Convex Optimization

Convex Problems
Today

- optimization problem in standard form
- convex optimization problems
- quasiconvex optimization
- linear optimization
- quadratic optimization
Optimization problem in standard form

\begin{align*}
\text{minimize} & \quad f_0(x) \\
\text{subject to} & \quad f_i(x) \leq 0, \quad i = 1, \ldots, m \\
& \quad h_i(x) = 0, \quad i = 1, \ldots, p
\end{align*}

- $x \in \mathbb{R}^n$ is the optimization variable
- $f_0 : \mathbb{R}^n \rightarrow \mathbb{R}$ is the objective or cost function
- $f_i : \mathbb{R}^n \rightarrow \mathbb{R}$, $i = 1, \ldots, m$, are the inequality constraint functions
- $h_i : \mathbb{R}^n \rightarrow \mathbb{R}$ are the equality constraint functions

optimal value:

\[ p^* = \inf \{ f_0(x) \mid f_i(x) \leq 0, \ i = 1, \ldots, m, \ h_i(x) = 0, \ i = 1, \ldots, p \} \]

- $p^* = \infty$ if problem is infeasible (no $x$ satisfies the constraints)
- $p^* = -\infty$ if problem is unbounded below
Optimal and locally optimal points

$x$ is **feasible** if $x \in \text{dom} \, f_0$ and it satisfies the constraints

A feasible $x$ is **optimal** if $f_0(x) = p^*$; $X_{\text{opt}}$ is the set of optimal points

$x$ is **locally optimal** if there is an $R > 0$ such that $x$ is optimal for

\[
\begin{align*}
\text{minimize (over } z) \quad & f_0(z) \\
\text{subject to} \quad & f_i(z) \leq 0, \quad i = 1, \ldots, m, \quad h_i(z) = 0, \quad i = 1, \ldots, p \\
& \|z - x\|_2 \leq R
\end{align*}
\]

**Examples** (with $n = 1, m = p = 0$)

- $f_0(x) = 1/x$, $\text{dom} \, f_0 = \mathbb{R}_{++}$: $p^* = 0$, no optimal point
- $f_0(x) = -\log x$, $\text{dom} \, f_0 = \mathbb{R}_{++}$: $p^* = -\infty$
- $f_0(x) = x \log x$, $\text{dom} \, f_0 = \mathbb{R}_{++}$: $p^* = -1/e$, $x = 1/e$ is optimal
- $f_0(x) = x^3 - 3x$, $p^* = -\infty$, local optimum at $x = 1$
**Implicit constraints**

the standard form optimization problem has an **implicit constraint**

\[ x \in \mathcal{D} = \bigcap_{i=0}^{m} \text{dom } f_i \cap \bigcap_{i=1}^{p} \text{dom } h_i, \]

- we call \( \mathcal{D} \) the **domain** of the problem
- the constraints \( f_i(x) \leq 0, h_i(x) = 0 \) are the explicit constraints
- a problem is **unconstrained** if it has no explicit constraints \( (m = p = 0) \)

**example:**

\[
\text{minimize } \quad f_0(x) = -\sum_{i=1}^{k} \log(b_i - a_i^T x) \\
\]

is an unconstrained problem with implicit constraints \( a_i^T x < b_i \)
Feasibility problem

\[
\begin{align*}
\text{find} & \quad x \\
\text{subject to} & \quad f_i(x) \leq 0, \quad i = 1, \ldots, m \\
& \quad h_i(x) = 0, \quad i = 1, \ldots, p
\end{align*}
\]

can be considered a special case of the general problem with \( f_0(x) = 0 \):

\[
\begin{align*}
\text{minimize} & \quad 0 \\
\text{subject to} & \quad f_i(x) \leq 0, \quad i = 1, \ldots, m \\
& \quad h_i(x) = 0, \quad i = 1, \ldots, p
\end{align*}
\]

- \( p^* = 0 \) if constraints are feasible; any feasible \( x \) is optimal
- \( p^* = \infty \) if constraints are infeasible
Convex optimization problem

standard form convex optimization problem

\[
\begin{align*}
\text{minimize} & \quad f_0(x) \\
\text{subject to} & \quad f_i(x) \leq 0, \quad i = 1, \ldots, m \\
& \quad a_i^T x = b_i, \quad i = 1, \ldots, p
\end{align*}
\]

