Convex Optimization M2

Lecture 7

Large Scale Optimization

Outline

- First-order methods: introduction
- Exploiting structure
- First order algorithms
  - Subgradient methods
  - Gradient methods
  - Accelerated gradient methods
- **Other algorithms**
  - Coordinate descent methods
  - Localization methods
  - Franke-Wolfe
  - Dykstra, alternating projection
  - Stochastic optimization

Coordinate Descent

Coordinate Descent

We seek to solve

$$\begin{array}{ll} \text{minimize} & f(x) \\ \text{subject to} & x \in C \end{array}$$

in the variable $x \in \mathbb{R}^n$, with $C \subset \mathbb{R}^n$ convex.

- Our main assumption here is that C **is a product of simpler sets**. We rewrite the problem

$$\begin{array}{ll} \text{minimize} & f(x_1, \ldots, x_p) \\ \text{subject to} & x_i \in C_i, \quad i = 1, \ldots, p \end{array}$$

  where $C = C_1 \times \ldots \times C_p$.

- This helps if the minimization subproblems

$$\min_{x_i \in C_i} f(x_1, \ldots, x_i, \ldots, x_p)$$

  can be solved very efficiently (or in closed-form).

Coordinate Descent

Algorithm. The algorithm simply computes the iterates $x^{(k+1)}$ as

$$\begin{aligned} x_i^{(k+1)} &= \operatorname*{argmin}_{x_i \in C_i} f(x_1^{(k)}, \ldots, x_i^{(k)}, \ldots, x_p^{(k)}) \\ x_j^{(k+1)} &= x_j^{(k)}, \quad j \neq i \end{aligned}$$

for a certain $i \in [1, p]$, cycling over all indices in $[1, p]$.

Convergence.

- Complexity analysis similar to coordinate-wise gradient descent (or steepest descent in ℓ_1 norm).
- Need $f(x)$ strongly convex to get linear complexity bound.
- Few clean results outside of this setting.

Coordinate Descent

Example.

- Consider the box constrained minimization problem

$$\begin{array}{ll} \text{minimize} & x^T A x + b^T x \\ \text{subject to} & \|x\|_\infty \leq 1 \end{array}$$

  in the variable $x \in \mathbb{R}^n$. We assume $A \succ 0$.
- The set $\|x\|_\infty \leq 1$ is a box, i.e. a product of intervals.
- Each minimization subproblem means solving a second order equation.
- The dual is

$$\min_{y \in \mathbb{R}^n} \ (b+y)^T A^{-1} (b+y) - 4\|y\|_1$$

  which can be interpreted as a penalized regression problem in the variable $y \in \mathbb{R}^n$.

Localization methods

Localization methods

- Function $f : \mathbb{R}^n \to \mathbb{R}$ convex (and for now, differentiable)
- **problem:** minimize f
- **oracle model:** for any x we can evaluate f and $\nabla f(x)$ (at some cost)

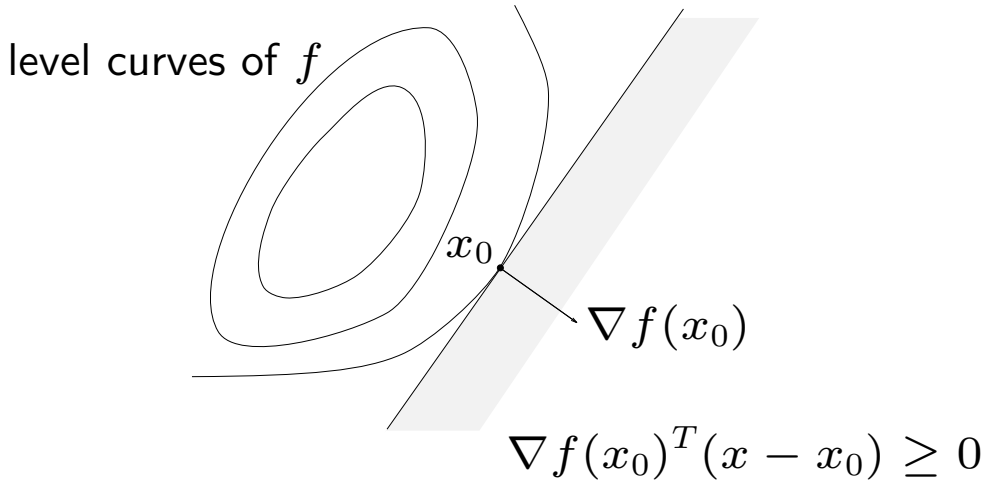
Main assumption: **evaluating the gradient is very expensive**.

from $f(x) \geq f(x_0) + \nabla f(x_0)^T (x - x_0)$ we conclude

$$\nabla f(x_0)^T (x - x_0) \geq 0 \quad \Longrightarrow \quad f(x) \geq f(x_0)$$

i.e., all points in halfspace $\nabla f(x_0)^T (x - x_0) \geq 0$ are **worse** than x_0

Localization methods



- by evaluating ∇f we rule out a halfspace in our search for $x^\star$:

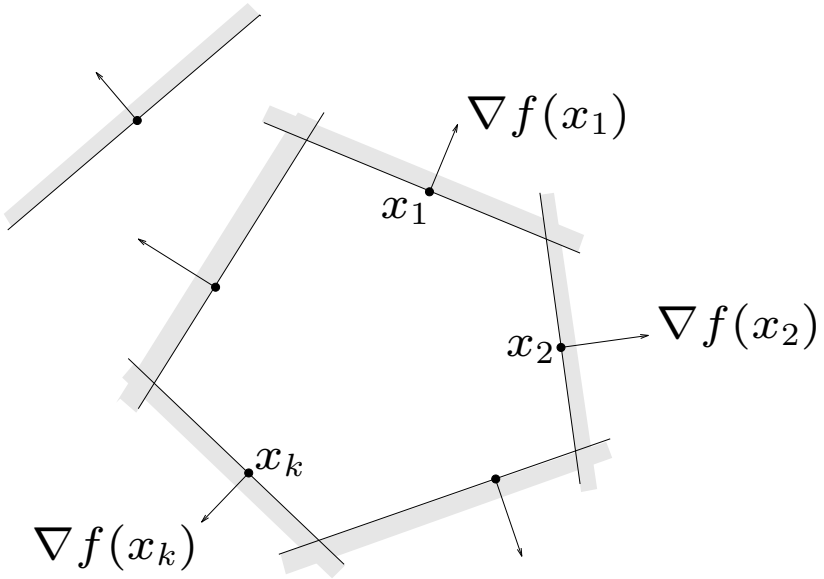
$$x^\star \in \{x \mid \nabla f(x_0)^T(x - x_0) \leq 0\}$$

- **idea:** get one bit of info (on location of $x^\star$) by evaluating ∇f
- for nondifferentiable f, can replace $\nabla f(x_0)$ with any subgradient $g \in \partial f(x_0)$

Localization methods

suppose we have evaluated $\nabla f(x_1), \ldots, \nabla f(x_k)$ then we know

$x^\star \in \{x \mid \nabla f(x_i)^T(x - x_i) \leq 0\}$



on the basis of $\nabla f(x_1), \ldots, \nabla f(x_k)$, we have **localized** $x^\star$ to a polyhedron

question: what is a 'good' point x_{k+1} at which to evaluate ∇f?

