Calibration of BGM models by semidefinite programming.

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1.1 Interest rate model calibration

- All Heath, Jarrow & Morton (1992) based models are fully parametrized by the *curve today* and a *covariance* function.

- If we discretize this covariance function, the natural variable in the calibration problem is a covariance matrix, i.e. a *positive semidefinite matrix*.

- Classic calibration methods are *heavily parametrized* and only describe a small, often non-convex subset of the set of semidefinite matrices.

- When using these techniques, sensitivity analysis has to be done by recalibrating.

1.2 Results on the BGM model calibration

- We can express the swap rate as a *basket of Forwards* with very stable coefficients.

- European Caplets and Swaptions can be priced using the Black (1976) market formula with an appropriately chosen variance.

- This market variance is *linear* in the coefficients of the Forward rates covariance matrix.

- This allows us to solve the calibration problem as a *semidefinite program*.

1.3 Related literature

- Works by Nesterov & Nemirovskii (1994) and Vandenberghe & Boyd (1996) on semidefinite programming

- Brace, Gatarek & Musiela (1997) and Musiela & Rutkowski (1997) on the Libor market model.

- Rebonato (1998), Brace, Dun & Barton (1999) and Singleton & Umantsev (2001) on Swaps as baskets of Forwards. Rebonato (1999) on a calibration method parametrized by factors.

- Parallel work by Brace & Womersley (2000) on the calibration of the BGM by semidefinite programming and the evaluation of the Bermudan Swaption.

2 Swaption pricing

2.1 The Swap rate

The Swap is defined here as the rate that equals the PV of a fixed and a floating leg:

$$swap(t, T_0, T_n) = \frac{B(t, T_0^{floating}) - B(t, T_{n+1}^{floating})}{Level(t, T_0^{fixed}, T_n^{fixed})}$$

where

$$Level(t, T_0^{fixed}, T_n^{fixed}) = \sum_{i=1}^{n+1} coverage(T_{i-1}^{fixed}, T_i^{fixed}) B(t, T_i^{fixed})$$

This rate can again be written:

$$swap(t, T_0, T_n) = \sum_{i=0}^{n} \omega_i(t) K(t, T_i)$$

where $K(t, T_i)$ are the Forward Rates with maturities T_i , $i = 1, ..., n$ and the weights $\omega_i(t)$ are given by

$$\omega_i(t) = \frac{coverage(T_i^{float}, T_{i+1}^{float}) B(t, T_{i+1}^{float})}{Level(t, T_0^{fixed}, T_n^{fixed})}$$

In practice, these weights are very stable (see Rebonato (1998)).

This stability has been studied in Hamy (1999) of which we report, with the author's permission, some summary statistics:

Currency	USD	USD	GBP	GBP	EUR	EUR
Swap	$2Y$	$5Y$	$2Y$	$5Y$	$2Y$	$5Y$
Min ratio	712	842	885	981	148	333
Max ratio	7629	7927	6575	3473	5006	4322
Variance	.023	.020	.017	.007	.005	.004

Sample ratio of volatility between weights and corresponding Forwards.

Here, *Min ratio* and *Max ratio* are the minimum (resp. maximum) volatility ratio among the weights of a particular Swap. Computed using the standard quadratic variation estimator with exponentially decaying weights (1998-1999 period, market data courtesy of BNP-Paribas London).

2.2 BGM Swaption price

Following Jamshidian (1997), we can write the price of the Swaption with strike k as a that of a Call on a Swap rate:

$$Ps(t) = Level(t, T, T_N) E_t^{Q_{LVL}} \left[\left(\sum_{i=0}^{n} \omega_i(T) K(T, T_i) - k \right)^+ \right]$$

where Q_{LVL} is the swap forward martingale probability measure. In what follows, we will make two approximations:

- We replace the weights $\omega_i(T)$ by their value today $\omega_i(t)$.
- We suppose that $\sum_{i=0}^{n} \omega_i(t) K(T, T_i)$ is a sum of Q_{LVL} lognormal martingale.

2.3 (A remark on the) Gaussian HJM Swaption price

We can also express the price of the Swaption as that of a Bond Put:

$$Ps(t) = B(t,T)E_t^{Q_T}\left[\left(1 - B(t,T_{N+1}) - k\delta \sum_{i=i_T}^{N} B(t,T_i)\right)^+\right]$$

In the Gaussian H.J.M. model (see El Karoui & Lacoste (1992), Musiela & Rutkowski (1997) or Duffie & Kan (1996)), this expression defines the price of a Swaption as that of a Put on a basket of lognormal zero-coupon prices.

2.4 Basket option pricing

We have seen that we can reduce the problem of pricing a Swaption to that of pricing a classic Black & Scholes (1973) basket option. In generic terms, the problem becomes that of computing:

$$C = E\left[(S_T^\omega - k)^+ \right]$$

with

$$S_T^\omega = \sum_{i=1}^n \omega_i K_s^i \quad \text{and} \quad dK_s^i = K_s^i \sigma_s^i dW_s$$

where W_t is a n-dimensional $\mathbf{Q}^T$-BM and $\sigma_s = \left(\sigma_s^i\right)_{i=1,\ldots,n} \in \mathrm{R}^{n\times n}$ describes the volatility matrix.

We can write the dynamics of the basket as:

$$\begin{cases} \frac{dS_u^\omega}{S_u^\omega} = \left(\sum_{i=1}^n \widehat{\omega}_{i,u}\sigma_u^i\right) dW_u \\ \frac{d\widehat{\omega}_{i,s}}{\widehat{\omega}_{i,s}} = \left(\sum_{j=1}^n \widehat{\omega}_{j,s}\left(\sigma_s^i - \sigma_s^j\right)\right)\left(dW_s + \sum_{j=1}^n \widehat{\omega}_{j,s}\sigma_s^j ds\right) \end{cases}$$

where we have noted:

$$\widehat{\omega}_{i,s} = \frac{\omega_i K_s^i}{\sum_{i=1}^n \omega_i K_s^i}$$

We notice that $0 \leq \widehat{\omega}_{i,s} \leq 1$ with $\sum_{j=1}^n \widehat{\omega}_{i,s} = 1$. We also set:

$$\tilde{\sigma}_s^i = \sigma_s^i - \sigma_s^\omega \quad \text{with} \quad \sigma_s^\omega = \sum_{j=1}^n \widehat{\omega}_{i,t}\sigma_s^j$$

note that $\sigma_s^\omega = \sum_{j=1}^n \widehat{\omega}_{i,t}\sigma_s^j$ is F_t-measurable.