- \( f_0, f_1, \ldots, f_m \) are convex; equality constraints are affine
- problem is quasiconvex if \( f_0 \) is quasiconvex (and \( f_1, \ldots, f_m \) convex)

often written as

\[
\begin{align*}
\text{minimize} & \quad f_0(x) \\
\text{subject to} & \quad f_i(x) \leq 0, \quad i = 1, \ldots, m \\
& \quad Ax = b
\end{align*}
\]

important property: feasible set of a convex optimization problem is convex
example

\[
\begin{align*}
\text{minimize} & \quad f_0(x) = x_1^2 + x_2^2 \\
\text{subject to} & \quad f_1(x) = \frac{x_1}{1 + x_2^2} \leq 0 \\
& \quad h_1(x) = (x_1 + x_2)^2 = 0
\end{align*}
\]

- $f_0$ is convex; feasible set $\{(x_1, x_2) \mid x_1 = -x_2 \leq 0\}$ is convex
- not a convex problem (according to our definition): $f_1$ is not convex, $h_1$ is not affine
- equivalent (but not identical) to the convex problem

\[
\begin{align*}
\text{minimize} & \quad x_1^2 + x_2^2 \\
\text{subject to} & \quad x_1 \leq 0 \\
& \quad x_1 + x_2 = 0
\end{align*}
\]
any locally optimal point of a convex problem is (globally) optimal

**proof:** suppose $x$ is locally optimal and $y$ is optimal with $f_0(y) < f_0(x)$

$x$ locally optimal means there is an $R > 0$ such that

$$z \text{ feasible, } \|z - x\|_2 \leq R \implies f_0(z) \geq f_0(x)$$

consider $z = \theta y + (1 - \theta)x$ with $\theta = R/(2\|y - x\|_2)$

- $\|y - x\|_2 > R$, so $0 < \theta < 1/2$
- $z$ is a convex combination of two feasible points, hence also feasible
- $\|z - x\|_2 = R/2$ and

$$f_0(z) \leq \theta f_0(x) + (1 - \theta)f_0(y) < f_0(x)$$

which contradicts our assumption that $x$ is locally optimal
$x$ is optimal if and only if it is feasible and

$$\nabla f_0(x)^T(y - x) \geq 0 \quad \text{for all feasible } y$$

\[\nabla f_0(x)^T(y - x) \text{ for all } y \in X, \text{ means that } \nabla f_0(x) \neq 0 \text{ defines a supporting hyperplane to feasible set } X \text{ at } x\]
- **unconstrained problem**: $x$ is optimal if and only if

\[ x \in \text{dom } f_0, \quad \nabla f_0(x) = 0 \]

- **equality constrained problem**

\[
\begin{align*}
\text{minimize} & \quad f_0(x) \\
\text{subject to} & \quad Ax = b
\end{align*}
\]

$x$ is optimal if and only if there exists a $\nu$ such that

\[ x \in \text{dom } f_0, \quad Ax = b, \quad \nabla f_0(x) + A^T \nu = 0 \]

- **minimization over nonnegative orthant**

\[
\begin{align*}
\text{minimize} & \quad f_0(x) \\
\text{subject to} & \quad x \succeq 0
\end{align*}
\]

$x$ is optimal if and only if

\[ x \in \text{dom } f_0, \quad x \succeq 0, \quad \left\{ \begin{array}{ll} \nabla f_0(x)_i \geq 0 & x_i = 0 \\ \nabla f_0(x)_i = 0 & x_i > 0 \end{array} \right. \]
two problems are (informally) **equivalent** if the solution of one is readily obtained from the solution of the other, and vice-versa

some common transformations that preserve convexity:

- **eliminating equality constraints**

  \[
  \text{minimize} \quad f_0(x) \\
  \text{subject to} \quad f_i(x) \leq 0, \quad i = 1, \ldots, m \\
  Ax = b
  \]

  is equivalent to

  \[
  \text{minimize (over } z) \quad f_0(Fz + x_0) \\
  \text{subject to} \quad f_i(Fz + x_0) \leq 0, \quad i = 1, \ldots, m
  \]

  where \( F \) and \( x_0 \) are such that

  \[
  Ax = b \iff x = Fz + x_0 \text{ for some } z
  \]
introducing equality constraints

\[
\begin{align*}
\text{minimize} & \quad f_0(A_0x + b_0) \\
\text{subject to} & \quad f_i(A_ix + b_i) \leq 0, \quad i = 1, \ldots, m
\end{align*}
\]

