Risk Management Methods for the Libor Market Model Using Semidefinite Programming.

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1.1 Interest rate model calibration

- All Heath, Jarrow & Morton (1992) based models are fully parametrized by the *curve today* and a *covariance* matrix.

- The natural variable in the calibration problem is a covariance matrix, i.e. a *positive semidefinite matrix*.

- Classic calibration methods are *heavily parametrized* and only describe a small, often non-convex subset of the set of semidefinite matrices.

- When using these techniques, sensitivity analysis has to be done by bumping the data and recalibrating.

1.2 Results on the LMM model calibration

- We can express the swap rate as a *basket of Forwards* with very stable coefficients.

- European Caplets and Swaptions can be priced using the Black (1976) market formula with a variance that is *linear* in the coefficients of the Forward rates covariance matrix..

- The calibration problem is a *semidefinite program* and the dual solution naturally provides the *local sensitivity to all market movements*.

- The dual solution also provides the coefficients of an upper (lower) hedging portfolio in the sense of El Karoui & Quenez (1991) and Avellaneda & Paras (1996).

1.3 Related literature

- Works by Nesterov & Nemirovskii (1994) and Vandenberghe & Boyd (1996) on semidefinite programming

- Brace, Gatarek & Musiela (1997) and Musiela & Rutkowski (1997) on the Libor market model.

- Rebonato (1998), Brace, Dun & Barton (1999) and Singleton & Umantsev (2001) on Swaps as baskets of Forwards. Rebonato (1999) on a calibration method parametrized by factors.

- Parallel work by Brace & Womersley (2000) on the calibration of the BGM by semidefinite programming and the evaluation of the Bermudan Swaption.

2 Swaption pricing

2.1 The Swap rate

We write the swap rate as a basket of Forwards:

$$swap(t, T_0, T_n) = \sum_{i=0}^{n} \omega_i(t) K(t, T_i)$$

where $K(t, T_i)$ are the Forward Rates with maturities T_i , $i = 1, ..., n$ and the weights $\omega_i(t)$ are given by

$$\omega_i(t) = \frac{coverage(T_i^{float}, T_{i+1}^{float}) B(t, T_{i+1}^{float})}{Level(t, T_0^{fixed}, T_n^{fixed})}$$

2.2 BGM Swaption price

In practice, the weights $\omega_i(t)$ are very stable (see Rebonato (1998)) and following Jamshidian (1997), we can write the price of the Swaption with strike k as a that of a Call on a Swap rate:

$$Ps(t) = Level(t, T, T_N) E_t^{Q_{LVL}} \left[\left(\sum_{i=0}^{n} \omega_i(T) K(T, T_i) - k \right)^+ \right]$$

where Q_{LVL} is the swap forward martingale probability measure.

In fact, as detailed in Huynh (1994) or Brace et al. (1999) the Swaption price can be very efficiently approximated by the Black (1976) formula:

$$Swaption = Level(t,T,T_N)\left(swap(t,T,T_N)N(h) - \kappa N(h - V_T^{1/2})\right)$$

with

$$h = \frac{\left(\ln\left(\frac{swap(t,T,T_N)}{\kappa}\right) + \frac{1}{2}V_T\right)}{V_T^{1/2}}$$

where the cumulative variance is computed by:

$$V_T = \int_t^T \left\| \sum_{i=1}^N \hat{\omega}_i(t)\gamma(s,T_i - s) \right\|^2 ds \text{ and } \hat{\omega}_i(t) = \omega_i(t)\frac{K(t,T_i)}{swap(t,T,T_N)}$$

with $dK(s,T_i) = \gamma(s,T_i - s)K(s,T_i)dW_s^{Q_{T_{i+1}}}$.

2.3 BGM approximation precision

- We plot the difference between two distinct sets of Swaption prices in the Libor Market Model. One is obtained by Monte-Carlo simulation using enough steps to make the 95% confidence margin of error always less than 1bp. The second set of prices is computed using the order zero approximation.