We can develop these dynamics around small values of $\tilde{\sigma}_s^i$ and $\sum_{j=1}^n \widehat{\omega}_{j,s}\tilde{\sigma}_s^j$ in particular. For some $\varepsilon > 0$, we write:

$$\begin{cases} dS_s^{\omega,\varepsilon} = S_s^{\omega,\varepsilon}\left(\sigma_s^\omega + \varepsilon \sum_{j=1}^n \widehat{\omega}_{j,s}\tilde{\sigma}_s^j\right) dW_s \\ d\widehat{\omega}_{i,s}^\varepsilon = \widehat{\omega}_{i,s}^\varepsilon \left(\tilde{\sigma}_s^i - \varepsilon \sum_{j=1}^n \widehat{\omega}_{j,s}^\varepsilon \tilde{\sigma}_s^j\right)\left(dW_s + \sigma_s^\omega ds + \varepsilon \sum_{j=1}^n \widehat{\omega}_{j,s}\tilde{\sigma}_s^j ds\right) \end{cases}$$

As in Fournie, Lebuchoux & Touzi (1997) and Lebuchoux & Musiela (1999) we compute:

$$C^\varepsilon = E\left[\left(S_T^{\omega,\varepsilon} - k\right)^+ \mid (S_t^\omega, \widehat{\omega}_t)\right]$$

and approximate it around $\varepsilon = 0$ by:

$$C^\varepsilon = C^0 + C^{(1)}\varepsilon + o(\varepsilon)$$

Both C^0 and $C^{(1)}$ (as well as $C^{(2)}$) can be computed explicitly.

In fact, C^0 is given by the BS formula:

$$C^0 = BS(T, S_t^\omega, V_T) = S_t^\omega N(h(V_T)) - \kappa N\left(h(V_T) - V_T^{1/2}\right)$$

with

$$h(V_T) = \frac{\left(\ln\left(\frac{S_t^\omega}{\kappa}\right) + \frac{1}{2}V_T\right)}{V_T^{1/2}} \quad \text{and } V_T = \int_t^T \|\sigma_s^\omega\|^2 \, ds$$

and we get $C^{(1)}$ as:

$$\begin{aligned} C^{(1)} &= S_t^\omega \int_t^T \sum_{j=1}^n \widehat{\omega}_{j,t} \frac{\left\langle \tilde{\sigma}_s^j, \sigma_s^\omega \right\rangle}{V_T^{1/2}} \exp\left(2 \int_t^s \left\langle \tilde{\sigma}_u^j, \sigma_u^\omega \right\rangle du\right) \\ & \quad n\left(\frac{\ln \frac{S_t^\omega}{K} + \int_t^s \left\langle \tilde{\sigma}_u^j, \sigma_u^\omega \right\rangle du + \frac{1}{2}V_T}{V_T^{1/2}}\right) ds \end{aligned}$$

2.5 The BGM Swaption pricing formula

We can write the order zero price approximation for Swaptions:

$$Swaption = Level(t,T,T_N)\left(swap(t,T,T_N)N(h) - \kappa N(h - V_T^{1/2})\right)$$

with

$$h = \frac{\left(\ln\left(\frac{swap(t,T,T_N)}{\kappa}\right) + \frac{1}{2}V_T\right)}{V_T^{1/2}}$$

where

$$V_T = \int_t^T \left\| \sum_{i=1}^{N} \hat{\omega}_i(t)\gamma(s, T_i - s) \right\|^2 ds \text{ and } \hat{\omega}_i(t) = \omega_i(t)\frac{K(t,T_i)}{swap(t,T,T_N)}$$

and $dK(s,T_i) = \gamma(s, T_i - s)K(s,T_i)dW_s^{Q_{T_{i+1}}}$.

2.6 BGM approximation precision

- We plot the difference between two distinct sets of Swaption prices in the Libor Market Model. One is obtained by Monte-Carlo simulation using enough steps to make the 95% confidence margin of error always less than 1bp. The second set of prices is computed using the order zero approximation.

- The plots are based on the prices obtained by calibrating the model to EURO Swaption prices on November 6 2000. We have used all Cap volatilities and the following Swaptions: 2Y into 5Y, 5Y into 5Y, 5Y into 2Y, 10Y into 5Y, 7Y into 5Y, 10Y into 2Y, 10Y into 7Y, 2Y into 2Y, 1Y into 9Y.

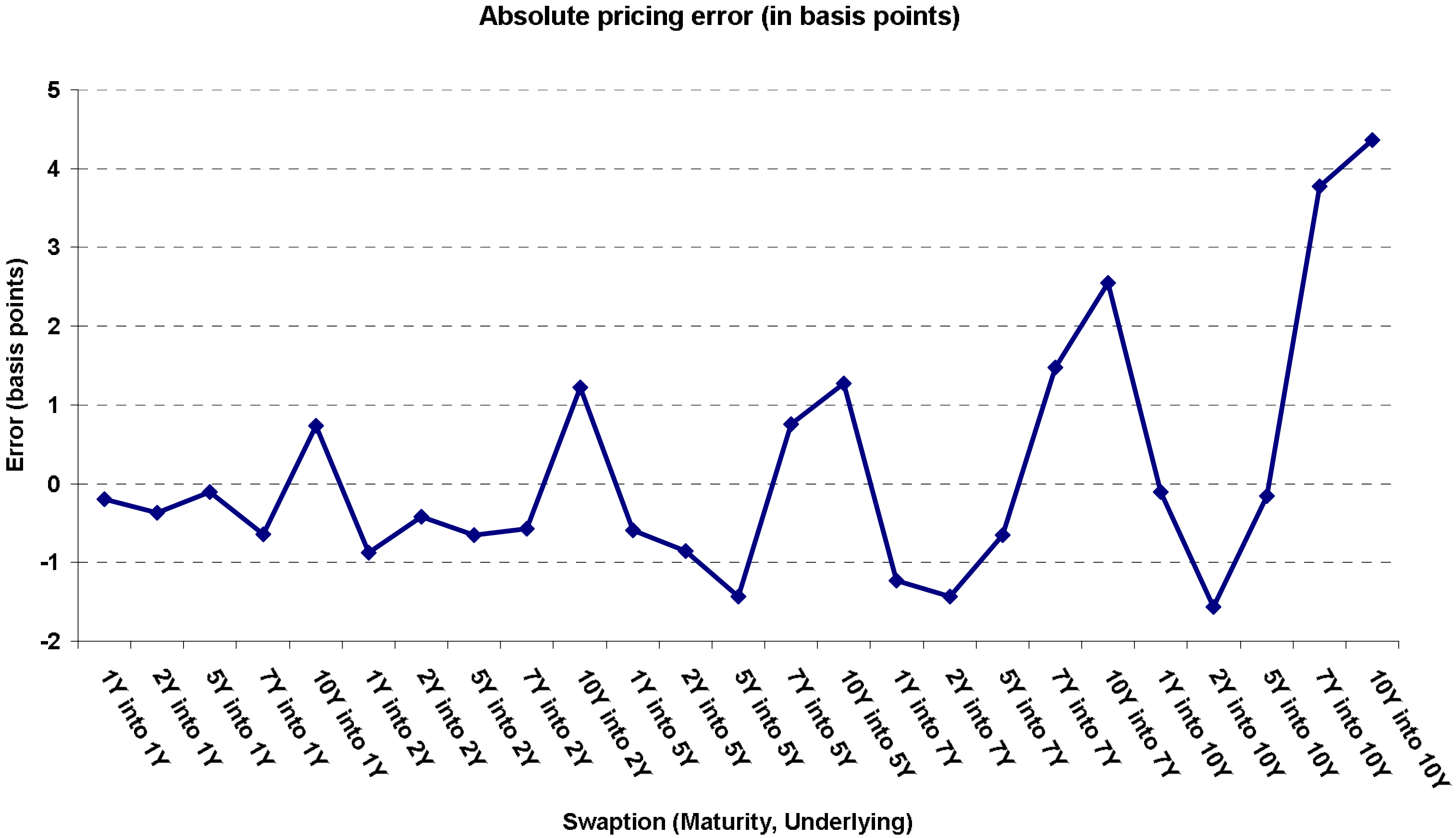

Figure 1: Absolute error (in bp) for various ATM Swaptions.