Localization methods

Basic **localization** (or cutting-plane) algorithm:

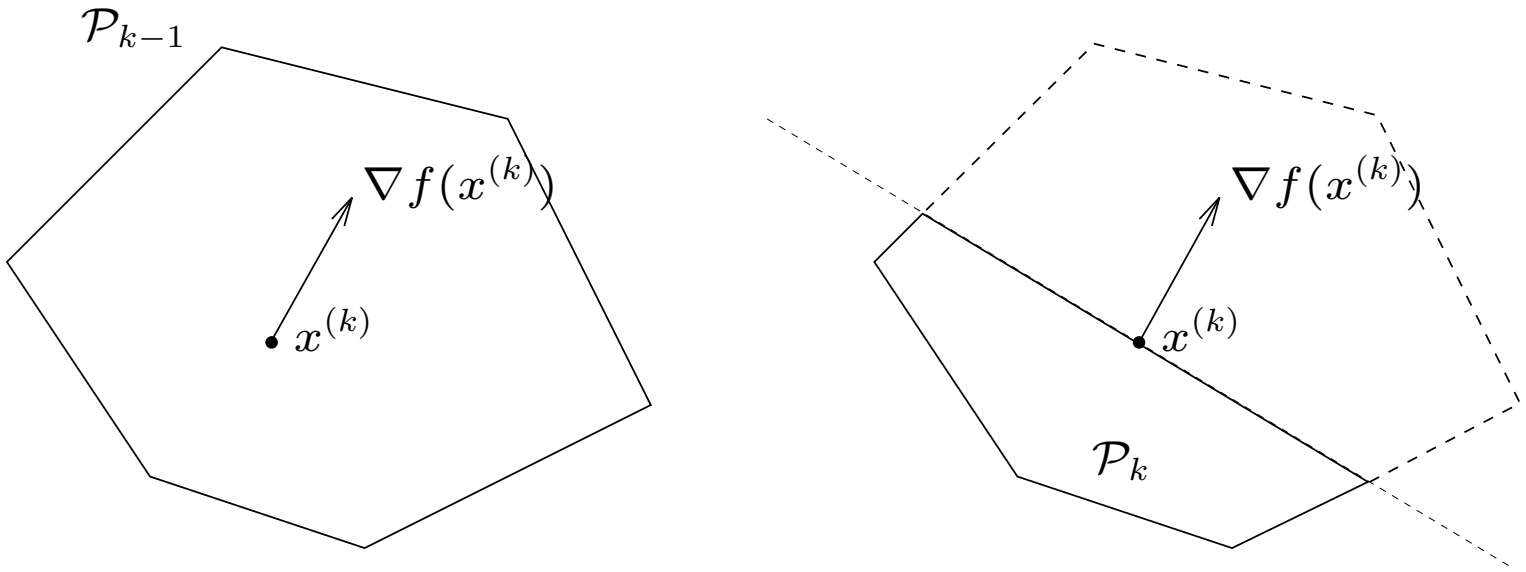
1. after iteration $k-1$ we know $x^\star \in \mathcal{P}_{k-1}$:

$$\mathcal{P}_{k-1} = \{x \mid \nabla f(x^{(i)})^T (x - x^{(i)}) \leq 0,\ i = 1, \ldots, k-1\}$$

2. evaluate $\nabla f(x^{(k)})$ (or $g \in \partial f(x^{(k)})$) for some $x^{(k)} \in \mathcal{P}_{k-1}$

3. $\mathcal{P}_k := \mathcal{P}_{k-1} \cap \{x \mid \nabla f(x^{(k)})^T (x - x^{(k)}) \leq 0\}$

Localization methods



- $\mathcal{P}_k$ gives our uncertainty of $x^\star$ at iteration k
- want to pick $x^{(k)}$ so that $\mathcal{P}_{k+1}$ is as small as possible
- clearly want $x^{(k)}$ near center of $C^{(k)}$

Example: bisection on $\mathbb{R}$

- $f : \mathbb{R} \to \mathbb{R}$
- $\mathcal{P}_k$ is interval
- obvious choice: $x^{(k+1)} := \text{midpoint}(\mathcal{P}_k)$

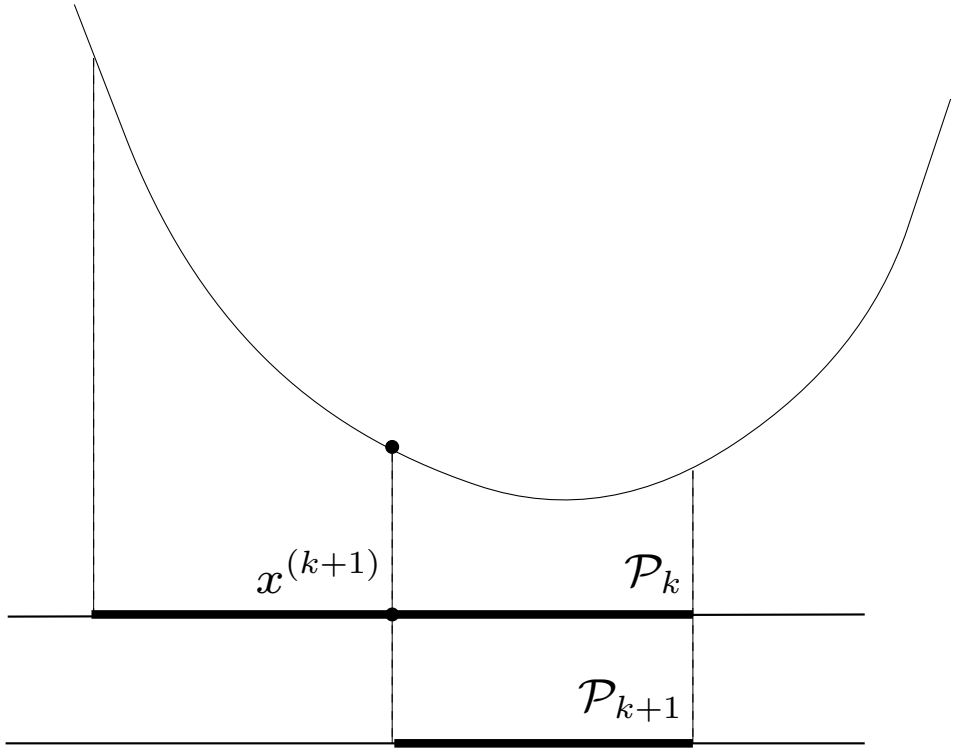
bisection algorithm

given interval $C = [l, u]$ containing $x^\star$

repeat

1. $x := (l + u)/2$
2. evaluate $f'(x)$
3. if $f'(x) < 0$, $l := x$; else $u := x$