- The plots are based on the prices obtained by calibrating the model to EURO Swaption prices on November 6 2000. We have used all Cap volatilities and the following Swaptions: 2Y into 5Y, 5Y into 5Y, 5Y into 2Y, 10Y into 5Y, 7Y into 5Y, 10Y into 2Y, 10Y into 7Y, 2Y into 2Y, 1Y into 9Y.

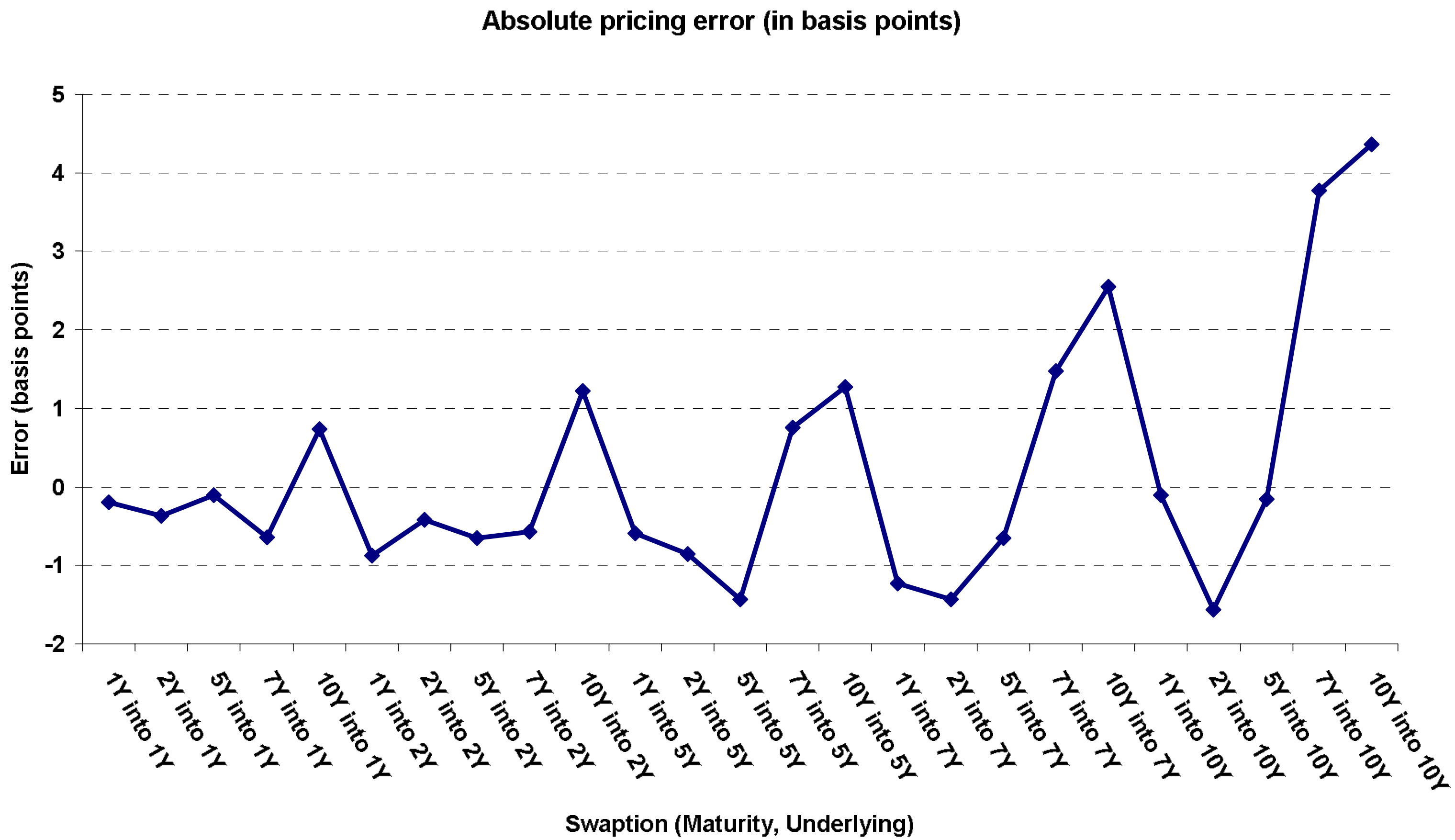

Figure 1: Absolute error (in bp) for various ATM Swaptions.

