# Convex Optimization

## Convex Functions

# Today

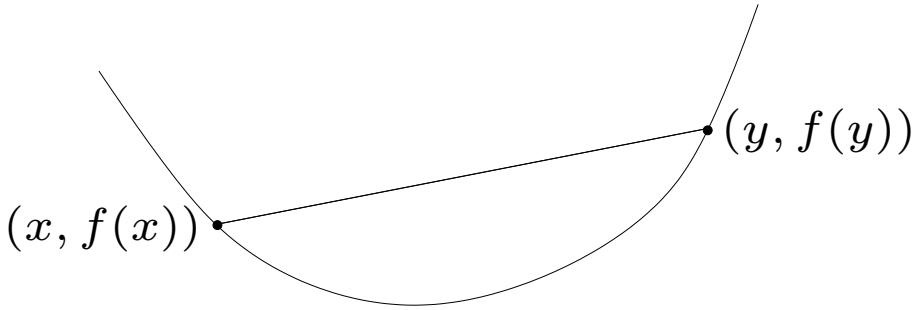
- basic properties and examples
- operations that preserve convexity
- the conjugate function
- quasiconvex functions
- log-concave and log-convex functions
- convexity with respect to generalized inequalities

# Definition

$f : \mathbf{R}^n \to \mathbf{R}$ is convex if $\mathbf{dom}\, f$ is a convex set and

$$f(\theta x + (1-\theta)y) \leq \theta f(x) + (1-\theta)f(y)$$

for all $x, y \in \mathbf{dom}\, f$, $0 \leq \theta \leq 1$




- $f$ is concave if $-f$ is convex
- $f$ is strictly convex if $\mathbf{dom}\, f$ is convex and

$$f(\theta x + (1-\theta)y) < \theta f(x) + (1-\theta)f(y)$$

for $x, y \in \mathbf{dom}\, f$, $x \neq y$, $0 < \theta < 1$

# Examples on R

convex:

- affine: $ax + b$ on $\mathbf{R}$, for any $a, b \in \mathbf{R}$
- exponential: $e^{ax}$, for any $a \in \mathbf{R}$
- powers: $x^\alpha$ on $\mathbf{R}_{++}$, for $\alpha \geq 1$ or $\alpha \leq 0$
- powers of absolute value: $|x|^p$ on $\mathbf{R}$, for $p \geq 1$
- negative entropy: $x \log x$ on $\mathbf{R}_{++}$

concave:

- affine: $ax + b$ on $\mathbf{R}$, for any $a, b \in \mathbf{R}$
- powers: $x^\alpha$ on $\mathbf{R}_{++}$, for $0 \leq \alpha \leq 1$
- logarithm: $\log x$ on $\mathbf{R}_{++}$

# Examples on $\mathbf{R}^n$ and $\mathbf{R}^{m\times n}$

affine functions are convex and concave; all norms are convex

**examples on $\mathbf{R}^n$**

- affine function $f(x) = a^T x + b$
- norms: $\|x\|_p = (\sum_{i=1}^n |x_i|^p)^{1/p}$ for $p \geq 1$; $\|x\|_\infty = \max_k |x_k|$

**examples on $\mathbf{R}^{m\times n}$** ($m \times n$ matrices)

- affine function

$$f(X) = \mathbf{Tr}(A^T X) + b = \sum_{i=1}^m \sum_{j=1}^n A_{ij} X_{ij} + b$$

- spectral (maximum singular value) norm

$$f(X) = \|X\|_2 = \sigma_{\max}(X) = (\lambda_{\max}(X^T X))^{1/2}$$

# Restriction of a convex function to a line

$f : \mathbf{R}^n \to \mathbf{R}$ is convex if and only if the function $g : \mathbf{R} \to \mathbf{R}$,

$$g(t) = f(x + tv), \qquad \mathbf{dom}\, g = \{t \mid x + tv \in \mathbf{dom}\, f\}$$

is convex (in $t$) for any $x \in \mathbf{dom}\, f$, $v \in \mathbf{R}^n$

can check convexity of $f$ by checking convexity of functions of one variable

**example.** $f : \mathbf{S}^n \to \mathbf{R}$ with $f(X) = \log\det X$, $\mathbf{dom}\, X = \mathbf{S}^n_{++}$

$$\begin{aligned} g(t) = \log\det(X + tV) &= \log\det X + \log\det(I + tX^{-1/2}VX^{-1/2}) \\ &= \log\det X + \sum_{i=1}^{n} \log(1 + t\lambda_i) \end{aligned}$$