is equivalent to

\[
\begin{align*}
\text{minimize} \ (\text{over } x, y_i) & \quad f_0(y_0) \\
\text{subject to} & \quad f_i(y_i) \leq 0, \quad i = 1, \ldots, m \\
y_i & = A_ix + b_i, \quad i = 0, 1, \ldots, m
\end{align*}
\]

introducing slack variables for linear inequalities

\[
\begin{align*}
\text{minimize} & \quad f_0(x) \\
\text{subject to} & \quad a_i^Tx \leq b_i, \quad i = 1, \ldots, m
\end{align*}
\]

is equivalent to

\[
\begin{align*}
\text{minimize} \ (\text{over } x, s) & \quad f_0(x) \\
\text{subject to} & \quad a_i^Tx + s_i = b_i, \quad i = 1, \ldots, m \\
s_i & \geq 0, \quad i = 1, \ldots m
\end{align*}
\]


epigraph form: standard form convex problem is equivalent to

\[
\begin{align*}
\text{minimize (over } x, t) & \quad t \\
\text{subject to} & \quad f_0(x) - t \leq 0 \\
& \quad f_i(x) \leq 0, \quad i = 1, \ldots, m \\
& \quad Ax = b
\end{align*}
\]

minimizing over some variables

\[
\begin{align*}
\text{minimize} & \quad f_0(x_1, x_2) \\
\text{subject to} & \quad f_i(x_1) \leq 0, \quad i = 1, \ldots, m
\end{align*}
\]

is equivalent to

\[
\begin{align*}
\text{minimize} & \quad \tilde{f}_0(x_1) \\
\text{subject to} & \quad f_i(x_1) \leq 0, \quad i = 1, \ldots, m
\end{align*}
\]

where \( \tilde{f}_0(x_1) = \inf_{x_2} f_0(x_1, x_2) \)
Quasiconvex optimization

\[
\begin{align*}
\text{minimize} & \quad f_0(x) \\
\text{subject to} & \quad f_i(x) \leq 0, \quad i = 1, \ldots, m \\
& \quad Ax = b
\end{align*}
\]

with \( f_0 : \mathbb{R}^n \rightarrow \mathbb{R} \) quasiconvex, \( f_1, \ldots, f_m \) convex

can have locally optimal points that are not (globally) optimal
convex representation of sublevel sets of $f_0$

if $f_0$ is quasiconvex, there exists a family of functions $\phi_t$ such that:

- $\phi_t(x)$ is convex in $x$ for fixed $t$
- $t$-sublevel set of $f_0$ is 0-sublevel set of $\phi_t$, i.e.,

$$f_0(x) \leq t \iff \phi_t(x) \leq 0$$

example

$$f_0(x) = \frac{p(x)}{q(x)}$$

with $p$ convex, $q$ concave, and $p(x) \geq 0$, $q(x) > 0$ on $\text{dom } f_0$

can take $\phi_t(x) = p(x) - tq(x)$:

- for $t \geq 0$, $\phi_t$ convex in $x$
- $p(x)/q(x) \leq t$ if and only if $\phi_t(x) \leq 0$
quasiconvex optimization via convex feasibility problems

\[
\phi_t(x) \leq 0, \quad f_i(x) \leq 0, \quad i = 1, \ldots, m, \quad Ax = b
\] (1)

- for fixed \( t \), a convex feasibility problem in \( x \)
- if feasible, we can conclude that \( t \geq p^* \); if infeasible, \( t \leq p^* \)

*Bisection method for quasiconvex optimization*

**given** \( l \leq p^*, u \geq p^* \), tolerance \( \epsilon > 0 \).

**repeat**

1. \( t := (l + u)/2 \).
2. Solve the convex feasibility problem (1).
3. **if** (1) is feasible, \( u := t \); **else** \( l := t \).