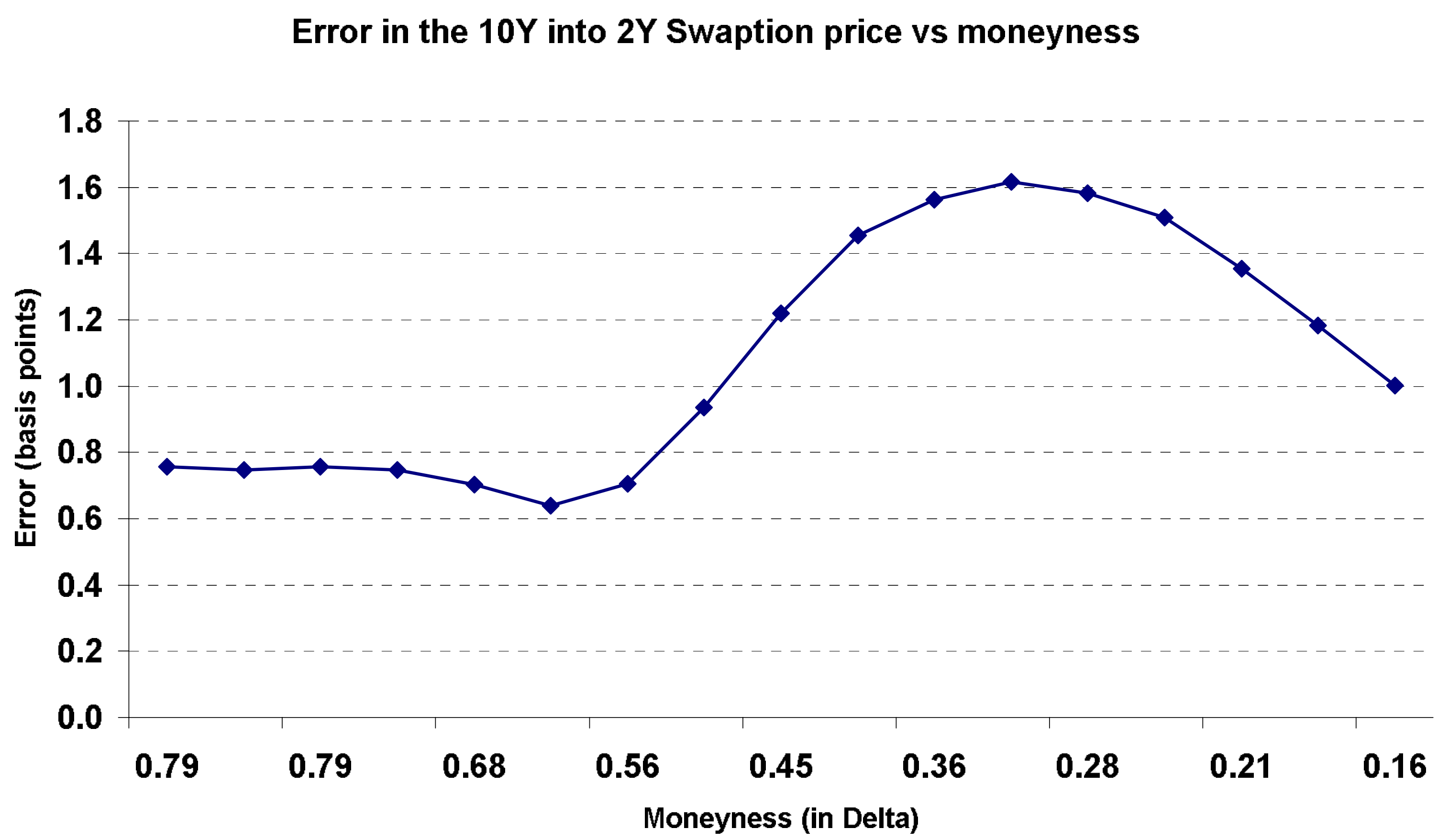

Figure 2: Absolute error (in bp) on the 10Y into 2Y.

