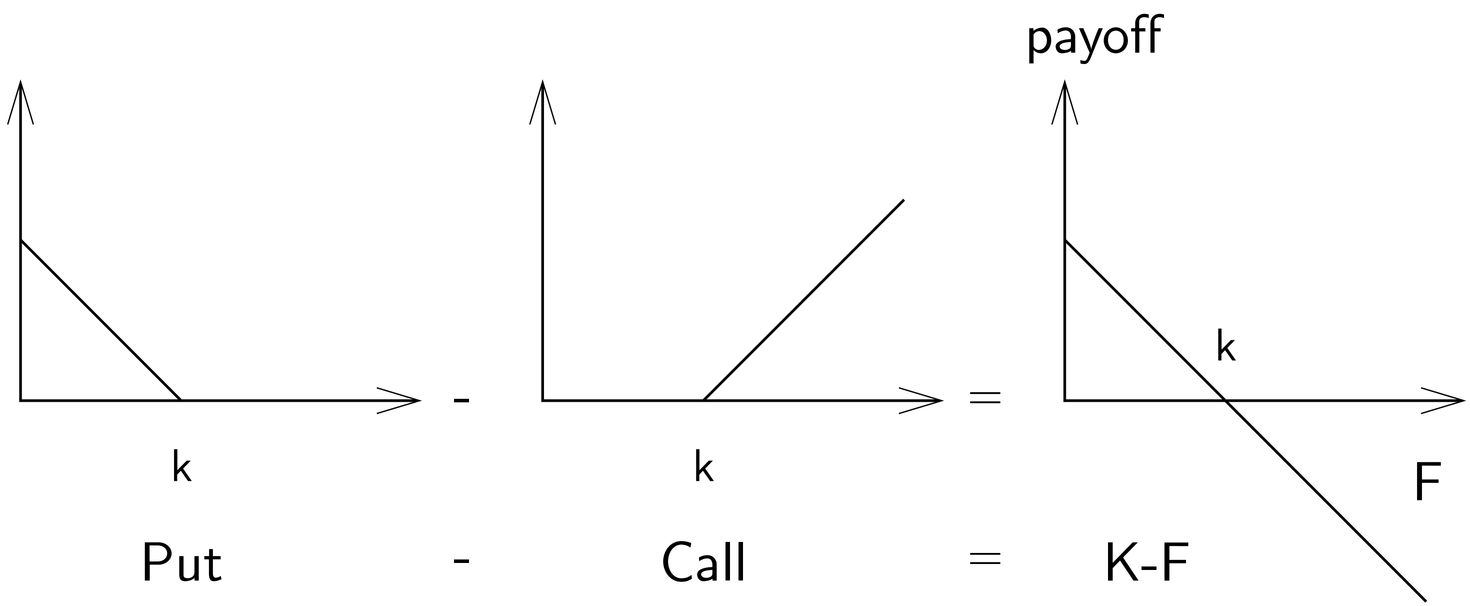
Arbitrage in this simple setting:

- form a portfolio at no cost today with a strictly positive payoff at maturity
- no trading involved between today and the option's maturity