**until** \( u - l \leq \epsilon \).

requires exactly \( \lceil \log_2((u - l)/\epsilon) \rceil \) iterations (where \( u, l \) are initial values)
Linear program (LP)

minimize \( c^T x + d \)
subject to \( Gx \preceq h \)
\( Ax = b \)

- convex problem with affine objective and constraint functions
- feasible set is a polyhedron
Examples

**diet problem:** choose quantities $x_1, \ldots, x_n$ of $n$ foods

- one unit of food $j$ costs $c_j$, contains amount $a_{ij}$ of nutrient $i$
- healthy diet requires nutrient $i$ in quantity at least $b_i$

to find cheapest healthy diet,

\[
\begin{align*}
\text{minimize} & \quad c^T x \\
\text{subject to} & \quad Ax \succeq b, \quad x \succeq 0
\end{align*}
\]

**piecewise-linear minimization**

\[
\begin{align*}
\text{minimize} & \quad \max_{i=1,\ldots,m}(a_i^T x + b_i) \\
\end{align*}
\]

equivalent to an LP

\[
\begin{align*}
\text{minimize} & \quad t \\
\text{subject to} & \quad a_i^T x + b_i \leq t, \quad i = 1, \ldots, m
\end{align*}
\]
Chebyshev center of a polyhedron

Chebyshev center of

$$\mathcal{P} = \{ x \mid a_i^T x \leq b_i, \ i = 1, \ldots, m \}$$

is center of largest inscribed ball

$$\mathcal{B} = \{ x_c + u \mid \|u\|_2 \leq r \}$$

- $a_i^T x \leq b_i$ for all $x \in \mathcal{B}$ if and only if

$$\sup \{ a_i^T (x_c + u) \mid \|u\|_2 \leq r \} = a_i^T x_c + r \|a_i\|_2 \leq b_i$$

- hence, $x_c$, $r$ can be determined by solving the LP

$$\begin{align*}
\text{maximize} & \quad r \\
\text{subject to} & \quad a_i^T x_c + r \|a_i\|_2 \leq b_i, \quad i = 1, \ldots, m
\end{align*}$$
(Generalized) linear-fractional program

\[ \begin{align*}
\text{minimize} & \quad f_0(x) \\
\text{subject to} & \quad Gx \preceq h \\
& \quad Ax = b
\end{align*} \]

linear-fractional program

\[ f_0(x) = \frac{c^T x + d}{e^T x + f}, \quad \text{dom} \ f_0(x) = \{ x \mid e^T x + f > 0 \} \]

- a quasiconvex optimization problem; can be solved by bisection
- also equivalent to the LP (variables \( y, z \))

\[ \begin{align*}
\text{minimize} & \quad c^T y + dz \\
\text{subject to} & \quad G y \preceq h z \\
& \quad A y = b z \\
& \quad e^T y + f z = 1 \\
& \quad z \geq 0
\end{align*} \]
generalized linear-fractional program

\[ f_0(x) = \max_{i=1,\ldots,r} \frac{c_i^T x + d_i}{e_i^T x + f_i}, \quad \text{dom } f_0(x) = \{x \mid e_i^T x + f_i > 0, \ i = 1, \ldots, r\} \]

a quasiconvex optimization problem; can be solved by bisection

**example:** Von Neumann model of a growing economy

maximize (over \(x, x^+\)) \[\min_{i=1,\ldots,n} \frac{x^+_i}{x_i}\]

subject to \[x^+ \succeq 0, \quad Bx^+ \preceq Ax\]

- \(x, x^+ \in \mathbb{R}^n\): activity levels of \(n\) sectors, in current and next period
- \((Ax)_i, (Bx^+)_i\): produced, resp. consumed, amounts of good \(i\)
- \(x^+_i/x_i\): growth rate of sector \(i\)

allocate activity to maximize growth rate of slowest growing sector
Quadratic program (QP)

\[
\begin{align*}
\text{minimize} \quad & \frac{1}{2} x^T P x + q^T x + r \\
\text{subject to} \quad & Gx \preceq h \\
\quad & Ax = b
\end{align*}
\]

- \( P \in S^n_+ \), so objective is convex quadratic
- minimize a convex quadratic function over a polyhedron
Examples

least-squares

\[
\text{minimize} \quad \|Ax - b\|^2_2
\]

- analytical solution \(x^* = A^\dagger b\) (\(A^\dagger\) is pseudo-inverse)
- can add linear constraints, e.g., \(l \leq x \leq u\)

linear program with random cost

\[
\begin{align*}
\text{minimize} \quad & \bar{c}^T x + \gamma x^T \Sigma x = E c^T x + \gamma \text{var}(c^T x) \\
\text{subject to} \quad & Gx \preceq h, \quad Ax = b
\end{align*}
\]