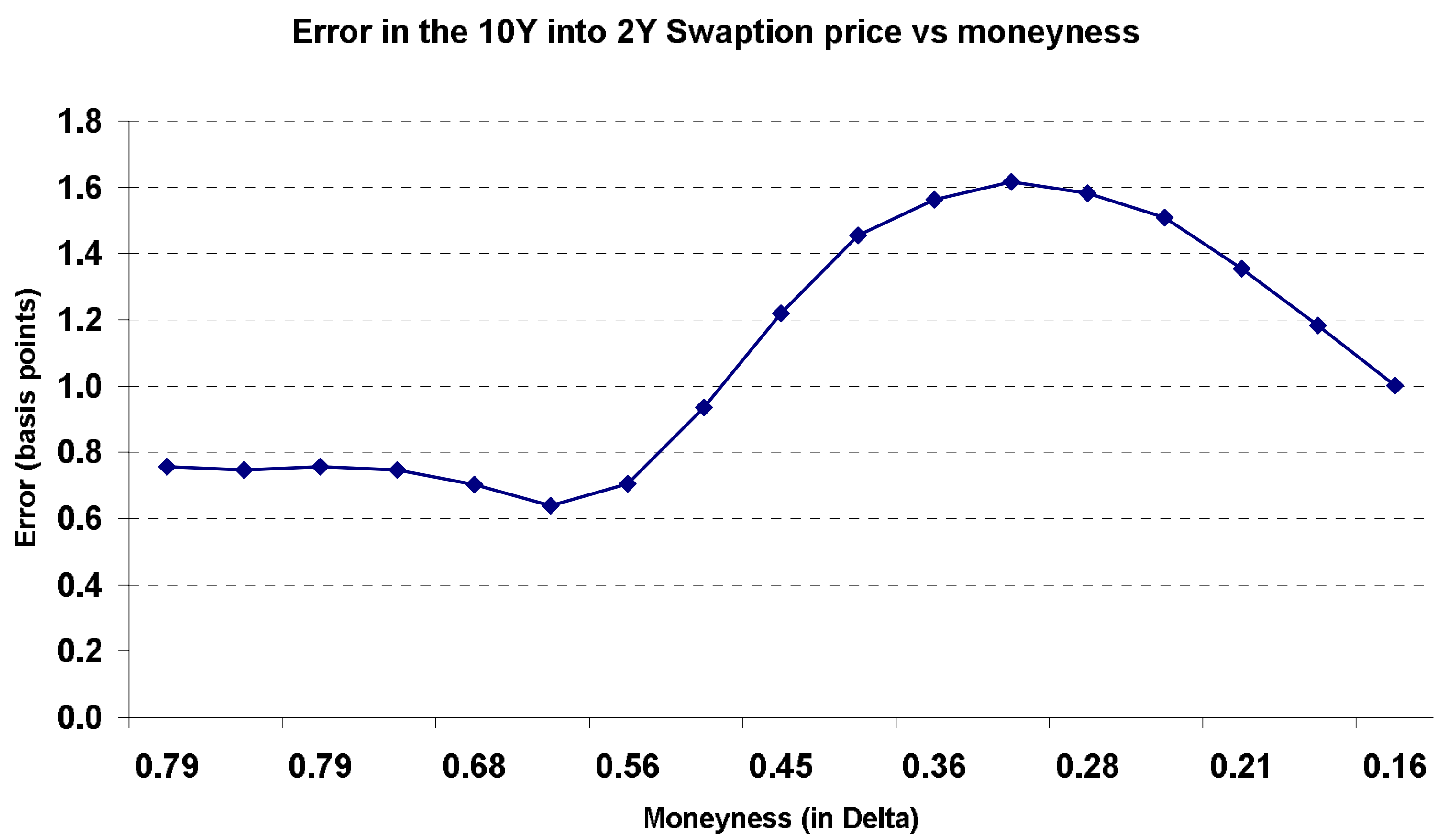

Figure 2: Absolute error (in bp) on the 10Y into 2Y.