Example: bisection on $\mathbb{R}$



$$\mathsf{length}(\mathcal{P}_{k+1}) = u_{k+1} - l_{k+1} = \frac{u_k - l_k}{2} = (1/2)\mathsf{length}(\mathcal{P}_k)$$

and so $\mathsf{length}(\mathcal{P}_k) = 2^{-k}\mathsf{length}(\mathcal{P}_0)$

Example: bisection on $\mathbb{R}$

interpretation:

- length$(\mathcal{P}_k)$ measures our uncertainty in $x^\star$
- uncertainty is halved at each iteration; get exactly one bit of info about $x^\star$ per iteration
- # steps required for uncertainty (in $x^\star$) $\leq \epsilon$:

$$\log_2 \frac{\text{length}(\mathcal{P}_0)}{\epsilon} = \log_2 \frac{\text{initial uncertainty}}{\text{final uncertainty}}$$

question:

- can bisection be extended to $\mathbb{R}^n$?
- or is it special since $\mathbb{R}$ is linear ordering?

Center of gravity algorithm

Take $x^{(k+1)} = \mathsf{CG}(\mathcal{P}_k)$ (center of gravity)

$$\mathsf{CG}(\mathcal{P}_k) = \int_{\mathcal{P}_k} x \, dx \Big/ \int_{\mathcal{P}_k} dx$$

theorem. if $C \subseteq \mathbb{R}^n$ convex, $x_{\mathsf{cg}} = \mathsf{CG}(C)$, $g \neq 0$,

$$\mathbf{vol}\left(C \cap \{x \mid g^T(x - x_{\mathsf{cg}}) \leq 0\}\right) \leq (1 - 1/e)\,\mathbf{vol}(C) \approx 0.63\ \mathbf{vol}(C)$$

(independent of dimension n)

hence in CG algorithm, $\mathbf{vol}(\mathcal{P}_k) \leq 0.63^k\,\mathbf{vol}(\mathcal{P}_0)$

Center of gravity algorithm

- $\mathbf{vol}(\mathcal{P}_k)^{1/n}$ measures uncertainty (in $x^\star$) at iteration k
- uncertainty reduced at least by $0.63^{1/n}$ each iteration
- from this can prove $f(x^{(k)}) \to f(x^\star)$ (later)
- max. # steps required for uncertainty $\leq \epsilon$:

$$1.51n \log_2 \frac{\text{initial uncertainty}}{\text{final uncertainty}}$$

(cf. bisection on $\mathbb{R}$)

Center of gravity algorithm

advantages of CG-method

- guaranteed convergence
- number of steps proportional to dimension n, log of uncertainty reduction

disadvantages

- finding $x^{(k+1)} = \mathrm{CG}(\mathcal{P}_k)$ is **harder** than original problem
- $\mathcal{P}_k$ becomes more complex as k increases
  (removing redundant constraints is harder than solving original problem)

(but, can modify CG-method to work)

Analytic center cutting-plane method

analytic center of polyhedron $\mathcal{P} = \{z \mid a_i^T z \preceq b_i,\ i = 1, \ldots, m\}$ is

$$\mathsf{AC}(\mathcal{P}) = \operatorname*{argmin}_z -\sum_{i=1}^{m} \log(b_i - a_i^T z)$$

ACCPM is localization method with next query point $x^{(k+1)} = \mathsf{AC}(\mathcal{P}_k)$ (found by Newton's method)

Outer ellipsoid from analytic center

- let x^* be analytic center of $\mathcal{P} = \{z \mid a_i^T z \preceq b_i,\ i = 1, \ldots, m\}$
- let H^* be Hessian of barrier at x^*,

$$H^* = -\nabla^2 \sum_{i=1}^m \log(b_i - a_i^T z)\Big|_{z=x^*} = \sum_{i=1}^m \frac{a_i a_i^T}{(b_i - a_i^T x^*)^2}$$

- then, $\mathcal{P} \subseteq \mathcal{E} = \{z \mid (z - x^*)^T H^* (z - x^*) \leq m^2\}$ (not hard to show)

Lower bound in ACCPM

- let $\mathcal{E}^{(k)}$ be outer ellipsoid associated with $x^{(k)}$
- a lower bound on optimal value $p^\star$ is

$$\begin{aligned} p^\star &\geq \inf_{z \in \mathcal{E}^{(k)}} \left(f(x^{(k)}) + g^{(k)T}(z - x^{(k)}) \right) \\ &= f(x^{(k)}) - m_k \sqrt{g^{(k)T} H^{(k)-1} g^{(k)}} \end{aligned}$$

  (m_k is number of inequalities in $\mathcal{P}_k$)

- gives simple stopping criterion $\sqrt{g^{(k)T} H^{(k)-1} g^{(k)}} \leq \epsilon / m_k$

Best objective and lower bound

since ACCPM isn't a descent a method, we keep track of best point found, and best lower bound

best function value so far: $u_k = \min_{i=1,\ldots,k} f(x^{(k)})$

best lower bound so far: $l_k = \max_{i=1,\ldots,k} f(x^{(k)}) - m_k \sqrt{g^{(k)T} H^{(k)-1} g^{(k)}}$

can stop when $u_k - l_k \leq \epsilon$

Basic ACCPM

given polyhedron $\mathcal{P}$ containing $x^\star$

repeat

1. compute x^*, the analytic center of $\mathcal{P}$, and H^*
2. compute $f(x^*)$ and $g \in \partial f(x^*)$
3. $u := \min\{u, f(x^*)\}$
   $l := \max\{l, f(x^*) - m\sqrt{g^T H^{*-1} g}\}$
4. add inequality $g^T(z - x^*) \leq 0$ to $\mathcal{P}$

until $u - l < \epsilon$

here m is number of inequalities in $\mathcal{P}$

Dropping constraints

ACCPM adds an inequality to $\mathcal{P}$ each iteration, so centering gets harder, more storage as algorithm progresses

schemes for **dropping constraints** from $\mathcal{P}^{(k)}$:

- remove all redundant constraints (expensive)
- remove some constraints known to be redundant
- remove constraints based on some relevance ranking

Dropping constraints in ACCPM

x^* is AC of $\mathcal{P} = \{x \mid a_i^T x \leq b_i,\ i = 1, \ldots, m\}$, H^* is barrier Hessian at x^*

define **(ir)relevance measure** $\eta_i = \dfrac{b_i - a_i^T x^*}{\sqrt{a_i^T H^{*-1} a_i}}$

- η_i/m is normalized distance from hyperplane $a_i^T x = b_i$ to outer ellipsoid
- if $\eta_i \geq m$, then constraint $a_i^T x \leq b_i$ is redundant