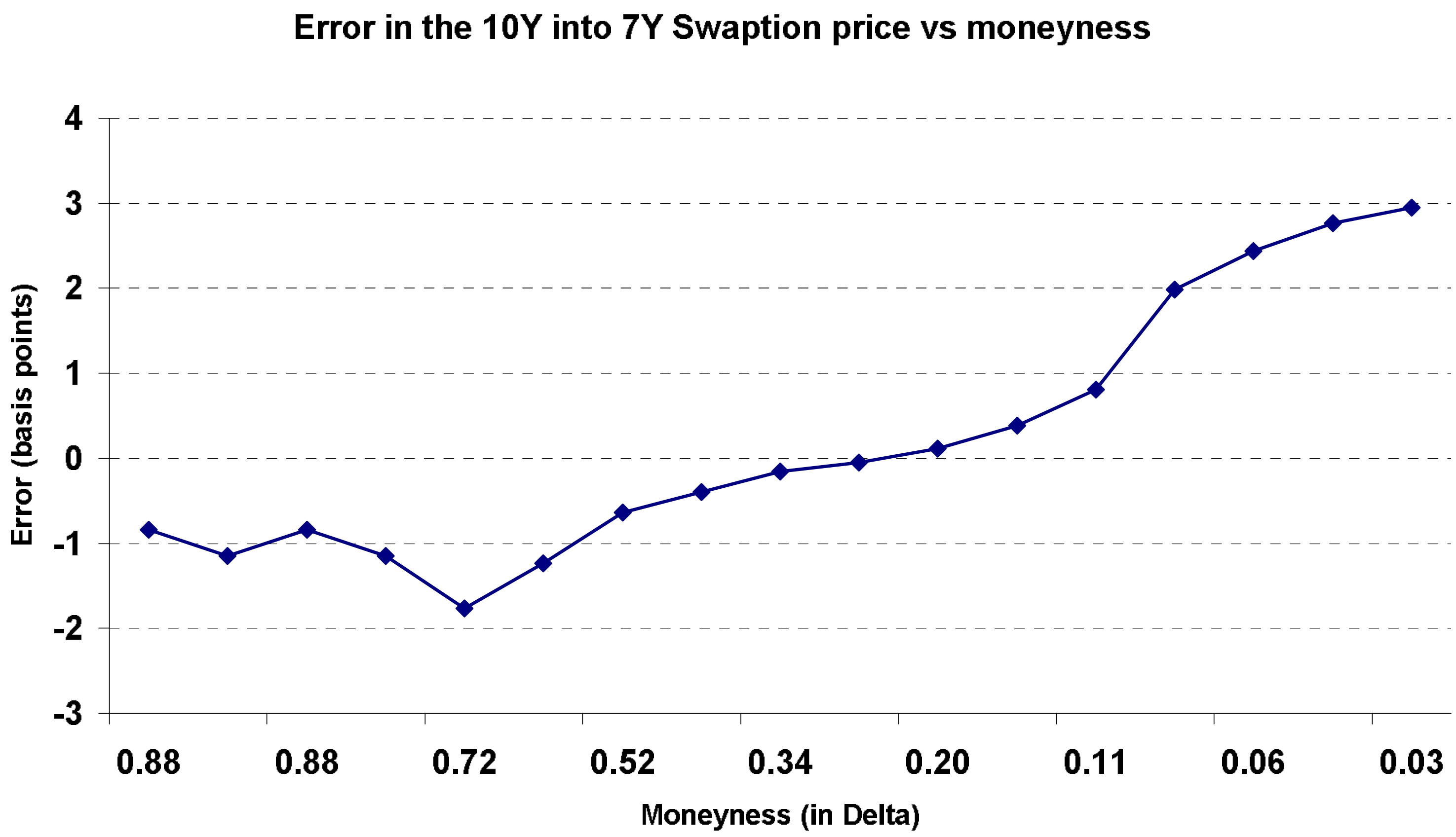

Figure 3: Absolute error (in bp) on the 10Y into 7Y.

3 Calibration

We have approximated the Swaption (T_m, T_{u+m}) price by:

$$P = Level(t, T_m, T_{u+m}) BS(T, swap(t, T_m, T_{u+m}), V(T_m, T_{u+m}))$$

where BS is the Black (1976) formula with

$$V(T_m, T_{u+m}) = \int_t^{T_m} \left\| \sum_{i=m}^{u} \hat{\omega}_i(t) \gamma(s, T_i - s) \right\|^2 ds$$

Suppose that we need to impose a sequence of M market pricing constraints. We express these constraints in terms of the market variance inputs σ_k^2:

$$V(T_{m_k}, T_{u_k+m_k}) = \sigma_k^2 T_{m_k} \quad \text{for } k = 1, ..., M$$

We can rewrite the cumulative variance:

$$\begin{aligned} & \int_t^{T_m} \left\| \sum_{i=m}^{u} \hat{\omega}_i(t)\gamma(s, T_i - s) \right\|^2 ds \\ = & \int_t^{T_m} \sum_{i=m}^{u} \sum_{j=m}^{u} \hat{\omega}_i(t)\hat{\omega}_j(t) \left\langle \gamma(s, T_i - s), \gamma(s, T_j - s) \right\rangle ds \\ = & \int_t^{T_m} Tr\left(\Omega X_s\right) ds \end{aligned}$$

where Tr is the trace, X_s is the Forward rate covariance matrix, with

$$\left(X_s\right)_{i,j} = \left\langle \gamma(s, T_i - s), \gamma(s, T_j - s) \right\rangle \text{ and}$$