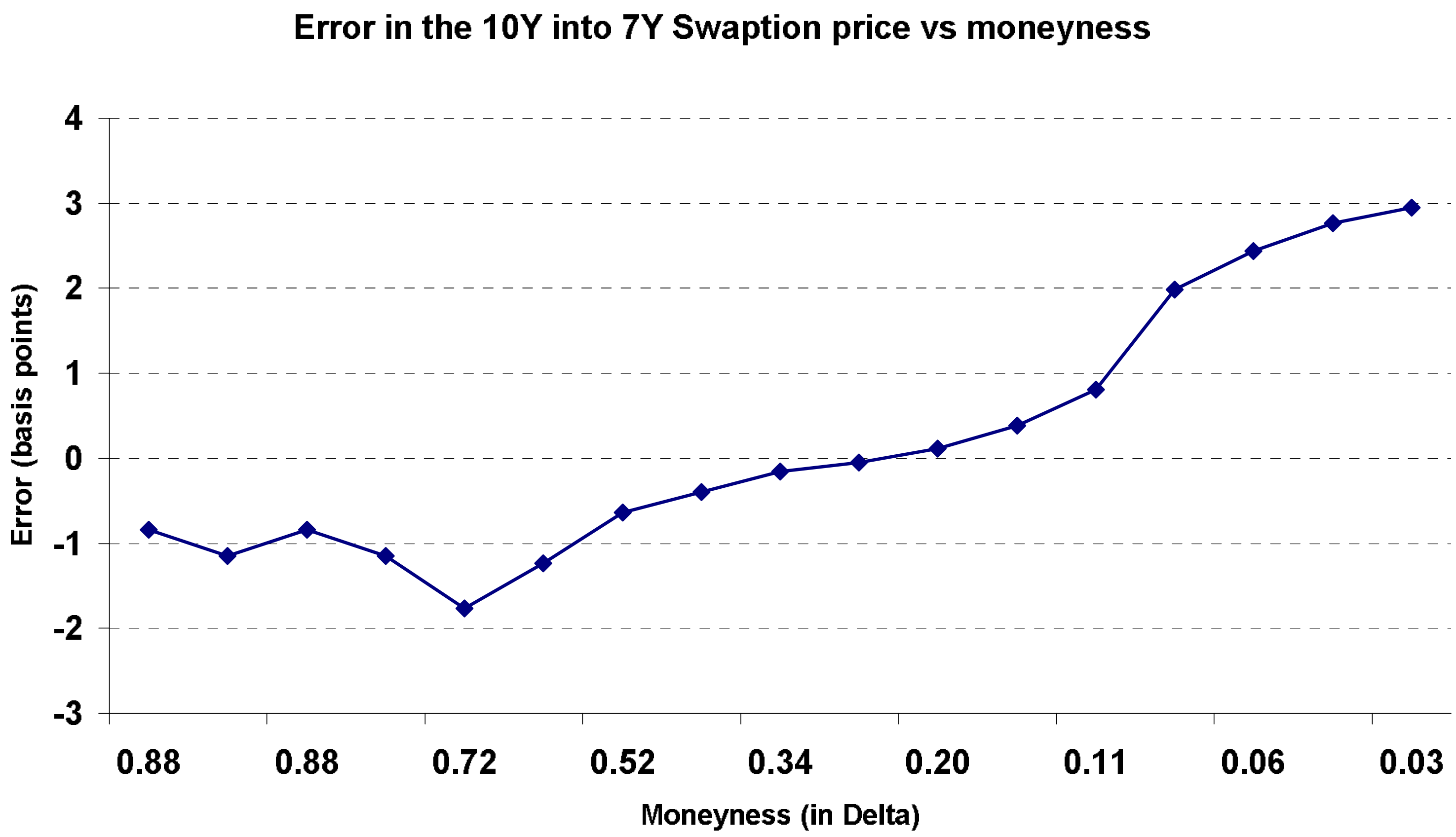

Figure 3: Absolute error (in bp) on the 10Y into 7Y.

3 Calibration

We have approximated the Swaption (T_m, T_{u+m}) price by:

$$P = Level(t, T_m, T_{u+m}) BS(T, swap(t, T_m, T_{u+m}), V(T_m, T_{u+m}))$$

where BS is the Black (1976) formula with

$$V(T_m, T_{u+m}) = \int_t^{T_m} \left\| \sum_{i=m}^{u} \hat{\omega}_i(t) \gamma(s, T_i - s) \right\|^2 ds$$

Suppose that we need to impose a sequence of M market pricing constraints. We express these constraints in terms of the market variance inputs σ_k^2:

$$V(T_{m_k}, T_{u_k+m_k}) = \sigma_k^2 T_{m_k} \quad \text{for } k = 1, \ldots, M$$

We can rewrite the cumulative variance:

$$\begin{aligned}
&\int_t^{T_m} \left\| \sum_{i=m}^{u} \hat{\omega}_i(t)\gamma(s,T_i-s) \right\|^2 ds \\
=&\int_t^{T_m} \sum_{i=m}^{u}\sum_{j=m}^{u} \hat{\omega}_i(t)\hat{\omega}_j(t) \left\langle \gamma(s,T_i-s),\gamma(s,T_j-s)\right\rangle ds \\
=&\int_t^{T_m} Tr\left(\Omega X_s\right) ds
\end{aligned}$$

where Tr is the trace, X_s is the Forward rate covariance matrix, with

$$(X_s)_{i,j} = \left\langle \gamma(s,T_i-s),\gamma(s,T_j-s)\right\rangle \text{ and}$$

and $\left(\hat{\omega}(t)\hat{\omega}^T(t)\right)$ is a rank one matrix with $\left(\hat{\omega}(t)\hat{\omega}^T(t)\right)_{i,j} = \hat{\omega}_i(t)\hat{\omega}_j(t)$.

This means that the calibration constraints are linear in G_s and can be written:

$$\int_t^{T_{m_k}} Tr\left(\Omega_k X_s\right) ds = \sigma_k^2 T_{m_k} \quad \text{for } k = 1, ..., M$$

Suppose, for example, that the volatility of the sliding maturity Libors is stationary and discretized yearly, with $\gamma(s, T_i - s) = \gamma(\lfloor T_i - s \rfloor)$, we can rewrite the pricing constraints:

$$Tr\left(\Omega_k X\right) = \sigma_k^2 T_{m_k} \quad \text{for } k = 1, ..., M$$

where $\Omega_k = \sum_{i=1}^{T_{m_k}} \Omega_{k,i}$, with $\Omega_{k,i}$ a matrix equal to zero everywhere except for the submatrix Ω_k starting at position (i, i).

3.1 Semidefinite programming

The calibration problem can finally be stated as:

$$\begin{array}{ll} \text{find} & X \\ \text{s.t.} & Tr(\Omega_k X) = \sigma_k^2 T_{m_k} \quad \text{for } k = 1, ..., M \\ & X \succeq 0 \end{array}$$

where $X \succeq 0$ stands for "X semidefinite positive". If we choose an objective matrix Ω_0, this becomes a semidefinite program:

$$\begin{array}{ll} \min & Tr\left(\Omega_0 X\right) \\ \text{s.t.} & Tr(\Omega_k X) = \sigma_k^2 T_{m_k} \quad \text{for } k = 1, ..., M \\ & X \succeq 0 \end{array}$$

which can be solved very efficiently (see Nesterov & Nemirovskii (1994), Vandenberghe & Boyd (1996) for the theory and Sturm (1999) for a MATLAB code).