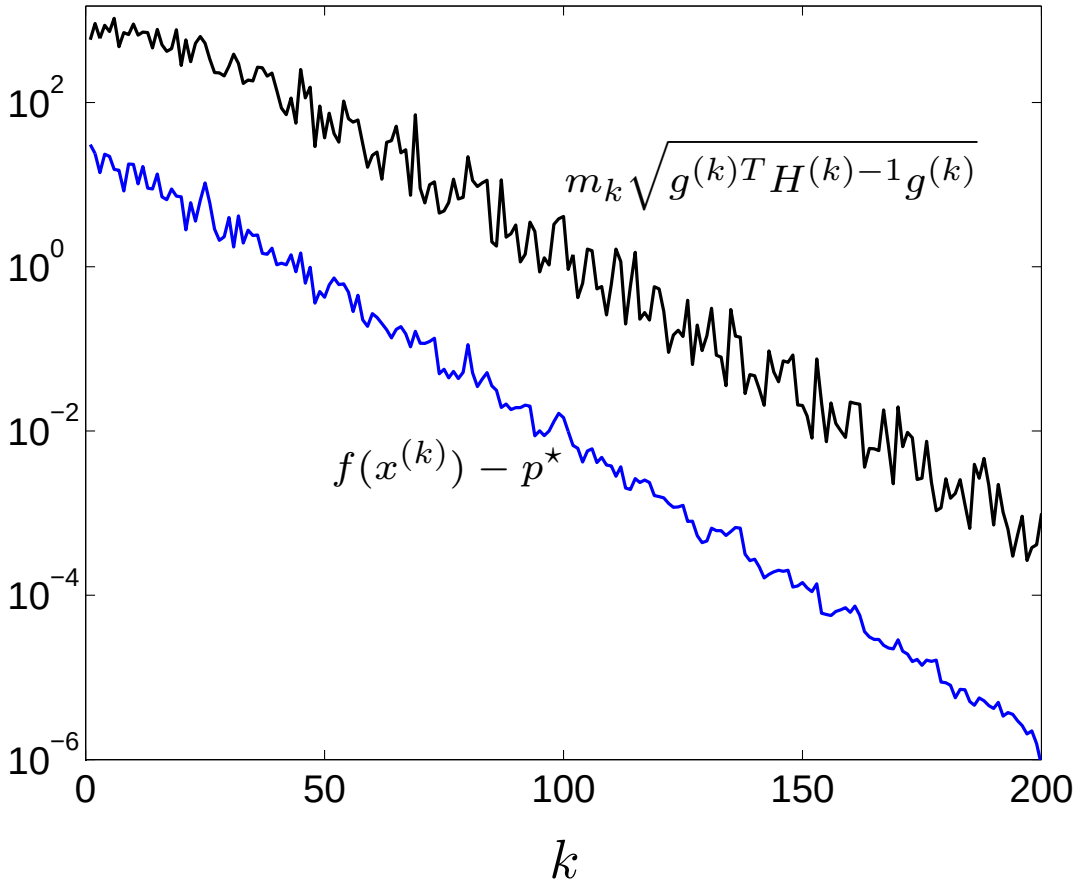
common ACCPM constraint dropping schemes:

- drop all constraints with $\eta_i \geq m$ (guaranteed to not change $\mathcal{P}$)
- drop constraints in order of irrelevance, keeping constant number, usually $3n$ – $5n$

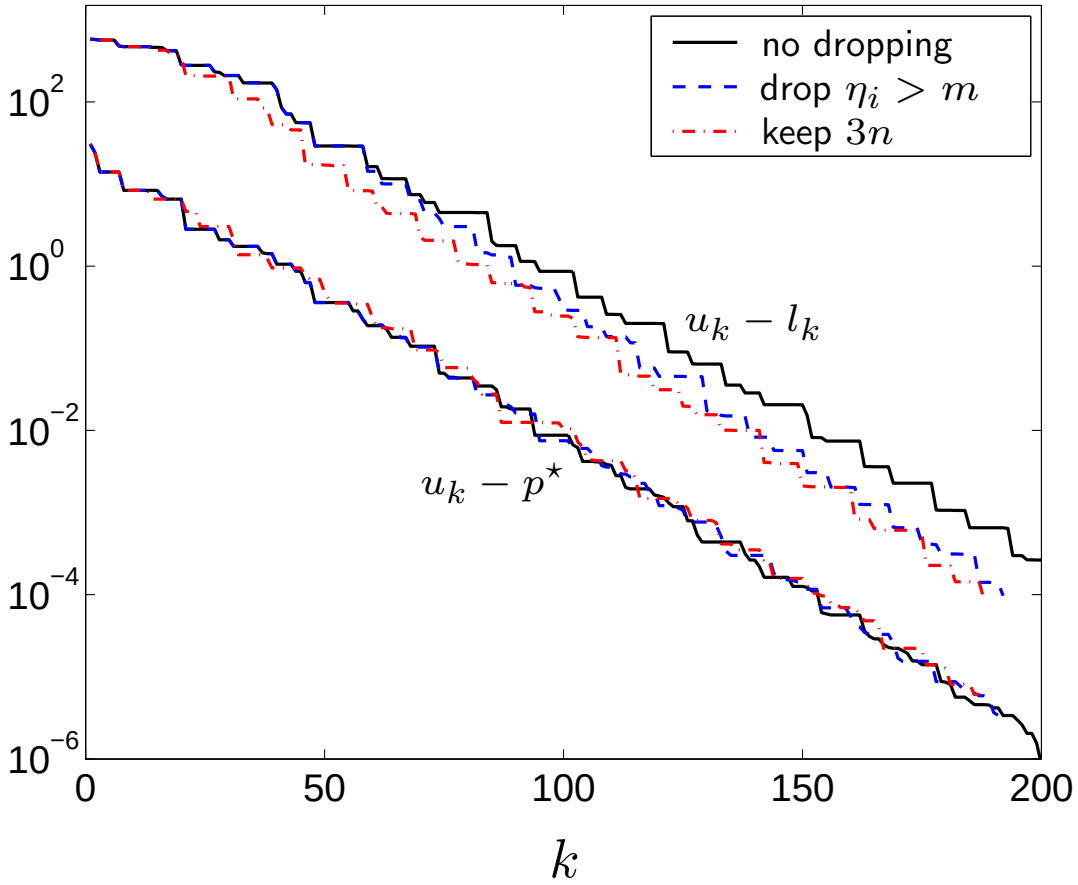
Example

PWL objective, $n = 10$ variables, $m = 100$ terms

simple ACCPM: $f(x^{(k)})$ and lower bound $f(x^{(k)}) - m\sqrt{g^{(k)T} H^{(k)-1} g^{(k)}}$