where $\lambda_i$ are the eigenvalues of $X^{-1/2}VX^{-1/2}$

$g$ is concave in $t$ (for any choice of $X \succ 0$, $V$); hence $f$ is concave

# Extended-value extension

extended-value extension $\tilde{f}$ of $f$ is

$$\tilde{f}(x) = f(x), \quad x \in \mathbf{dom}\, f, \qquad \tilde{f}(x) = \infty, \quad x \notin \mathbf{dom}\, f$$

often simplifies notation; for example, the condition

$$0 \leq \theta \leq 1 \quad \Longrightarrow \quad \tilde{f}(\theta x + (1-\theta)y) \leq \theta \tilde{f}(x) + (1-\theta)\tilde{f}(y)$$

(as an inequality in $\mathbf{R} \cup \{\infty\}$), means the same as the two conditions

- $\mathbf{dom}\, f$ is convex
- for $x, y \in \mathbf{dom}\, f$,

$$0 \leq \theta \leq 1 \quad \Longrightarrow \quad f(\theta x + (1-\theta)y) \leq \theta f(x) + (1-\theta)f(y)$$

# First-order condition

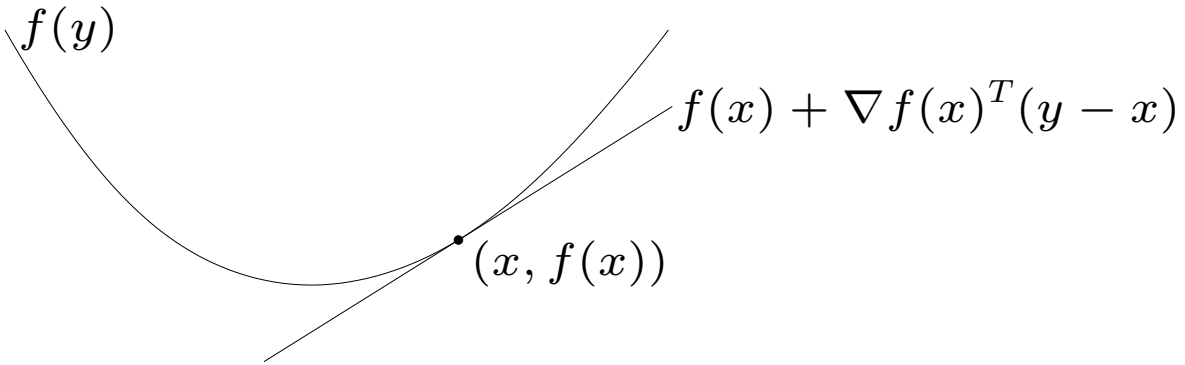
$f$ is **differentiable** if $\mathbf{dom}\, f$ is open and the gradient

$$\nabla f(x) = \left(\frac{\partial f(x)}{\partial x_1}, \frac{\partial f(x)}{\partial x_2}, \ldots, \frac{\partial f(x)}{\partial x_n}\right)$$

exists at each $x \in \mathbf{dom}\, f$

**1st-order condition:** differentiable $f$ with convex domain is convex iff

$$f(y) \geq f(x) + \nabla f(x)^T (y - x) \quad \text{for all } x, y \in \mathbf{dom}\, f$$




first-order approximation of $f$ is global underestimator

# Second-order conditions

$f$ is **twice differentiable** if $\mathbf{dom}\, f$ is open and the Hessian $\nabla^2 f(x) \in \mathbf{S}^n$,

$$\nabla^2 f(x)_{ij} = \frac{\partial^2 f(x)}{\partial x_i \partial x_j}, \quad i, j = 1, \ldots, n,$$

exists at each $x \in \mathbf{dom}\, f$

**2nd-order conditions:** for twice differentiable $f$ with convex domain

- $f$ is convex if and only if

$$\nabla^2 f(x) \succeq 0 \quad \text{for all } x \in \mathbf{dom}\, f$$

- if $\nabla^2 f(x) \succ 0$ for all $x \in \mathbf{dom}\, f$, then $f$ is strictly convex