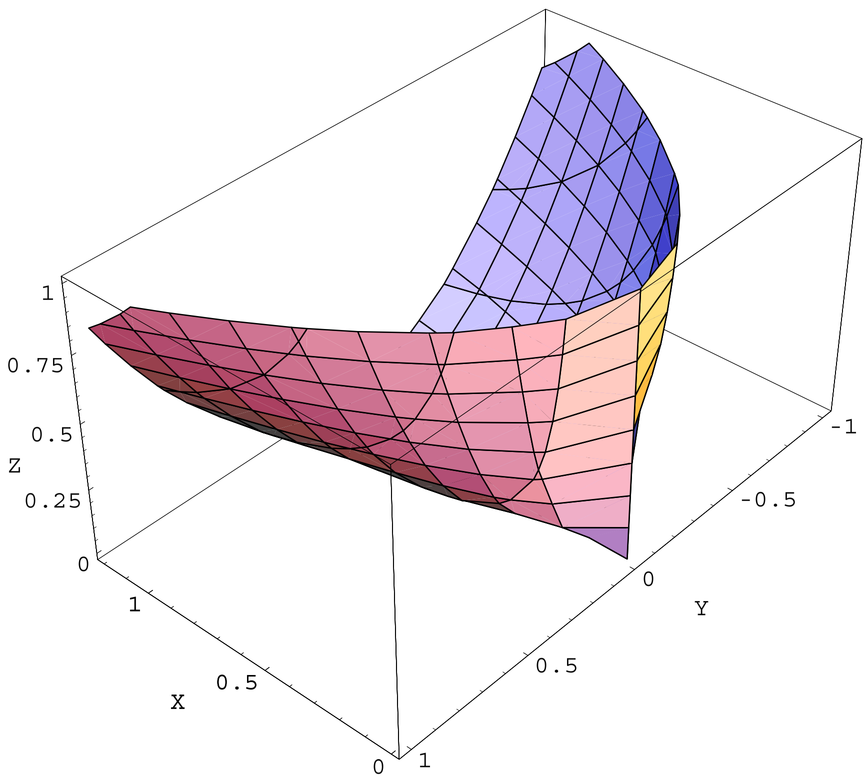

Figure 4: The semidefinite cone in dim 3: $\{\min(\text{eig}[x,y;y,z])=0\}$

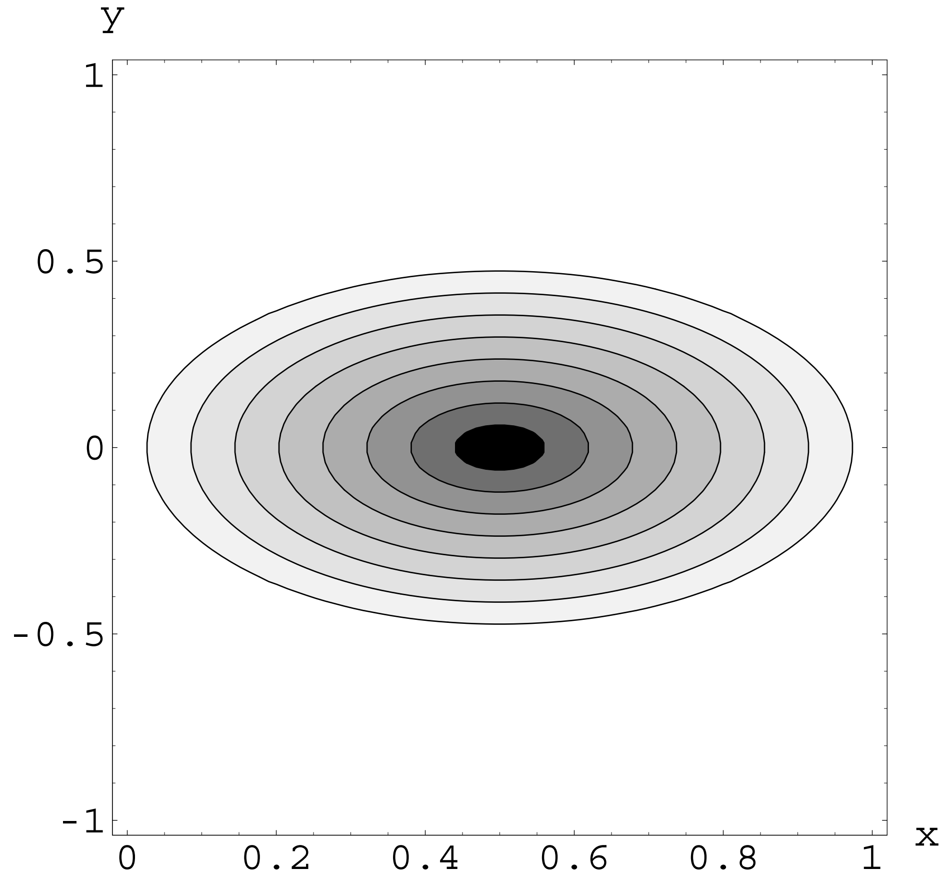

Figure 5: A typical SDP feasible set in dimension 3.

3.2 Definite advantages

- The calibration program has a unique solution computed in polynomial time, with a certificate of optimality or infeasibility.

- The dual solution provides the local sensitivity (see Todd & Yildirim (1999)) to all market price movements (no more "bump and recalibrate").

- Bid-Ask spread data, smoothness or other prices can be included in the inputs and objective.

- As in Cont (2001), we can use Tikhonov regularization to stabilize the solution (hence reduce hedging transaction costs).

3.3 Example: Swaption price bounds

We can use a Swaption matrix as the objective and compute its maximum or minimum price given a set of other Caplet and Swaption prices:

$$\begin{array}{ll} \text{min/max} & Tr\,(\Omega_0 X) \\ \text{s.t.} & Tr(\Omega_k X) = \sigma_k^2 T_{m_k} \quad \text{for } k = 1, ..., M \\ & X \succeq 0 \end{array}$$

In the next figure, we look at the evolution of these price bounds on the 5Y into 3Y Swaption as more and more Swaptions are added into the calibration set (which includes all Caplet prices). We use the same stationary sliding dynamics, with

$$dK(s, T_i) = \gamma(s, T_i - s) K(s, T_i) dW_s^{Q_{T_{i+1}}}$$

where $\gamma(s, T_i - s) = \gamma(\lfloor T_i - s \rfloor)$.