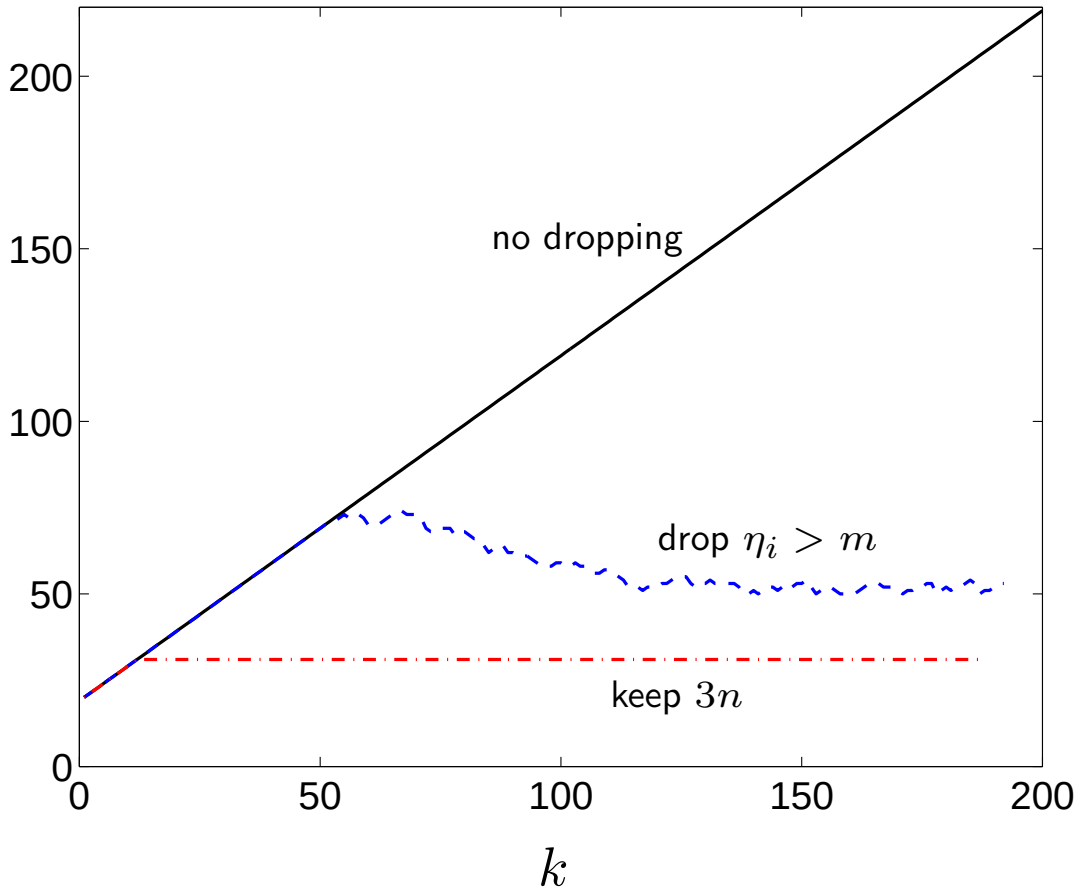


ACCPM with constraint dropping



ACCPM with constraint dropping

number of inequalities in $\mathcal{P}$:



. . . constraint dropping actually **improves** convergence (!)

The Ellipsoid Method

Challenges in cutting-plane methods:

- can be difficult to compute appropriate next query point
- localization polyhedron grows in complexity as algorithm progresses

can get around these challenges . . .

ellipsoid method is another approach

- developed in 70s by Shor and Yudin
- used in 1979 by Khachian to give polynomial time algorithm for LP

Ellipsoid algorithm

idea: localize $x^\star$ in an **ellipsoid** instead of a **polyhedron**

1. at iteration k we know $x^\star \in \mathcal{E}^{(k)}$

2. set $x^{(k+1)} := \text{center}(\mathcal{E}^{(k)})$; evaluate $\nabla f(x^{(k+1)})$ (or $g^{(k)} \in \partial f(x^{(k+1)})$)

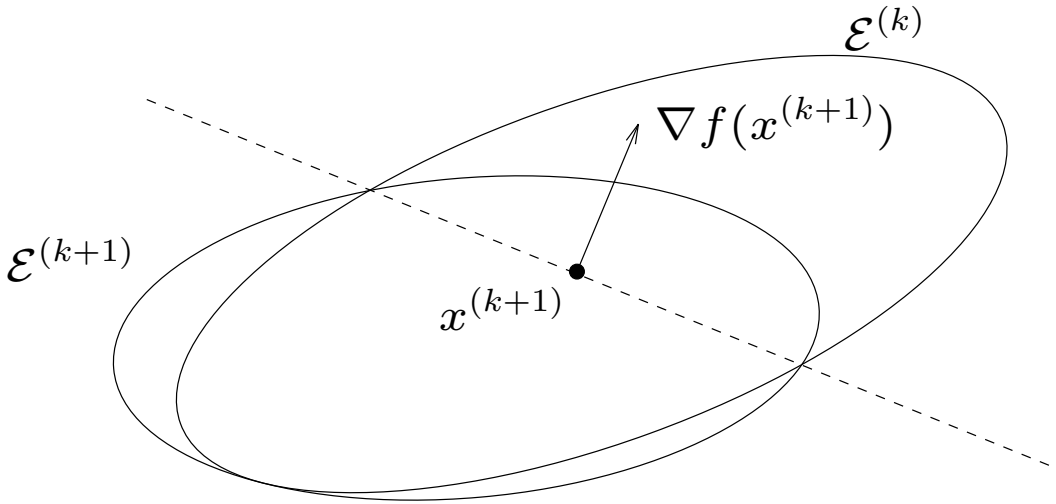
3. hence we know

$$x^\star \in \mathcal{E}^{(k)} \cap \{z \mid \nabla f(x^{(k+1)})^T (z - x^{(k+1)}) \leq 0\}$$

   (a half-ellipsoid)

4. set $\mathcal{E}^{(k+1)} :=$ minimum volume ellipsoid covering $\mathcal{E}^{(k)} \cap \{z \mid \nabla f(x^{(k+1)})^T (z - x^{(k+1)}) \leq 0\}$

Ellipsoid algorithm



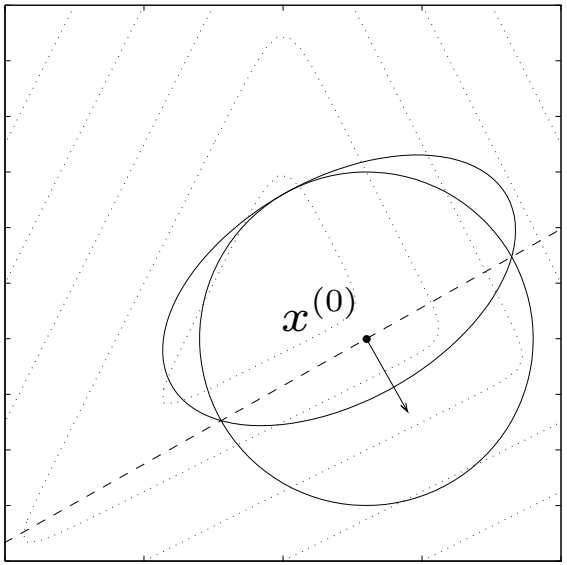
compared to cutting-plane method:

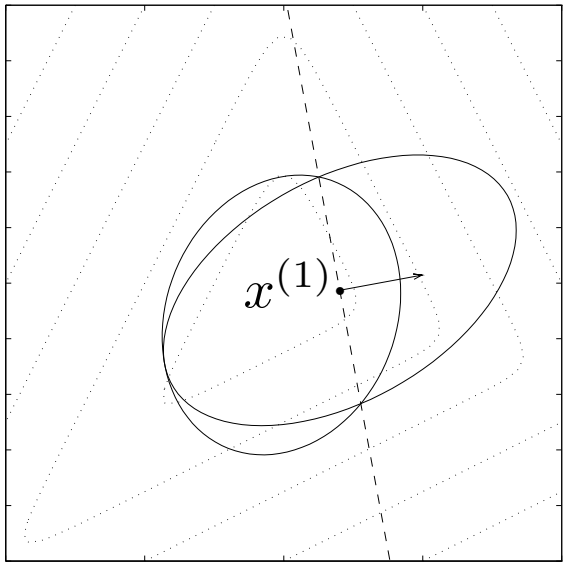
- localization set doesn't grow more complicated
- easy to compute query point
- but, we add unnecessary points in step 4

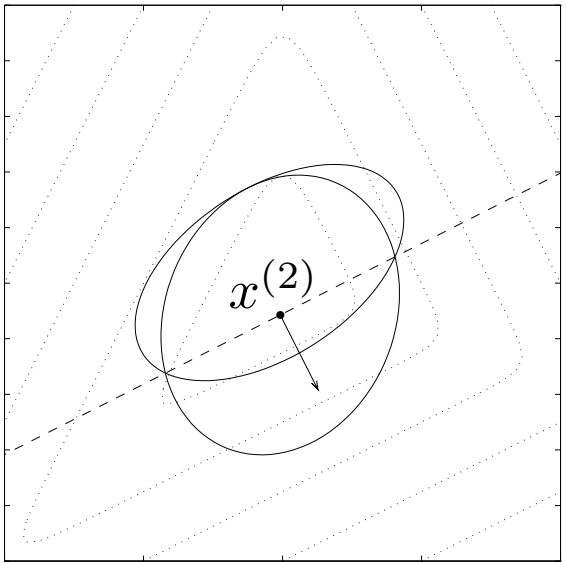
Properties of ellipsoid method

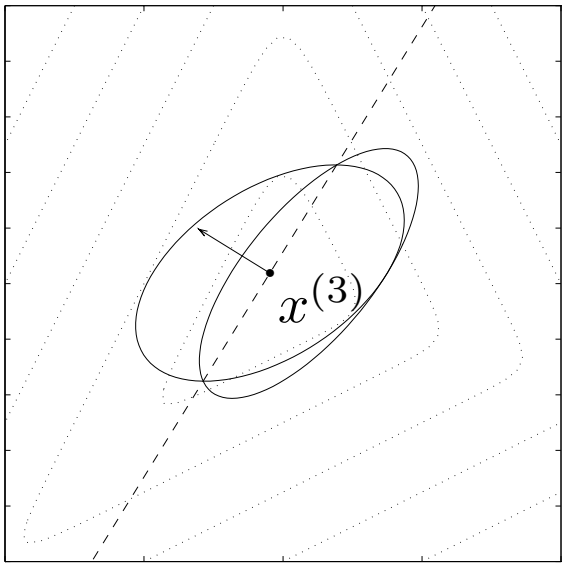
- reduces to bisection for $n = 1$
- simple formula for $\mathcal{E}^{(k+1)}$ given $\mathcal{E}^{(k)}$, $\nabla f(x^{(k+1)})$
- $\mathcal{E}^{(k+1)}$ can be larger than $\mathcal{E}^{(k)}$ in diameter (max semi-axis length), but is always smaller in volume
- $\mathbf{vol}(\mathcal{E}^{(k+1)}) < e^{-\frac{1}{2n}} \mathbf{vol}(\mathcal{E}^{(k)})$
  (note that volume reduction factor depends on n)