and $\left(\hat{\omega}(t)\hat{\omega}^T(t)\right)$ is a rank one matrix with $\left(\hat{\omega}(t)\hat{\omega}^T(t)\right)_{i,j} = \hat{\omega}_i(t)\hat{\omega}_j(t)$.

This means that the calibration constraints are linear in X_s and can be written:

$$\int_t^{T_{m_k}} Tr\left(\Omega_k X_s\right) ds = \sigma_k^2 T_{m_k} \quad \text{for } k = 1, ..., M$$

If we discretize in time we can write the above constraints as:

$$Tr\left(\Omega_k X\right) = \sigma_k^2 T_{m_k} \quad \text{for } k = 1, ..., M$$

where Ω_k is a block diagonal matrix.

3.1 Semidefinite programming

The calibration problem can finally be stated as:

$$\begin{array}{ll} \text{find} & X \\ \text{s.t.} & Tr(\Omega_k X) = \sigma_k^2 T_{m_k} \quad \text{for } k = 1, ..., M \\ & X \succeq 0 \end{array}$$

where $X \succeq 0$ stands for "X semidefinite positive". If we choose an objective matrix Ω_0, this becomes a semidefinite program:

$$\begin{array}{ll} \text{min} & Tr\,(\Omega_0 X) \\ \text{s.t.} & Tr(\Omega_k X) = \sigma_k^2 T_{m_k} \quad \text{for } k = 1, ..., M \\ & X \succeq 0 \end{array}$$

which can be solved very efficiently (see Nesterov & Nemirovskii (1994), Vandenberghe & Boyd (1996) for the theory and Sturm (1999) for a MATLAB code).