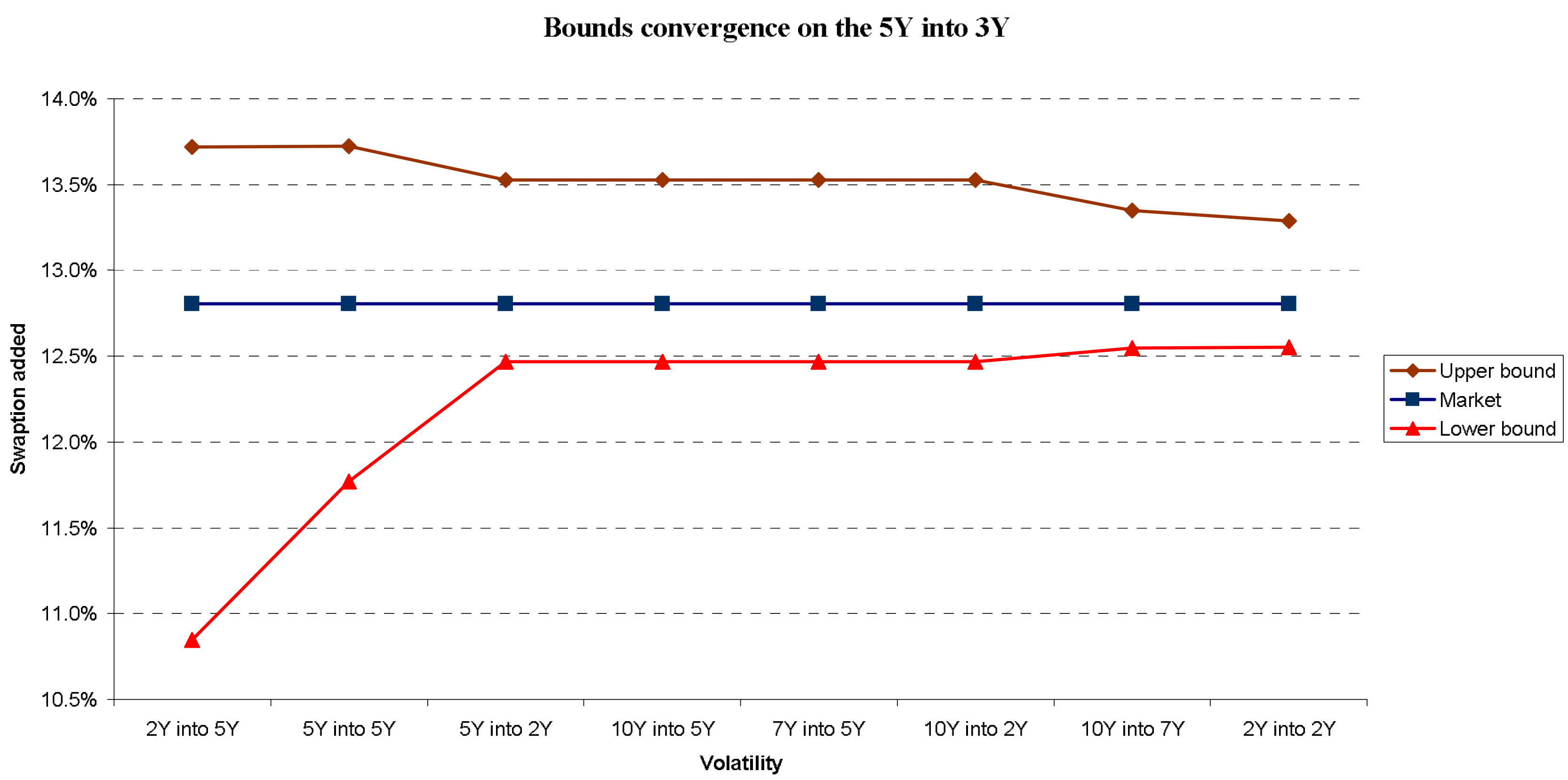

Figure 6: Upper and lower price bounds convergence as more Swaptions are included in the calibration set.

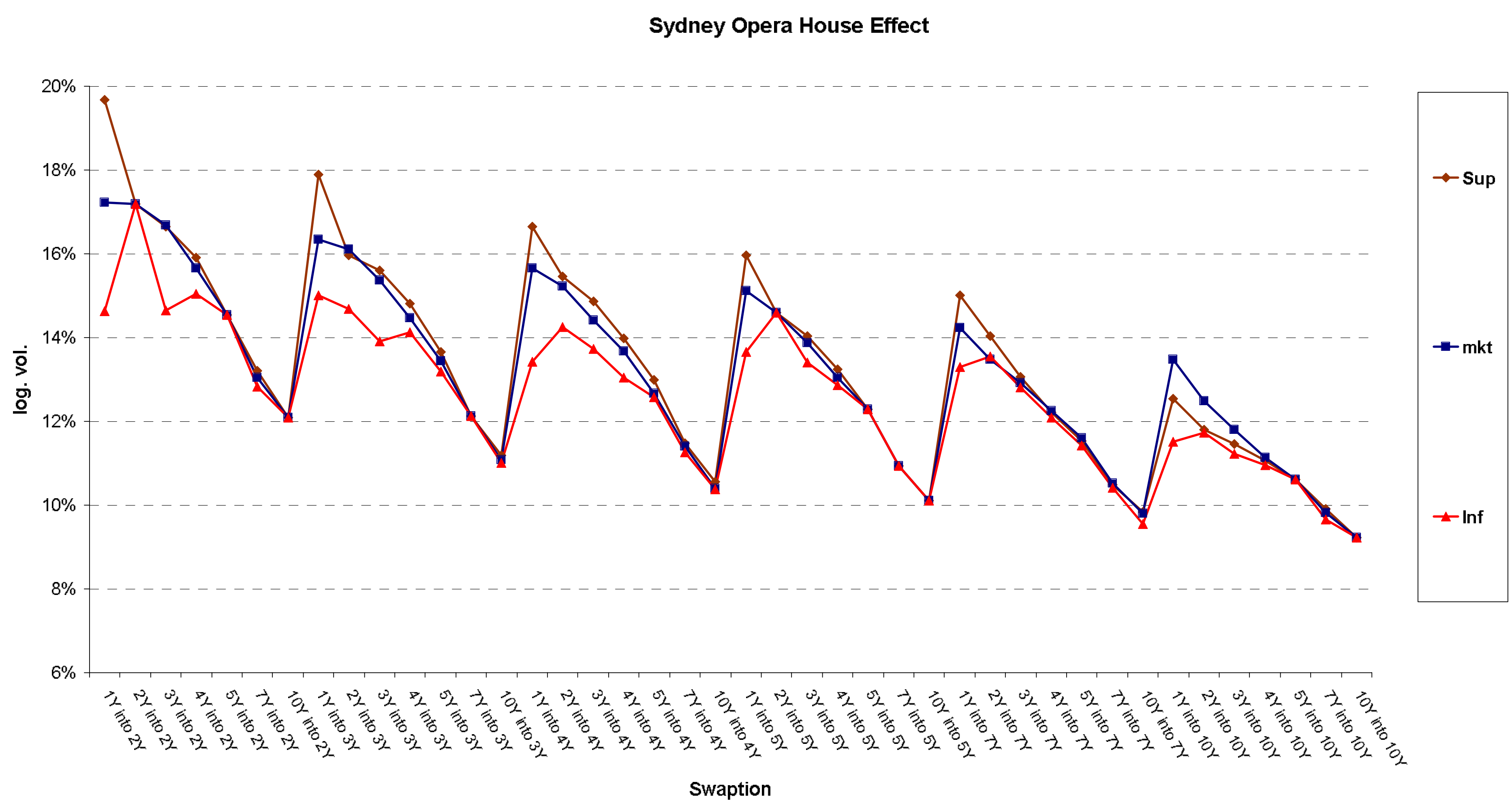

Figure 7: Upper and lower bounds for various Swaption (EUR, 11/6/2000)

3.4 Smooth calibration example

- We calibrate the model to EURO Swaption prices on November 6 2000.
- We use all Caplet volatilities and the following Swaptions: 2Y into 5Y, 5Y into 5Y, 5Y into 2Y, 10Y into 5Y, 7Y into 5Y, 10Y into 2Y, 10Y into 7Y, 2Y into 2Y, 1Y into 9Y (data courtesy of BNP Paribas, London).
- We add a smoothness constraint (minimum surface).