# Examples

**quadratic function:** $f(x) = (1/2)x^T Px + q^T x + r$ (with $P \in \mathbf{S}^n$)

$$\nabla f(x) = Px + q, \qquad \nabla^2 f(x) = P$$

convex if $P \succeq 0$

**least-squares objective**: $f(x) = \|Ax - b\|_2^2$

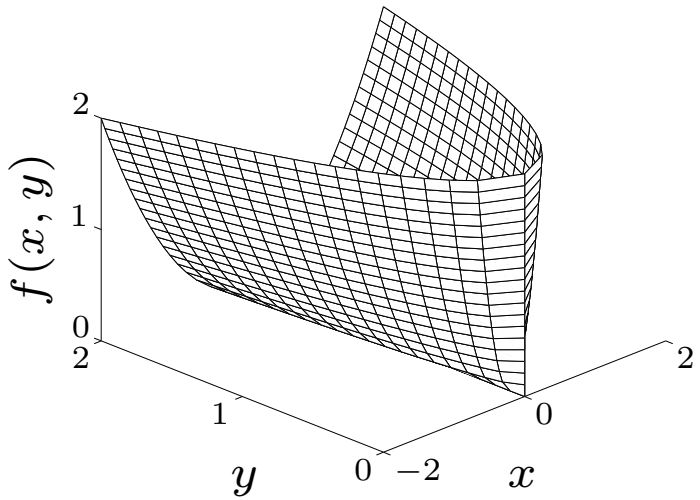
$$\nabla f(x) = 2A^T(Ax - b), \qquad \nabla^2 f(x) = 2A^T A$$

convex (for any $A$)

**quadratic-over-linear:** $f(x, y) = x^2/y$

$$\nabla^2 f(x,y) = \frac{2}{y^3} \begin{bmatrix} y \\ -x \end{bmatrix} \begin{bmatrix} y \\ -x \end{bmatrix}^T \succeq 0$$

convex for $y > 0$



**log-sum-exp**: $f(x) = \log \sum_{k=1}^n \exp x_k$ is convex

$$\nabla^2 f(x) = \frac{1}{\mathbf{1}^T z} \mathbf{diag}(z) - \frac{1}{(\mathbf{1}^T z)^2} z z^T \qquad (z_k = \exp x_k)$$

to show $\nabla^2 f(x) \succeq 0$, we must verify that $v^T \nabla^2 f(x) v \geq 0$ for all $v$:

$$v^T \nabla^2 f(x) v = \frac{(\sum_k z_k v_k^2)(\sum_k z_k) - (\sum_k v_k z_k)^2}{(\sum_k z_k)^2} \geq 0$$

since $(\sum_k v_k z_k)^2 \leq (\sum_k z_k v_k^2)(\sum_k z_k)$ (from Cauchy-Schwarz inequality)

**geometric mean**: $f(x) = (\prod_{k=1}^n x_k)^{1/n}$ on $\mathbf{R}^n_{++}$ is concave

(similar proof as for log-sum-exp)

# Epigraph and sublevel set

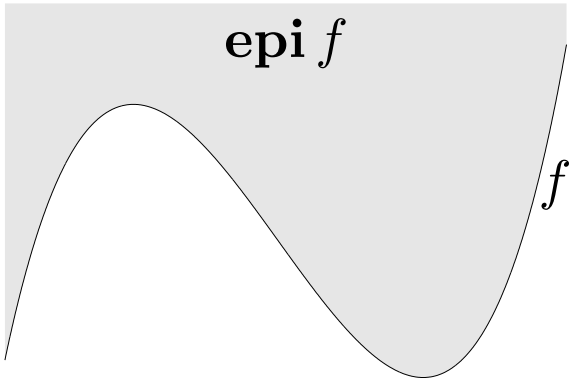
$\alpha$-**sublevel set** of $f : \mathbf{R}^n \to \mathbf{R}$:

$$C_\alpha = \{x \in \mathbf{dom}\, f \mid f(x) \leq \alpha\}$$

sublevel sets of convex functions are convex (converse is false)