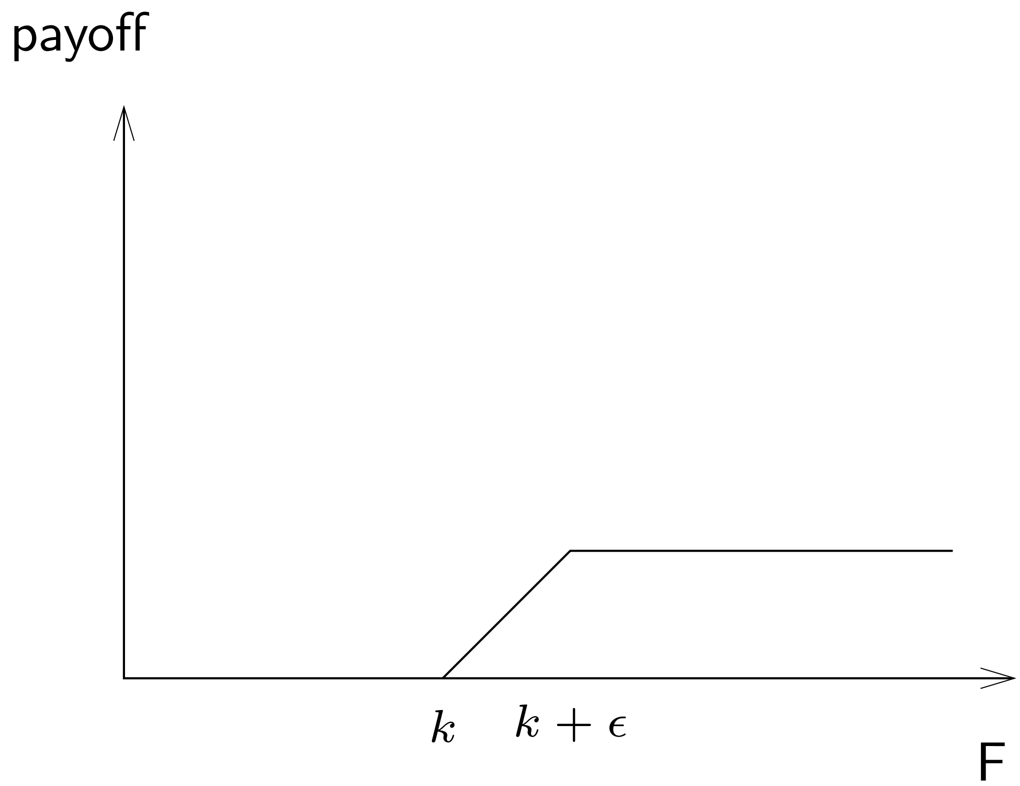
We note $C(K)$ the price of the call with payoff $(F-K)^+$

Simplest: Put Call Parity



If we know the forward prices (price of the asset F at maturity T), then we can deduce call prices from puts, ...