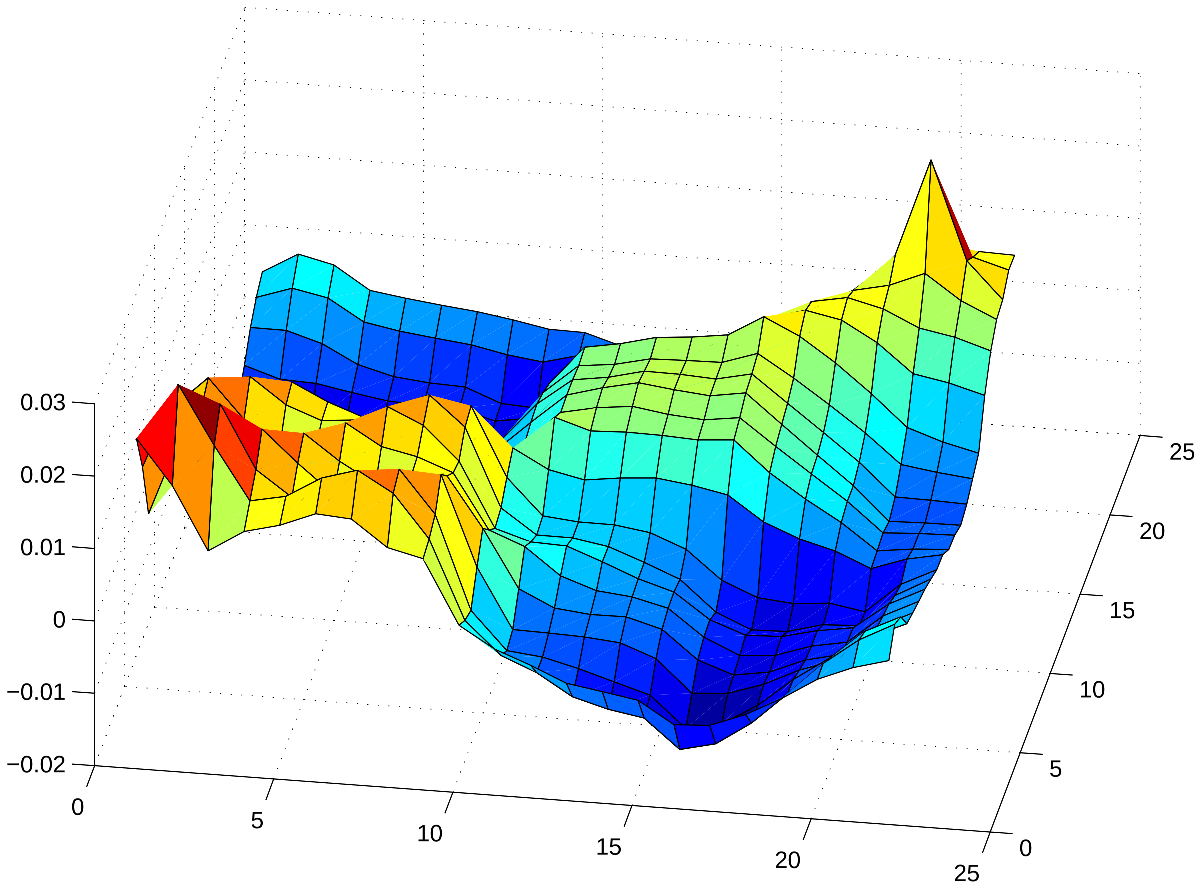

Figure 8: Forward rates covariance matrix

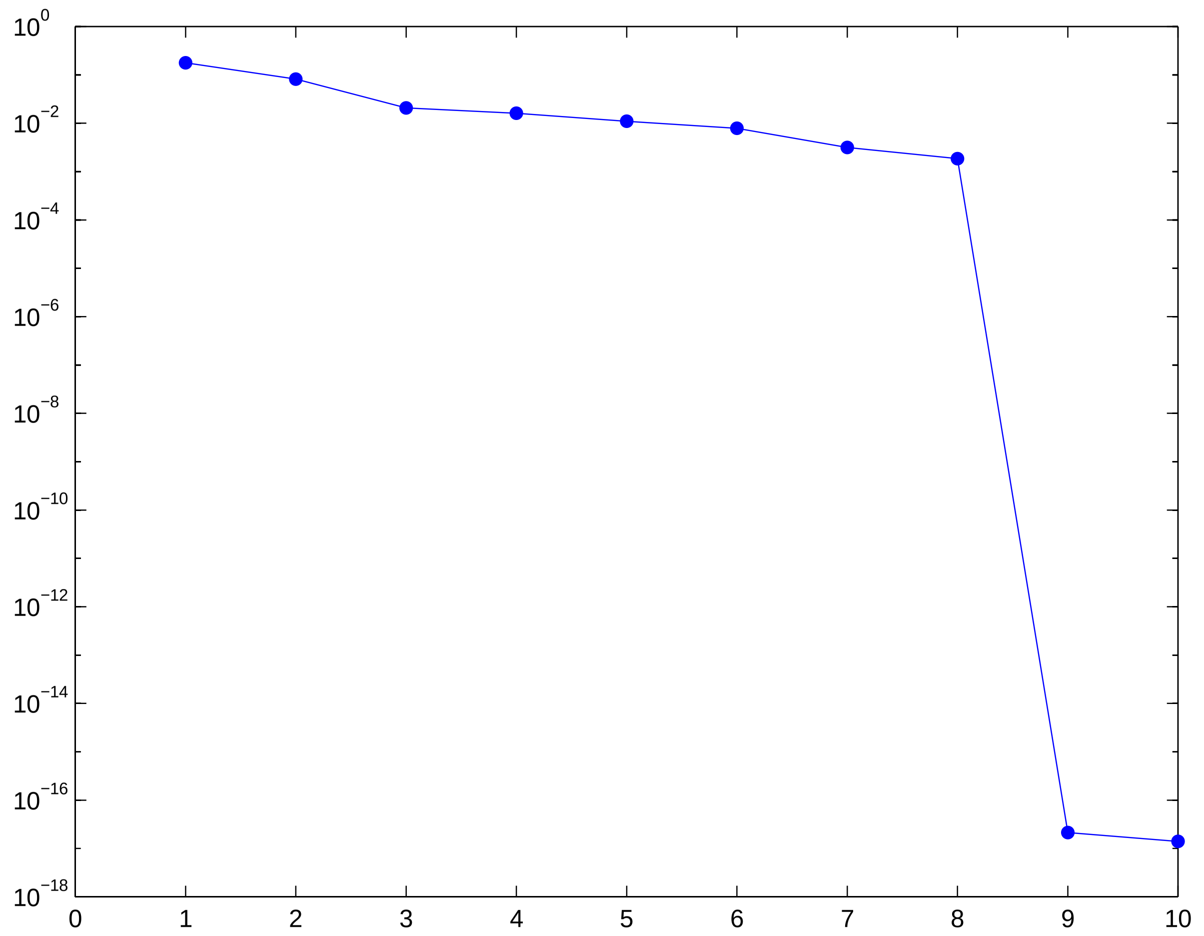

Figure 9: Eigenvalues of the smooth solution (semilog).

3.5 Low rank solution

There is no way to efficiently guarantee that the solution will be of given rank. But there are some excellent heuristical methods. For example, as in Boyd, Fazel & Hindi (2000), we can use another semidefinite positive matrix in the objective to get a low rank solution.