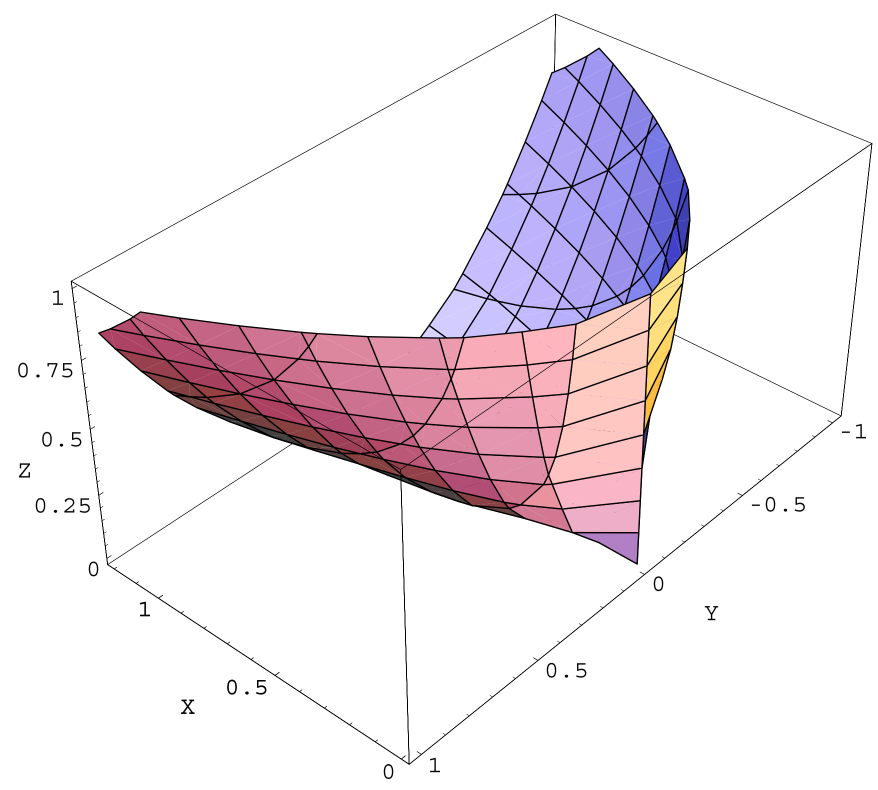

Figure 4: The semidefinite cone in dim 3: $\{\min(\text{eig}[x,y;y,z])=0\}$

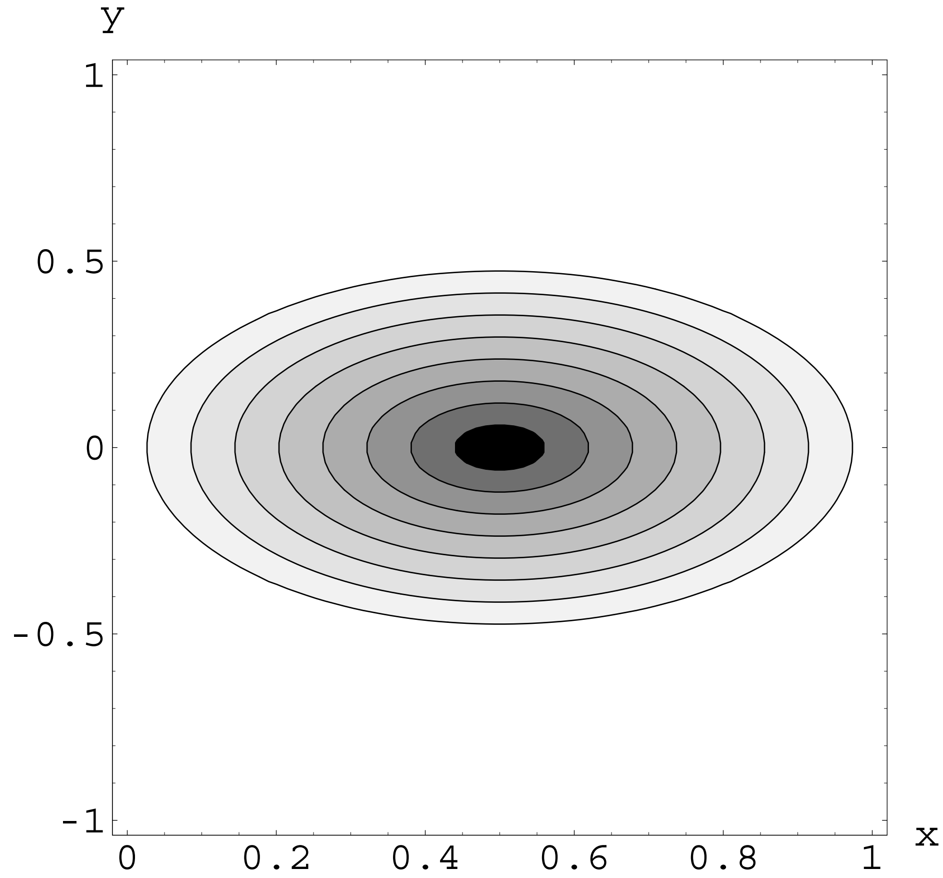

Figure 5: A typical SDP feasible set in dimension 3.

3.2 Definite advantages

- The calibration program has a unique solution computed in polynomial time, with a certificate of optimality or infeasibility.
- Bid-Ask spread data, smoothness or other prices can be included in the inputs and objective.
- The algorithms provide both primal and dual solutions. The primal gives the calibrated Forward rate covariance matrix, while the dual provides local sensitivity results.

3.3 Smooth calibration

- We calibrate the model to EURO Swaption prices on November 6 2000.

- We use all Caplet volatilities and the following Swaptions: 2Y into 5Y, 5Y into 5Y, 5Y into 2Y, 10Y into 5Y, 7Y into 5Y, 10Y into 2Y, 10Y into 7Y, 2Y into 2Y, 1Y into 9Y (data courtesy of BNP Paribas, London).

- We add a smoothness constraint (minimum surface), this acts as a Tikhonov (1963) stabilization of the solution and reduces purely the number of numerical hedging transactions.