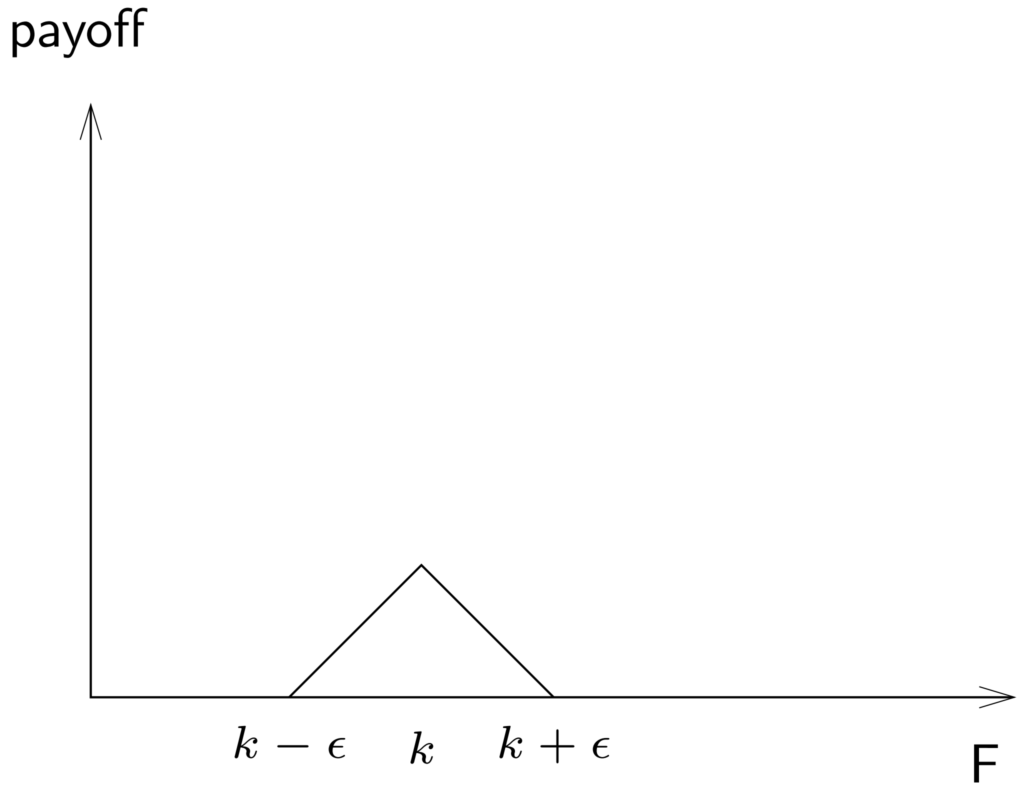
Call Spread



Here, Absence of Arbitrage implies that the price of a call spread be positive, hence call prices must be *decreasing* with strike

$$C(K + \epsilon) - C(K) \leq 0$$

Butterfly Spread



Absence of Arbitrage implies that the price of a butterfly spread be positive, hence call prices must be *convex* with strike

$$C(K+\epsilon)-2C(K)+C(K-\epsilon)\geq 0$$

Price Constraints

Absence of Arbitrage implies that if $C(K)$ is a function giving the price of an option of strike K, then $C(K)$ must satisfy:

- $C(K)$ positive
- $C(K)$ decreasing
- $C(K)$ convex

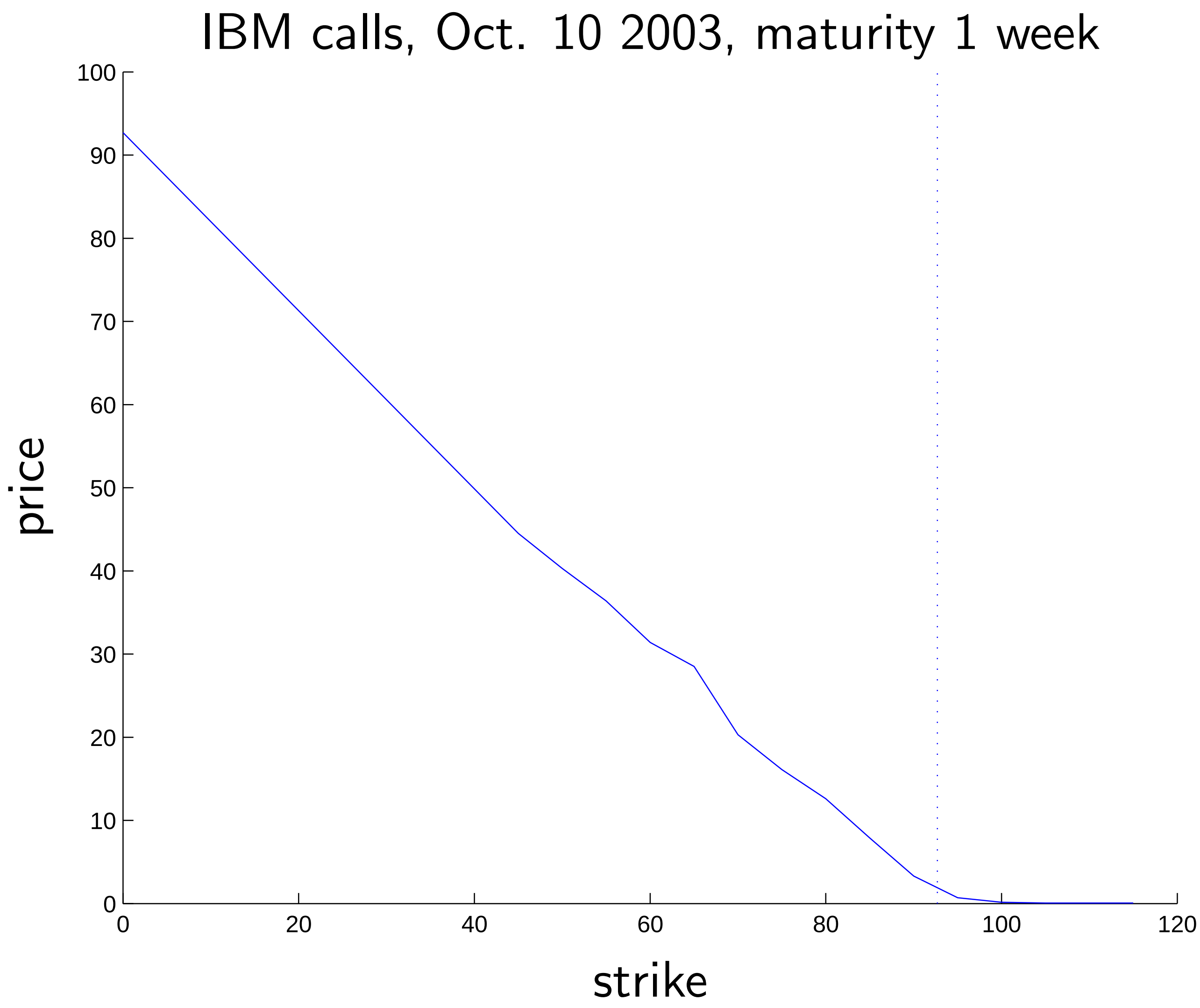
With $C(0) = F$, we have a set of *necessary* conditions for the absence of arbitrage