- \(c\) is random vector with mean \(\bar{c}\) and covariance \(\Sigma\)
- hence, \(c^T x\) is random variable with mean \(\bar{c}^T x\) and variance \(x^T \Sigma x\)
- \(\gamma > 0\) is risk aversion parameter; controls the trade-off between expected cost and variance (risk)
Quadratically constrained quadratic program (QCQP)

\[
\begin{align*}
\text{minimize} & \quad (1/2)x^T P_0 x + q_0^T x + r_0 \\
\text{subject to} & \quad (1/2)x^T P_i x + q_i^T x + r_i \leq 0, \quad i = 1, \ldots, m \\
& \quad Ax = b
\end{align*}
\]

- \( P_i \in \mathbb{S}^n_+ \); objective and constraints are convex quadratic
- if \( P_1, \ldots, P_m \in \mathbb{S}^n_{++} \), feasible region is intersection of \( m \) ellipsoids and an affine set
Second-order cone programming

\[
\begin{align*}
\text{minimize} & \quad f^T x \\
\text{subject to} & \quad \|A_i x + b_i\|_2 \leq c_i^T x + d_i, \quad i = 1, \ldots, m \\
& \quad Fx = g
\end{align*}
\]

\( (A_i \in \mathbb{R}^{n_i \times n}, F \in \mathbb{R}^{p \times n}) \)

- Inequalities are called second-order cone (SOC) constraints:
  \( (A_i x + b_i, c_i^T x + d_i) \in \text{second-order cone in } \mathbb{R}^{n_i+1} \)

- For \( n_i = 0 \), reduces to an LP; if \( c_i = 0 \), reduces to a QCQP

- More general than QCQP and LP
Generalized inequality constraints

convex problem with generalized inequality constraints

\[
\begin{align*}
\text{minimize} & \quad f_0(x) \\
\text{subject to} & \quad f_i(x) \preceq_{K_i} 0, \quad i = 1, \ldots, m \\
& \quad Ax = b
\end{align*}
\]

- \( f_0 : \mathbb{R}^n \to \mathbb{R} \) convex; \( f_i : \mathbb{R}^n \to \mathbb{R}^{k_i} \) \( K_i \)-convex w.r.t. proper cone \( K_i \)
- same properties as standard convex problem (convex feasible set, local optimum is global, etc.)

conic form problem: special case with affine objective and constraints

\[
\begin{align*}
\text{minimize} & \quad c^T x \\
\text{subject to} & \quad Fx + g \preceq_{K} 0 \\
& \quad Ax = b
\end{align*}
\]

extends linear programming \((K = \mathbb{R}^m_+)\) to nonpolyhedral cones
Semidefinite program (SDP)

\[
\begin{align*}
\text{minimize} & \quad c^T x \\
\text{subject to} & \quad x_1 F_1 + x_2 F_2 + \cdots + x_n F_n + G \preceq 0 \\
& \quad Ax = b
\end{align*}
\]

with \( F_i, G \in \mathbb{S}^k \)

- inequality constraint is called linear matrix inequality (LMI)
- includes problems with multiple LMI constraints: for example,

\[
x_1 \hat{F}_1 + \cdots + x_n \hat{F}_n + \hat{G} \preceq 0, \quad x_1 \tilde{F}_1 + \cdots + x_n \tilde{F}_n + \tilde{G} \preceq 0
\]

is equivalent to single LMI

\[
x_1 \begin{bmatrix} \hat{F}_1 & 0 \\ 0 & \tilde{F}_1 \end{bmatrix} + x_2 \begin{bmatrix} \hat{F}_2 & 0 \\ 0 & \tilde{F}_2 \end{bmatrix} + \cdots + x_n \begin{bmatrix} \hat{F}_n & 0 \\ 0 & \tilde{F}_n \end{bmatrix} + \begin{bmatrix} \hat{G} & 0 \\ 0 & \tilde{G} \end{bmatrix} \preceq 0
\]
LP and SOCP as SDP

LP and equivalent SDP

LP: minimize $c^T x$
subject to $Ax \preceq b$

SDP: minimize $c^T x$
subject to $\text{diag}(Ax - b) \preceq 0$

(note different interpretation of generalized inequality $\preceq$)

SOCM and equivalent SDP

SOCM: minimize $f^T x$
subject to $\|Ax + b\|_2 \leq c^T x + d_i$, $i = 1, \ldots, m$