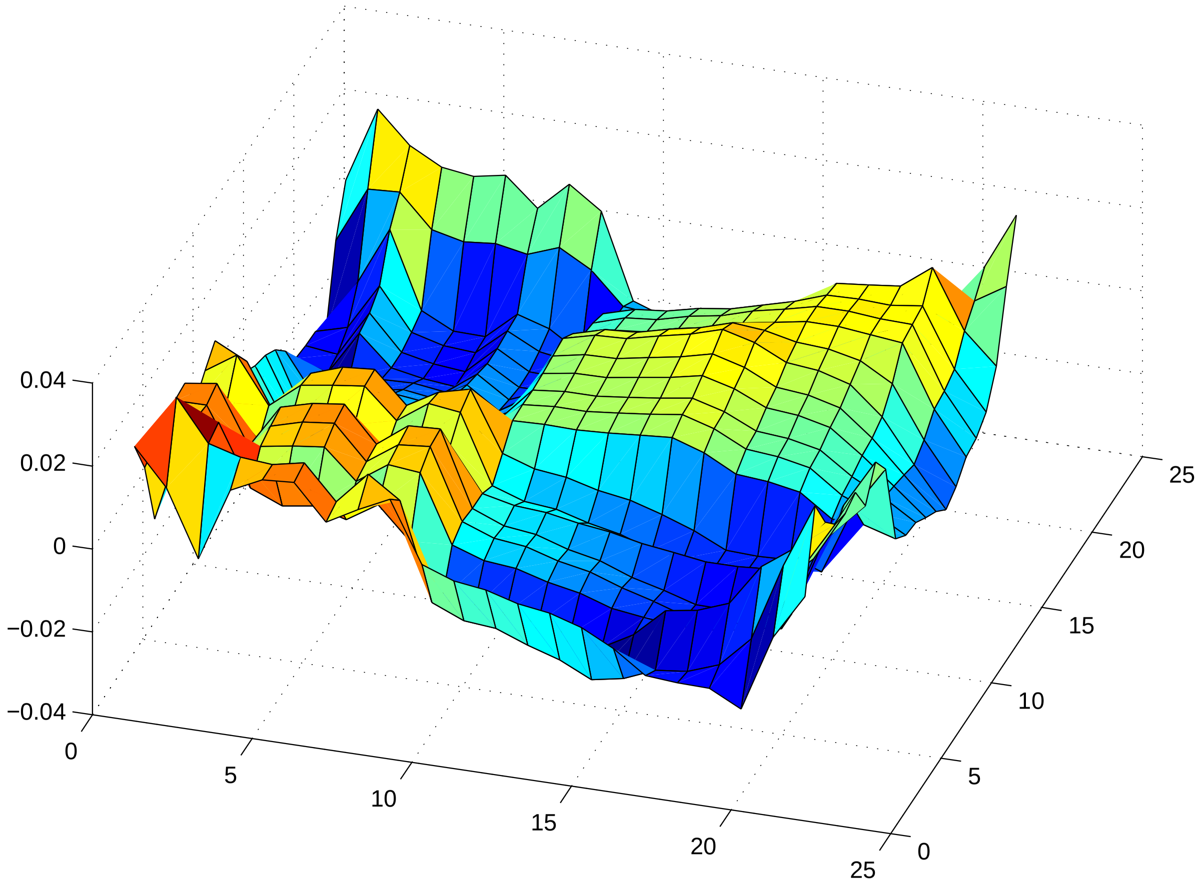

Figure 10: Low rank solution

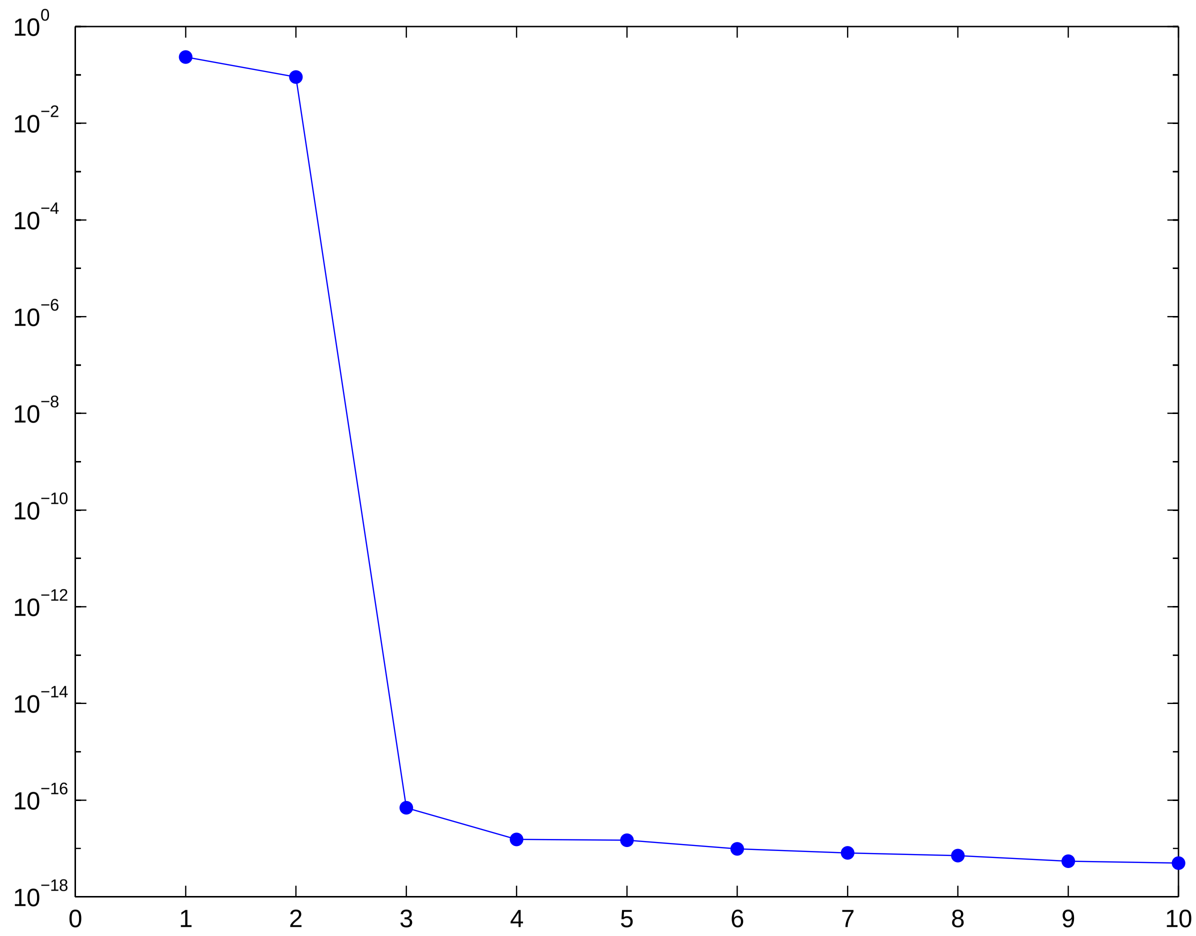

Figure 11: Eigenvalues of the low rank solution (semilog).

4 Conclusion

- We obtain a fast, reliable calibration method for the BGM model.
- The improvement in the solution's stability should reduce unnecessary hedging costs.
- The final trade-off in the calibration program becomes "low rank" vs. "stability".

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