Homework 1

Please make sure to underline your answers.

Exercises

1 - Convex Sets

Which of the following sets are convex?

1. A slab, i.e., a set of the form
$$\{x \in \mathbb{R}^n \mid \alpha \leq a^T x \leq \beta\}.$$

2. A rectangle, i.e., a set of the form
$$\{x \in \mathbb{R}^n \mid \alpha_i \leq x_i \leq \beta_i,\ i = 1, \ldots, n\}.$$
A rectangle is sometimes called a hyperrectangle when $n > 2$.

3. A wedge, i.e.,
$$\{x \in \mathbb{R}^n \mid a_1^T x \leq b_1,\ a_2^T x \leq b_2\}.$$

4. The set of points closer to a given point than a given set, i.e.,
$$\{x \mid \|x - x_0\|_2 \leq \|x - y\|_2 \text{ for all } y \in S\}, \quad S \subseteq \mathbb{R}^n.$$

5. The set of points closer to one set than another, i.e.,
$$\{x \mid \operatorname{dist}(x, S) \leq \operatorname{dist}(x, T)\}, \quad S, T \subseteq \mathbb{R}^n,$$
where
$$\operatorname{dist}(x, S) = \inf\{\|x - z\|_2 \mid z \in S\}.$$
$$\{x \mid x + S_2 \subseteq S_1\}, \quad S_1, S_2 \subseteq \mathbb{R}^n \text{ with } S_1 \text{ convex}.$$

6. The set of points whose distance to a does not exceed a fixed fraction θ of the distance to b, i.e.,
$$\{x \mid \|x - a\|_2 \leq \theta \|x - b\|_2\}, \quad a \neq b,\ 0 \leq \theta \leq 1.$$

2 - Pointwise Maximum and Supremum

Show that the following functions $f : \mathbb{R}^n \to \mathbb{R}$ are convex.

1. $f(x) = \max_{i=1,\ldots,k} \|A^{(i)} x - b^{(i)}\|$, where $A^{(i)} \in \mathbb{R}^{m \times n}$, $b^{(i)} \in \mathbb{R}^m$, and $\|\cdot\|$ is a norm on $\mathbb{R}^m$.

2. $f(x) = \sum_{i=1}^{r} |x|_{[i]}$ on $\mathbb{R}^n$, where $|x|$ denotes the vector with $|x|_i = |x_i|$, and $|x|_{[i]}$ is the ith largest component of $|x|$.

3 - Products and Ratios of Convex Functions

Prove the following result, where f and g are real functions.

- If f and g are convex, both nondecreasing (or nonincreasing), and positive functions on an interval, then fg is convex.

4 - Conjugates of Functions

Derive the conjugates of the following functions.

1. Max function: $f(x) = \max_{i=1,\dots,n} x_i$ on $\mathbb{R}^n$.
2. Sum of largest elements: $f(x) = \sum_{i=1}^{r} x_{[i]}$ on $\mathbb{R}^n$.
3. Piecewise-linear function on $\mathbb{R}$: $f(x) = \max_{i=1,\dots,m}(a_i x + b_i)$, with $a_1 \leq \cdots \leq a_m$ and no redundant function.