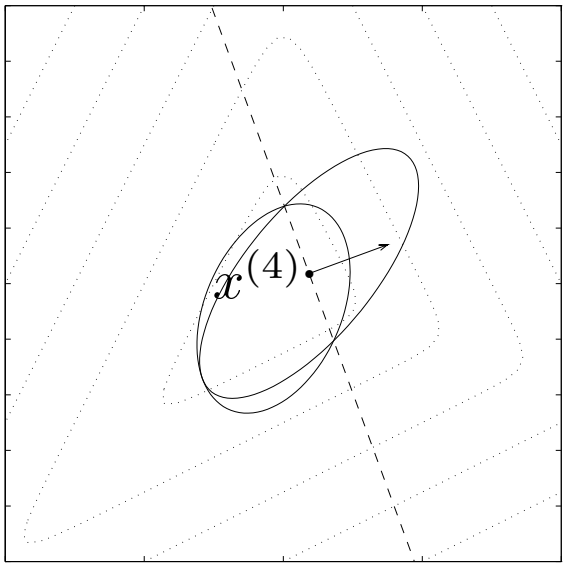
Example

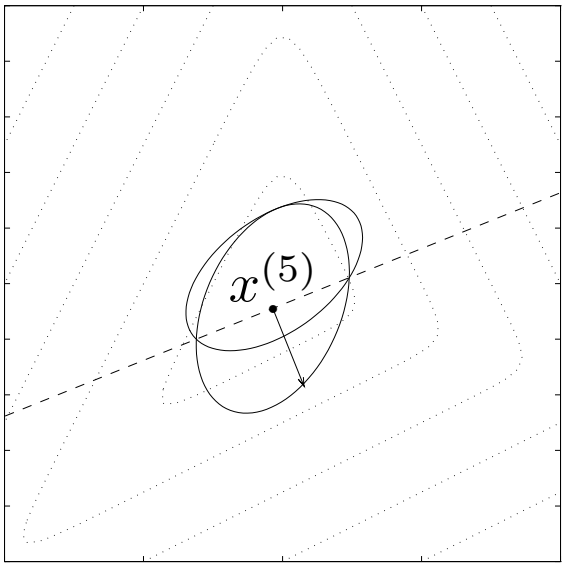






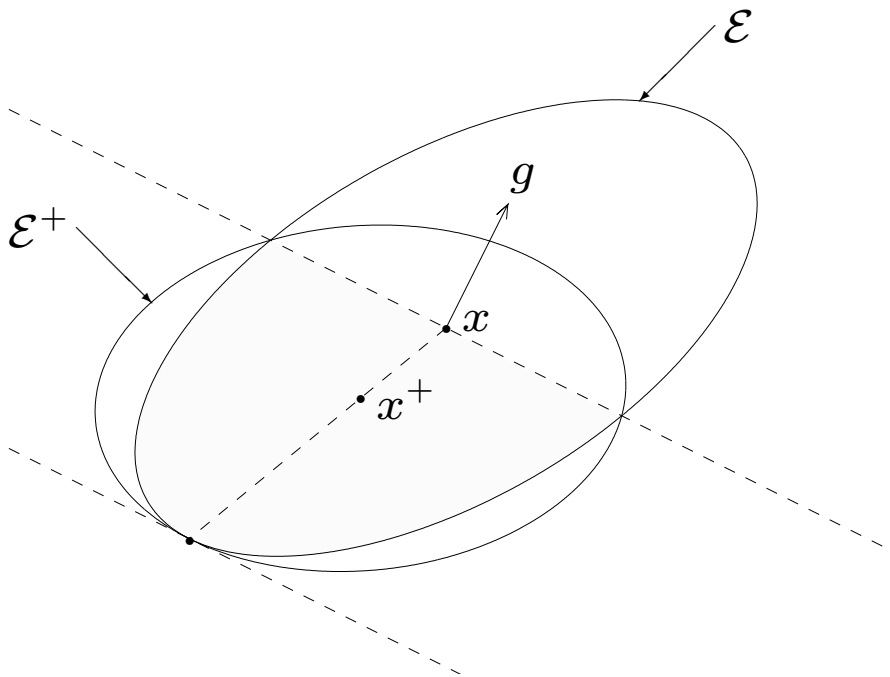






Updating the ellipsoid

$$\mathcal{E}(x, A) = \left\{ z \mid (z - x)^T A^{-1} (z - x) \leq 1 \right\}$$



Updating the ellipsoid

(for $n > 1$) minimum volume ellipsoid containing

$$\mathcal{E} \cap \left\{ z \mid g^T (z - x) \leq 0 \right\}$$

is given by

$$\begin{aligned} x^+ &= x - \frac{1}{n+1} A\tilde{g} \\ A^+ &= \frac{n^2}{n^2-1} \left(A - \frac{2}{n+1} A\tilde{g}\tilde{g}^T A \right) \end{aligned}$$

where $\tilde{g} \stackrel{\Delta}{=} g \Big/ \sqrt{g^T A g}$

Stopping criterion

As in the ACCPM case, we can get **error bounds** on the current iterate.

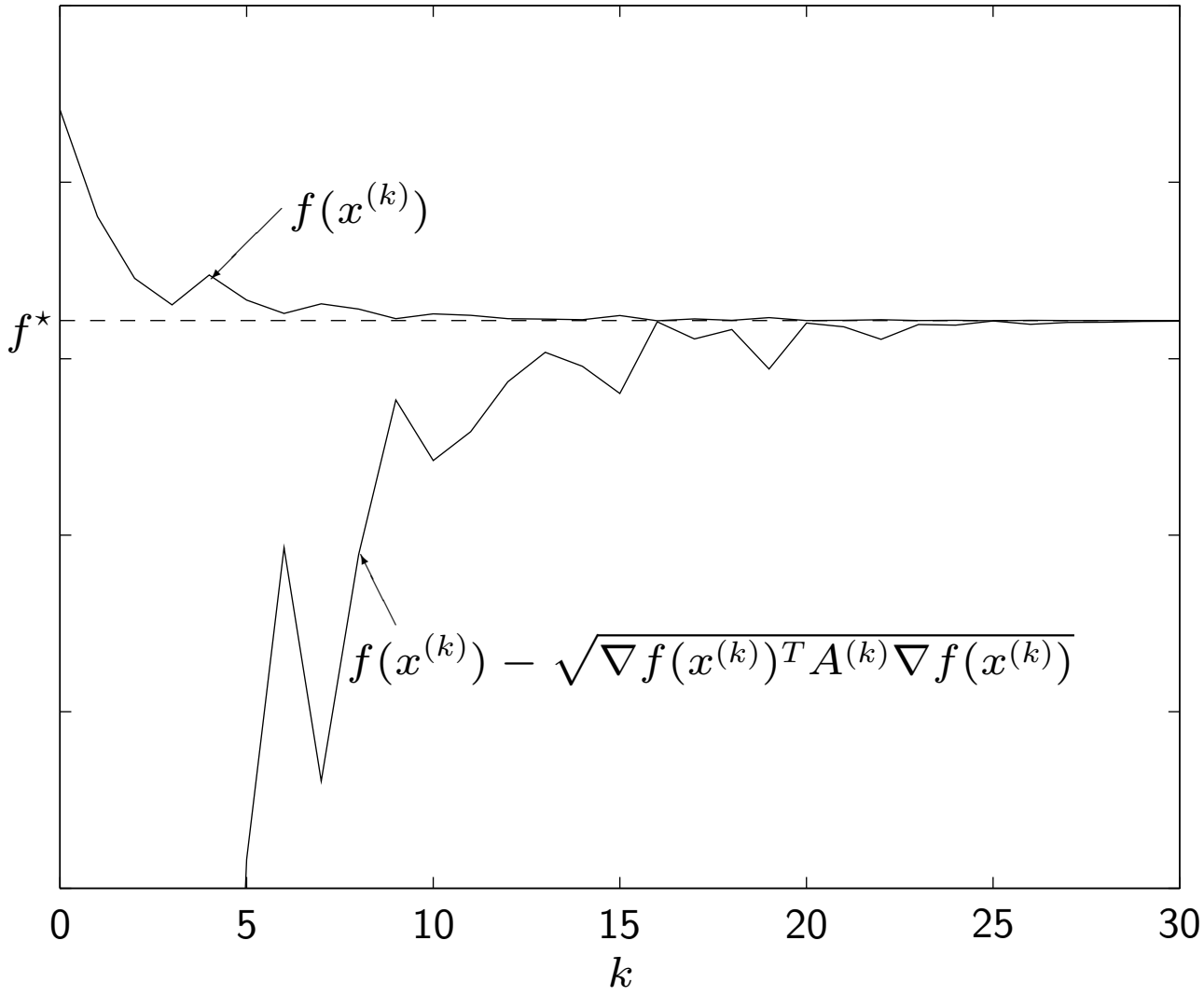
$x^\star \in \mathcal{E}_k$, so

$$\begin{aligned} f(x^\star) &\geq f(x^{(k)}) + \nabla f(x^{(k)})^T (x^\star - x^{(k)}) \\ &\geq f(x^{(k)}) + \inf_{x \in \mathcal{E}^{(k)}} \nabla f(x^{(k)})^T (x - x^{(k)}) \\ &= f(x^{(k)}) - \sqrt{\nabla f(x^{(k)})^T A^{(k)} \nabla f(x^{(k)})} \end{aligned}$$

simple stopping criterion:

$$\sqrt{\nabla f(x^{(k)})^T A^{(k)} \nabla f(x^{(k)})} \leq \epsilon$$

Stopping criterion



Basic ellipsoid algorithm

ellipsoid described as $\mathcal{E}(x, A) = \{\, z \mid (z-x)^T A^{-1}(z-x) \leq 1 \,\}$

given ellipsoid $\mathcal{E}(x, A)$ containing $x^\star$, accuracy $\epsilon > 0$

repeat

1. evaluate $\nabla f(x)$ (or $g \in \partial f(x)$)
2. if $\sqrt{\nabla f(x)^T A \nabla f(x)} \leq \epsilon$, return$(x)$
3. update ellipsoid