**epigraph** of $f : \mathbf{R}^n \to \mathbf{R}$:

$$\mathbf{epi}\, f = \{(x, t) \in \mathbf{R}^{n+1} \mid x \in \mathbf{dom}\, f,\ f(x) \leq t\}$$




$f$ is convex if and only if $\mathbf{epi}\, f$ is a convex set

# Jensen's inequality

**basic inequality:** if $f$ is convex, then for $0 \leq \theta \leq 1$,

$$f(\theta x + (1-\theta)y) \leq \theta f(x) + (1-\theta)f(y)$$

**extension:** if $f$ is convex, then

$$f(\mathbf{E}z) \leq \mathbf{E}f(z)$$

for any random variable $z$

basic inequality is special case with discrete distribution

$$\mathbf{Prob}(z = x) = \theta, \qquad \mathbf{Prob}(z = y) = 1 - \theta$$

# Operations that preserve convexity

practical methods for establishing convexity of a function

1. verify definition (often simplified by restricting to a line)

2. for twice differentiable functions, show $\nabla^2 f(x) \succeq 0$

3. show that $f$ is obtained from simple convex functions by operations that preserve convexity

   - nonnegative weighted sum
   - composition with affine function
   - pointwise maximum and supremum
   - composition
   - minimization
   - perspective

# Positive weighted sum & composition with affine function

**nonnegative multiple:** $\alpha f$ is convex if $f$ is convex, $\alpha \geq 0$

**sum:** $f_1 + f_2$ convex if $f_1, f_2$ convex (extends to infinite sums, integrals)

**composition with affine function**: $f(Ax + b)$ is convex if $f$ is convex

**examples**

- log barrier for linear inequalities

$$f(x) = -\sum_{i=1}^{m} \log(b_i - a_i^T x), \qquad \mathbf{dom}\, f = \{x \mid a_i^T x < b_i, i = 1, \ldots, m\}$$

- (any) norm of affine function: $f(x) = \|Ax + b\|$

# Pointwise maximum

if $f_1, \ldots, f_m$ are convex, then $f(x) = \max\{f_1(x), \ldots, f_m(x)\}$ is convex

**examples**

- piecewise-linear function: $f(x) = \max_{i=1,\ldots,m}(a_i^T x + b_i)$ is convex
- sum of $r$ largest components of $x \in \mathbf{R}^n$:

$$f(x) = x_{[1]} + x_{[2]} + \cdots + x_{[r]}$$

is convex ($x_{[i]}$ is $i$th largest component of $x$)

proof:

$$f(x) = \max\{x_{i_1} + x_{i_2} + \cdots + x_{i_r} \mid 1 \leq i_1 < i_2 < \cdots < i_r \leq n\}$$

# Pointwise supremum

if $f(x, y)$ is convex in $x$ for each $y \in \mathcal{A}$, then

$$g(x) = \sup_{y \in \mathcal{A}} f(x, y)$$

is convex

**examples**

- support function of a set $C$: $S_C(x) = \sup_{y \in C} y^T x$ is convex
- distance to farthest point in a set $C$:

$$f(x) = \sup_{y \in C} \|x - y\|$$

- maximum eigenvalue of symmetric matrix: for $X \in \mathbf{S}^n$,

$$\lambda_{\max}(X) = \sup_{\|y\|_2 = 1} y^T X y$$

# Composition with scalar functions

composition of $g : \mathbf{R}^n \to \mathbf{R}$ and $h : \mathbf{R} \to \mathbf{R}$:

$$f(x) = h(g(x))$$

$f$ is convex if
- $g$ convex, $h$ convex, $\tilde{h}$ nondecreasing
- $g$ concave, $h$ convex, $\tilde{h}$ nonincreasing

- proof (for $n = 1$, differentiable $g, h$)

$$f''(x) = h''(g(x))g'(x)^2 + h'(g(x))g''(x)$$

- note: monotonicity must hold for extended-value extension $\tilde{h}$

**examples**

- $\exp g(x)$ is convex if $g$ is convex
- $1/g(x)$ is convex if $g$ is concave and positive