Sufficient Conditions

In fact, these conditions are also *sufficient*, see Breeden & Litzenberger (1978), Laurent & Leisen (2000) and Bertsimas & Popescu (2002) among others.

Suppose we have a set of market prices for calls $C(K_i) = p_i$, then there is no arbitrage iff there is a function C(K):

- $C(K)$ positive
- $C(K)$ decreasing
- $C(K)$ convex
- $C(K_i) = p_i$ and $C(0) = F$



Source: reuters

Why?

data quality...

- all the prices are last quotes (not simultaneous)
- low volume
- some transaction costs

Problem: this data is used to calibrate models and price other derivatives...

Dimension n: Basket Options

- a basket call payoff is
$$\left(\sum_{i=1}^{k} w_i F_i - K\right)_+$$
where $w_1, \ldots, w_k$ are the basket's weights and K is the option's strike price

- examples include: Index options, spread options, swaptions...

- basket option prices are used to gather information on *correlation*

We note $C(w, K)$ the price of such an option, can we get conditions to test basket price data?

Sufficient Conditions

Similar to dimension one...

Suppose we have a set of market prices for calls $C(w_i, K_i) = p_i$, and there is no arbitrage, then the function $C(w, K)$ satisfies:

- $C(w, K)$ positive
- $C(w, K)$ decreasing
- $C(w, K)$ jointly convex in (w, K)
- $C(w_i, K_i) = p_i$ and $C(0) = F$

Is this *tractable*?

Tractable?

The problem can be formulated as:

$$\begin{array}{ll} \text{find} & z \\ \text{subject to} & Az \leq b, \ Cz = d \\ & z = \left[f(x_1), \ldots, f(x_k), g_1^T, \ldots, g_k^T\right]^T \\ & g_i \text{ subgradient of } f \text{ at } x_i \quad i = 1, \ldots, k \\ & f \text{ monotone, convex} \end{array}$$

in the variables $f \in C(\mathbf{R}^n)$, $z \in \mathbf{R}^{(n+1)k}$, $g_1, \ldots, g_k \in \mathbf{R}^n$

- *discretize* and sample the convexity constraints to get a polynomial size LP feasibility problem

- enforce the convexity and subgradient constraints at the points $(x_i)_{i=1,\ldots,k}$ (monotonicity is a simple inequality on g) to get:

$$\begin{array}{ll}\text{find} & z\\ \text{subject to} & Cz = d,\ Az \leq b\\ & z = \left[f(x_1), \ldots, f(x_k), g_1^T, \ldots, g_k^T\right]^T\\ & \langle g_i, x_j - x_i\rangle \leq f(x_j) - f(x_i) \quad i,j = 1, \ldots, k\end{array}$$

in the variables $f(x_i)_{i=1,\ldots,k}$ and g in $\mathbf{R}^n \times \mathbf{R}^{n\times k}$