   3a. $\tilde{g} := \nabla f(x) \Big/ \sqrt{\nabla f(x)^T A \nabla f(x)}$

   3b. $x := x - \frac{1}{n+1} A\tilde{g}$

   3c. $A := \frac{n^2}{n^2-1}\left(A - \frac{2}{n+1} A\tilde{g}\tilde{g}^T A\right)$

properties:

- can propagate Cholesky factor of A; get $O(n^2)$ update
- **not** a descent method
- often slow but robust in practice

Franke-Wolfe

Franke-Wolfe

- Classical first order methods for solving

$$\begin{array}{ll} \text{minimize} & f(x) \\ \text{subject to} & x \in C, \end{array}$$

in $x \in \mathbb{R}^n$, with $C \subset \mathbb{R}^n$ convex, relied on the assumption that the following subproblem could be solved efficiently

$$\begin{array}{ll} \text{minimize} & y^T x + d(x) \\ \text{subject to} & x \in C, \end{array}$$

in the variable $x \in \mathbb{R}^n$, where $d(x)$ is a strongly convex function.

- The method detailed here assumes instead that the **affine minimization subproblem**

$$\begin{array}{ll} \text{minimize} & d^T x \\ \text{subject to} & x \in C \end{array}$$

can be solved efficiently for any $y \in \mathbb{R}^n$.

Franke-Wolfe

Algorithm.

- Choose $x_0 \in C$.
- **For** $k = 1, \ldots, k^{max}$ **iterate**

  1. Compute $d \in \partial f(y_k)$
  2. Solve

  $$\begin{array}{ll} \text{minimize} & d^T x \\ \text{subject to} & x \in C \end{array}$$

  in $x \in \mathbb{R}^n$, call the solution x_d.
  3. Update the current point

$$x_{k+1} = x_k + \frac{2}{k+2}(d - x_k)$$

Note that all iterates are feasible.

Franke-Wolfe

- **Complexity.** Assume that f is differentiable. Define the curvature C_f of the function $f(x)$ as

$$C_f \triangleq \sup_{\substack{s,x \in \mathcal{M},\ \alpha \in [0,1], \\ y = x + \alpha(s-x)}} \frac{1}{\alpha^2}(f(y) - f(x) - \langle y - x, \nabla f(x) \rangle).$$

  The Franke-Wolfe algorithm will then produce an ϵ solution after

$$N_{\max} = \frac{4C_f}{\epsilon}$$

  iterations.

Franke-Wolfe

- **Stopping criterion.** At each iteration, we get a lower bound on the optimum as a byproduct of the affine minimization step. By convexity

$$f(x_k) + \nabla f(x_k)^T (x_d - x_k) \leq f(x), \quad \text{for all } x \in C$$

and finally, calling f^* the optimal value of problem, we obtain

$$f(x_k) - f^* \leq \nabla f(x_k)^T (x_k - x_d).$$

This allows us to bound the suboptimality of iterate at no additional cost.

Dykstra, alternating projection

Dykstra, alternating projection

We focus on a simple **feasibility problem**

$$\text{find } x \in C_1 \cap C_2$$

in the variable $x \in \mathbb{R}^n$ with C_1, $C_2 \subset \mathbb{R}^n$ two convex sets.

We assume now that the projection problems on C_i are easier to solve

$$\begin{array}{ll} \text{minimize} & \|x - y\|_2 \\ \text{subject to} & x \in C_i \end{array}$$

in $x \in \mathbb{R}^n$.

Dykstra, alternating projection

Algorithm (alternating projection)

- Choose $x_0 \in \mathbb{R}^n$.
- **For** $k = 1, \ldots, k^{max}$ **iterate**

  1. Project on C_1
  $$x_{k+1/2} = \underset{x \in C_1}{\operatorname{argmin}} \|x - x_k\|_2$$
  2. Project on C_2
  $$x_{k+1} = \underset{x \in C_2}{\operatorname{argmin}} \|x - x_{k+1/2}\|_2$$

Convergence. We can show $\mathbf{dist}(x_k, C_1 \cap C_2) \to 0$. Linear convergence provided some additional regularity assumptions.

Dykstra, alternating projection

Algorithm (Dykstra)

- Choose $x_0, z_0 \in \mathbb{R}^n$.
- **For** $k = 1, \ldots, k^{max}$ **iterate**

1. Project on C_1
$$x_{k+1/2} = \operatorname*{argmin}_{x \in C_1} \|x - z_k\|_2$$
2. Update
$$z_{k+1/2} = 2x_{k+1/2} - z_k$$
3. Project on C_2
$$x_{k+1} = \operatorname*{argmin}_{x \in C_2} \|x - z_{k+1/2}\|_2$$
4. Update
$$z_{k+1} = z_k + x_{k+1} - x_{k+1/2}$$

Convergence. Usually faster than simple alternating projection.

Stochastic Optimization

Stochastic Optimization

Solve

$$\begin{array}{ll} \text{minimize} & \mathbf{E}[f(x,\xi)] \\ \text{subject to} & x \in C, \end{array}$$

in $x \in \mathbb{R}^n$, where C is a simple convex set. The key difference here is that the function we are minimizing is **stochastic.**

Batch method. A simple option is to approximate the problem by

$$\begin{array}{ll} \text{minimize} & \sum_{i=1}^m f(x,\xi_m) \\ \text{subject to} & x \in C, \end{array}$$

where ξ_i are sampled from the distribution of ξ.

Sampling is costly, we can do better. . .

Stochastic Optimization

Let $p_C(\cdot)$ be the Euclidean projection operator on C.

Algorithm (Robust stochastic averaging)

- Choose $x_0 \in C$ and a step sequence $\gamma_j > 0$.
- **For** $k = 1, \ldots, k^{max}$ **iterate**

  1. Compute a subgradient
  $$g \in \partial f(x_k, \xi_k)$$
  2. Update the current point
  $$x_{k+1} = p_C(x_k - \gamma_k g)$$

Stochastic Optimization

Complexity.

- Call $\tilde{x}_k = \sum_{i=1}^k \gamma_i x_i$ and assume

$$\max_{x \in C} \mathbf{E}[\|g\|_2^2] \leq M^2, \quad \text{and} \quad D_C = \max_{x,y \in C} \|x - y\|_2$$

- If we set $\gamma_i = D_C/(M\sqrt{k})$, we have

$$\mathbf{E}[f(\tilde{x}_k) - f^*] \leq \frac{D_C M}{\sqrt{k}}$$

- Furthermore, if we assume

$$\mathbf{E}\left[\exp\left(\frac{\|g\|_2^2}{M^2}\right)\right] \leq e, \quad \text{for all } g \in \partial f(x_k, \xi) \text{ and } x \in C$$

we get

$$\mathbf{Prob}\left[f(\tilde{x}_k) - f^* \geq \frac{D_C M}{\sqrt{k}}(12 + 2t)\right] \leq 2\exp(-t).$$