# Vector composition

composition of $g : \mathbf{R}^n \to \mathbf{R}^k$ and $h : \mathbf{R}^k \to \mathbf{R}$:

$$f(x) = h(g(x)) = h(g_1(x), g_2(x), \ldots, g_k(x))$$

$f$ is convex if
- $g_i$ convex, $h$ convex, $\tilde{h}$ nondecreasing in each argument
- $g_i$ concave, $h$ convex, $\tilde{h}$ nonincreasing in each argument

proof (for $n = 1$, differentiable $g, h$)

$$f''(x) = g'(x)^T \nabla^2 h(g(x)) g'(x) + \nabla h(g(x))^T g''(x)$$

**examples**

- $\sum_{i=1}^m \log g_i(x)$ is concave if $g_i$ are concave and positive
- $\log \sum_{i=1}^m \exp g_i(x)$ is convex if $g_i$ are convex

# Minimization

if $f(x, y)$ is convex in $(x, y)$ and $C$ is a convex set, then

$$g(x) = \inf_{y \in C} f(x, y)$$

is convex

**examples**

- $f(x, y) = x^T A x + 2x^T B y + y^T C y$ with

$$\begin{bmatrix} A & B \\ B^T & C \end{bmatrix} \succeq 0, \qquad C \succ 0$$

  minimizing over $y$ gives $g(x) = \inf_y f(x, y) = x^T (A - BC^{-1}B^T)x$

  $g$ is convex, hence Schur complement $A - BC^{-1}B^T \succeq 0$

- distance to a set: $\mathbf{dist}(x, S) = \inf_{y \in S} \|x - y\|$ is convex if $S$ is convex

# Perspective

the **perspective** of a function $f : \mathbf{R}^n \to \mathbf{R}$ is the function $g : \mathbf{R}^n \times \mathbf{R} \to \mathbf{R}$,

$$g(x,t) = tf(x/t), \qquad \mathbf{dom}\, g = \{(x,t) \mid x/t \in \mathbf{dom}\, f,\ t > 0\}$$

$g$ is convex if $f$ is convex

**examples**

- $f(x) = x^T x$ is convex; hence $g(x,t) = x^T x/t$ is convex for $t > 0$
- negative logarithm $f(x) = -\log x$ is convex; hence relative entropy $g(x,t) = t\log t - t\log x$ is convex on $\mathbf{R}^2_{++}$
- if $f$ is convex, then

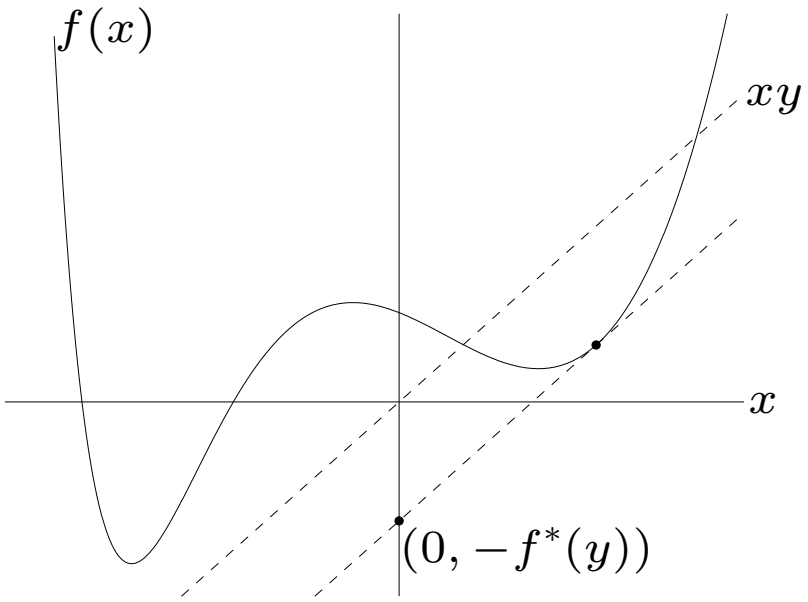
$$g(x) = (c^T x + d) f\left((Ax+b)/(c^T x + d)\right)$$

is convex on $\{x \mid c^T x + d > 0,\ (Ax+b)/(c^T x + d) \in \mathbf{dom}\, f\}$