- we note $z^{\mathrm{opt}} = \left[f^{\mathrm{opt}}(x_1), \ldots, f^{\mathrm{opt}}(x_k), (g_1^{\mathrm{opt}})^T, \ldots, (g_k^{\mathrm{opt}})^T\right]^T$ a solution to this problem

- from z^{opt}, we define:

$$s(x) = \max_{i=1,\ldots,k} \left\{ f^{\mathrm{opt}}(x_i) + \left\langle g_i^{\mathrm{opt}}, x - x_i \right\rangle \right\}$$

- by construction, $s(x_i)$ solves the finite LP with:

$$s(x_i) = f^{\mathrm{opt}}(x_i), \quad i = 1, \ldots, k$$

- $s(x)$ is convex and monotone as the pointwise maximum of monotone affine functions

- so $s(x)$ is also a feasible point of the original problem

this means that the price conditions *remain tractable* on basket options...

Sufficient?

key difference with dimension one, Bertsimas & Popescu (2002) show that the exact problem is NP-Hard

- the conditions are *only necessary*...
- here however, numerical cost is minimal (small LP)
- we can show *tightness* in some particular cases
- how sharp are these conditions?

Full Conditions

derived by Henkin & Shananin (1990). A function can be written

$$C(w,K)=\int_{\mathbf{R}_+^n}(w^Tx-K)_+d\pi(x)$$

with $w\in\mathbf{R}_+^n$ and $K>0$, if and only if:

- $C(w,K)$ is *convex* and *homogenous* of degree one;
- $\lim_{K\to\infty}C(w,K)=0$ and $\lim_{K\to 0^+}\frac{\partial C(w,K)}{\partial K}=-1$
- $F(w)=\displaystyle\int_0^\infty e^{-K}d\left(\frac{\partial C(w,K)}{\partial K}\right)$ belongs to $C_0^\infty(\mathbf{R}_+^n)$
- For some $\tilde{w}\in\mathbf{R}_+^n$ the inequalities: $(-1)^{k+1}D_{\xi_1}...D_{\xi_k}F(\lambda\tilde{w})\geq 0$, for all positive integers k and $\lambda\in\mathbf{R}_{++}$ and all $\xi_1,\ldots,\xi_k$ in $\mathbf{R}_+^n$.

Numerical Example

- two assets: x_1, x_2, we look for bounds on the price of $(x_1 + x_2 - K)^+$
- simple discrete model for the assets:

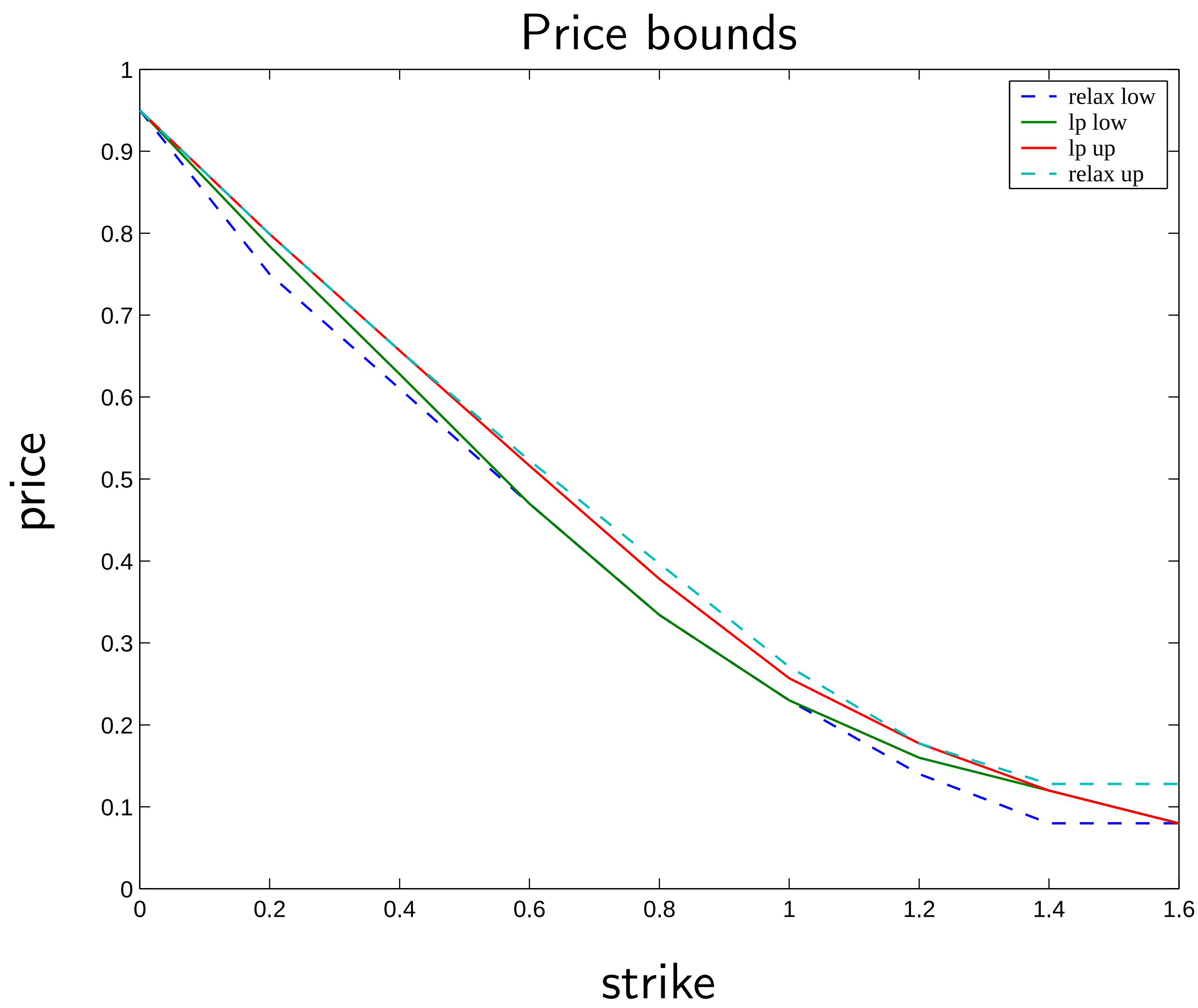
$$x = \{(0,0),(0,.8),(.8,.3),(.6,.6),(.1,.4),(1,1)\}$$

  with probability

$$p = (.2,.2,.2,.1,.1,.2)$$

- the forward prices are given, together with the following call prices:

$$(.2x_1 + x_2 - .1)^+, (.5x_1 + .8x_2 - .8)^+, (.5x_1 + .3x_2 - .4)^+,$$
$$(x_1 + .3x_2 - .5)^+, (x_1 + .5x_2 - .5)^+, (x_1 + .4x_2 - 1)^+, (x_1 + .6x_2 - 1.2)^+$$


Price bounds
relax low
lp low
lp up
relax up
price
strike
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
0
0.2
0.4
0.6
0.8
1
1.2
1.4
1.6

Cone Programs

market data

$\Downarrow$

model calibration

$\Downarrow$

pricing & hedging

$\Downarrow$

risk-management

Conclusion

- A number of important problems arising in the calibration & risk-management of multivariate models can be put in the form:

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax - b \succeq 0 \\ & \|Bx + d\| \leq d^T x + e \\ & \sum_{j=1}^n D_j x_j - D_0 \succeq 0 \end{array}$$

- These can be solved *very efficiently* using readily available software:

$SEDUMI$	http://fewcal.kub.nl/sturm/software/sedumi.html	GPL
$SDPT3$	http://www.math.nus.edu.sg/∼mattohkc/sdpt3.html	GPL
$MOSEK$	http://www.mosek.com	acad.

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