SDP: minimize $f^T x$
subject to $\begin{bmatrix} (c^T x + d_i)I & A_i x + b_i \\ (A_i x + b_i)^T & c^T x + d_i \end{bmatrix} \succeq 0$, $i = 1, \ldots, m$
Eigenvalue minimization

minimize $\lambda_{\text{max}}(A(x))$

where $A(x) = A_0 + x_1 A_1 + \cdots + x_n A_n$ (with given $A_i \in S^k$)

equivalent SDP

minimize $t$
subject to $A(x) \preceq tI$

- variables $x \in \mathbb{R}^n$, $t \in \mathbb{R}$
- follows from

$$\lambda_{\text{max}}(A) \leq t \iff A \preceq tI$$
Matrix norm minimization

\[
\text{minimize} \quad \|A(x)\|_2 = \left( \lambda_{\text{max}}(A(x)^T A(x)) \right)^{1/2}
\]

where \( A(x) = A_0 + x_1 A_1 + \cdots + x_n A_n \) (with given \( A_i \in \mathbb{S}^{p \times q} \))

equivalent SDP

\[
\begin{align*}
\text{minimize} & \quad t \\
\text{subject to} & \quad \begin{bmatrix} t I & A(x) \\ A(x)^T & t I \end{bmatrix} \succeq 0
\end{align*}
\]

- variables \( x \in \mathbb{R}^n, \ t \in \mathbb{R} \)
- constraint follows from

\[
\|A\|_2 \leq t \iff A^T A \preceq t^2 I, \quad t \geq 0
\]

\[
\iff \quad \begin{bmatrix} t I & A \\ A^T & t I \end{bmatrix} \succeq 0
\]
Multicriterion optimization

Vector optimization problem with $K = \mathbb{R}^q_+

\[ f_0(x) = (F_1(x), \ldots, F_q(x)) \]

- $q$ different objectives $F_i$; roughly speaking we want all $F_i$'s to be small
- Feasible $x^*$ is optimal if
  \[ y \text{ feasible} \implies f_0(x^*) \leq f_0(y) \]
  if there exists an optimal point, the objectives are noncompeting
- Feasible $x^{po}$ is Pareto optimal if
  \[ y \text{ feasible, } f_0(y) \leq f_0(x^{po}) \implies f_0(x^{po}) = f_0(y) \]
  if there are multiple Pareto optimal values, there is a trade-off between the objectives
Regularized least-squares

multicriterion problem with two objectives

\[ F_1(x) = \|Ax - b\|_2^2, \quad F_2(x) = \|x\|_2^2 \]

- example with \( A \in \mathbb{R}^{100 \times 10} \)
- shaded region is \( \mathcal{O} \)
- heavy line is formed by Pareto optimal points
Risk return trade-off in portfolio optimization

\[
\begin{align*}
\text{minimize (w.r.t. } & \mathbf{R}_+^2) \quad (-\bar{p}^T \mathbf{x}, \mathbf{x}^T \Sigma \mathbf{x}) \\
\text{subject to} \quad & \mathbf{1}^T \mathbf{x} = 1, \quad \mathbf{x} \succeq \mathbf{0}
\end{align*}
\]

\begin{itemize}
\item $\mathbf{x} \in \mathbb{R}^n$ is investment portfolio; $x_i$ is fraction invested in asset $i$
\item $\mathbf{p} \in \mathbb{R}^n$ is vector of relative asset price changes; modeled as a random variable with mean $\bar{p}$, covariance $\Sigma$
\item $\bar{p}^T \mathbf{x} = \mathbb{E} r$ is expected return; $\mathbf{x}^T \Sigma \mathbf{x} = \text{var } r$ is return variance
\end{itemize}

example

\begin{itemize}
\item \begin{tikzpicture}
\begin{axis}[
width=0.45\textwidth,
height=0.45\textwidth,
axis lines=left,
xlabel={standard deviation of return},
ylabel={mean return},

\addplot[domain=0:20,samples=100,smooth] {x^0.5} node[pos=0.5,above] {allocation $x$};
\end{axis}
\end{tikzpicture}
\end{itemize}
Scalarization

to find Pareto optimal points: choose $\lambda \gtrsim_{K^*} 0$ and solve scalar problem

\[
\begin{align*}
\text{minimize} & \quad \lambda^T f_0(x) \\
\text{subject to} & \quad f_i(x) \leq 0, \quad i = 1, \ldots, m \\
& \quad h_i(x) = 0, \quad i = 1, \ldots, p
\end{align*}
\]

if $x$ is optimal for scalar problem, then it is Pareto-optimal for vector optimization problem

for convex vector optimization problems, can find (almost) all Pareto optimal points by varying $\lambda \gtrsim_{K^*} 0$