# The conjugate function

the **conjugate** of a function $f$ is

$$f^*(y) = \sup_{x \in \mathbf{dom}\, f} (y^T x - f(x))$$




- $f^*$ is convex (even if $f$ is not)
- will be useful in chapter 5

## examples

- negative logarithm $f(x) = -\log x$

$$\begin{aligned} f^*(y) &= \sup_{x>0}(xy + \log x) \\ &= \begin{cases} -1 - \log(-y) & y < 0 \\ \infty & \text{otherwise} \end{cases} \end{aligned}$$

- strictly convex quadratic $f(x) = (1/2)x^T Q x$ with $Q \in \mathbf{S}^n_{++}$

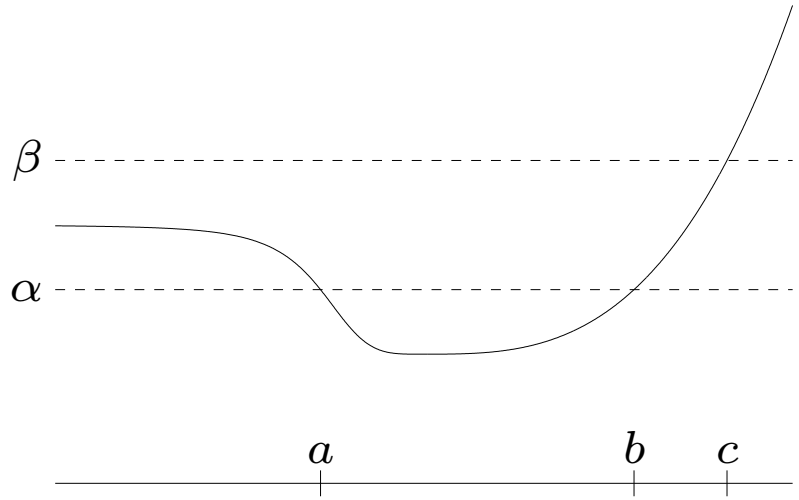
$$\begin{aligned} f^*(y) &= \sup_x (y^T x - (1/2)x^T Q x) \\ &= \frac{1}{2} y^T Q^{-1} y \end{aligned}$$

# Quasiconvex functions

$f : \mathbf{R}^n \to \mathbf{R}$ is quasiconvex if $\mathbf{dom}\, f$ is convex and the sublevel sets

$$S_\alpha = \{x \in \mathbf{dom}\, f \mid f(x) \leq \alpha\}$$

are convex for all $\alpha$




- $f$ is quasiconcave if $-f$ is quasiconvex
- $f$ is quasilinear if it is quasiconvex and quasiconcave

# Examples

- $\sqrt{|x|}$ is quasiconvex on $\mathbf{R}$
- $\mathrm{ceil}(x) = \inf\{z \in \mathbf{Z} \mid z \geq x\}$ is quasilinear
- $\log x$ is quasilinear on $\mathbf{R}_{++}$
- $f(x_1, x_2) = x_1 x_2$ is quasiconcave on $\mathbf{R}^2_{++}$
- linear-fractional function

$$f(x) = \frac{a^T x + b}{c^T x + d}, \qquad \mathbf{dom}\, f = \{x \mid c^T x + d > 0\}$$

  is quasilinear

- distance ratio

$$f(x) = \frac{\|x - a\|_2}{\|x - b\|_2}, \qquad \mathbf{dom}\, f = \{x \mid \|x - a\|_2 \leq \|x - b\|_2\}$$

  is quasiconvex

## internal rate of return

- cash flow $x = (x_0, \ldots, x_n)$; $x_i$ is payment in period $i$ (to us if $x_i > 0$)
- we assume $x_0 < 0$ and $x_0 + x_1 + \cdots + x_n > 0$
- present value of cash flow $x$, for interest rate $r$:

$$\mathrm{PV}(x, r) = \sum_{i=0}^{n} (1+r)^{-i} x_i$$

- internal rate of return is smallest interest rate for which $\mathrm{PV}(x, r) = 0$:

$$\mathrm{IRR}(x) = \inf\{r \geq 0 \mid \mathrm{PV}(x, r) = 0\}$$

$\mathrm{IRR}$ is quasiconcave: superlevel set is intersection of halfspaces

$$\mathrm{IRR}(x) \geq R \quad \Longleftrightarrow \quad \sum_{i=0}^{n} (1+r)^{-i} x_i \geq 0 \text{ for } 0 \leq r \leq R$$

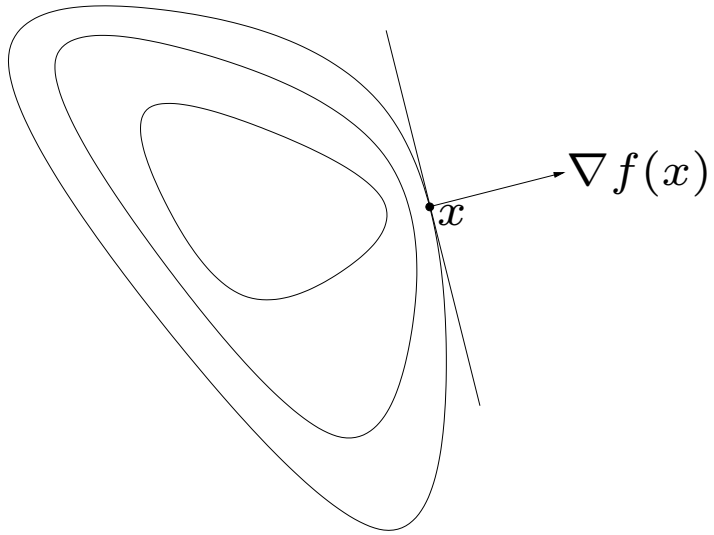
# Properties

**modified Jensen inequality:** for quasiconvex $f$

$$0 \leq \theta \leq 1 \quad \Longrightarrow \quad f(\theta x + (1-\theta)y) \leq \max\{f(x), f(y)\}$$

**first-order condition:** differentiable $f$ with cvx domain is quasiconvex iff

$$f(y) \leq f(x) \quad \Longrightarrow \quad \nabla f(x)^T (y - x) \leq 0$$




**sums** of quasiconvex functions are not necessarily quasiconvex

# Log-concave and log-convex functions

a positive function $f$ is log-concave if $\log f$ is concave:

$$f(\theta x + (1-\theta)y) \geq f(x)^\theta f(y)^{1-\theta} \quad \text{for } 0 \leq \theta \leq 1$$

$f$ is log-convex if $\log f$ is convex

- powers: $x^a$ on $\mathbf{R}_{++}$ is log-convex for $a \leq 0$, log-concave for $a \geq 0$
- many common probability densities are log-concave, $e.g.$, normal:

$$f(x) = \frac{1}{\sqrt{(2\pi)^n \det \Sigma}} e^{-\frac{1}{2}(x-\bar{x})^T \Sigma^{-1}(x-\bar{x})}$$

- cumulative Gaussian distribution function $\Phi$ is log-concave

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-u^2/2}\, du$$

# Properties of log-concave functions

- twice differentiable $f$ with convex domain is log-concave if and only if

$$f(x)\nabla^2 f(x) \preceq \nabla f(x)\nabla f(x)^T$$

  for all $x \in \mathbf{dom}\, f$

- product of log-concave functions is log-concave
- sum of log-concave functions is not always log-concave
- integration: if $f : \mathbf{R}^n \times \mathbf{R}^m \to \mathbf{R}$ is log-concave, then

$$g(x) = \int f(x, y)\, dy$$

  is log-concave (not easy to show)

## consequences of integration property

- convolution $f * g$ of log-concave functions $f$, $g$ is log-concave

$$(f * g)(x) = \int f(x-y)g(y)dy$$

- if $C \subseteq \mathbf{R}^n$ convex and $y$ is a random variable with log-concave pdf then

$$f(x) = \mathbf{Prob}(x + y \in C)$$

is log-concave

proof: write $f(x)$ as integral of product of log-concave functions

$$f(x) = \int g(x+y)p(y)\,dy, \qquad g(u) = \begin{cases} 1 & u \in C \\ 0 & u \notin C, \end{cases}$$

$p$ is pdf of $y$

## example: yield function

$$Y(x) = \mathbf{Prob}(x + w \in S)$$

- $x \in \mathbf{R}^n$: nominal parameter values for product
- $w \in \mathbf{R}^n$: random variations of parameters in manufactured product
- $S$: set of acceptable values

if $S$ is convex and $w$ has a log-concave pdf, then

- $Y$ is log-concave
- yield regions $\{x \mid Y(x) \geq \alpha\}$ are convex