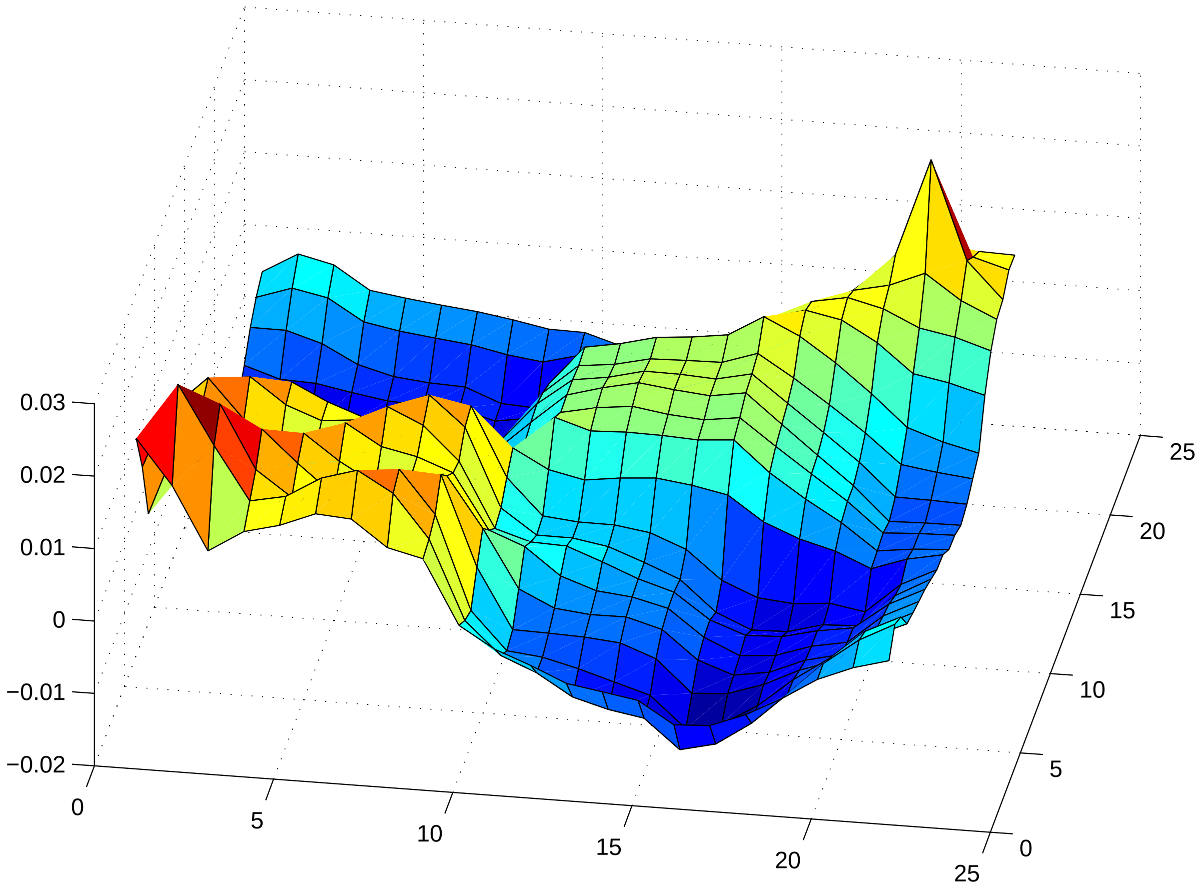

Figure 6: Forward rates covariance matrix

3.4 The dual program

When the original program is given by:

$$\begin{array}{ll} \text{max} & Tr\,(\Omega_0 X) \\ \text{s.t.} & Tr(\Omega_k X) = \sigma_k^2 T_{m_k} \quad \text{for } k = 1, ..., M \\ & X \succeq 0 \end{array}$$

the dual becomes:

$$\begin{array}{ll} \text{min} & -\sum_{k=1}^{M} y_k \sigma_k^2 T_{m_k} \\ \text{s.t.} & \Omega_0 \preceq \sum_{k=1}^{M} y_k \Omega_k \end{array}$$

most S.D.P. solvers (such as SEDUMI by Sturm (1999) for example) compute both solutions at the same time.

3.4.1 Local sensitivity

We can study the impact on the solution of a (small) change in market conditions given by u_k for $k = 1, ..., M$, the calibration program becomes:

$$\begin{array}{ll} \text{maximize} & Tr(CX) \\ \text{s.t.} & Tr(\Omega_k X) = \sigma_k^2 T_k + u_k \text{ for } k = 1, ..., M \\ & X \succeq 0 \end{array}$$

Using Todd & Yildirim (1999) we can compute the new calibrated matrix $X + \Delta X$ with:

$$\Delta X = E^{-1} F A^* \left[\left(A E^{-1} F A^* \right)^{-1} u \right]$$

where E,F and A are linear operators computed from (X^*, y^*) the primal and dual solutions to the original calibration program (with $u_k = 0$).

3.4.2 Super hedging price

As expected, we can interpret the dual solution to the calibration program in terms of hedging portfolio.

- As in Avellaneda & Paras (1996) we suppose that the volatility is uncertain and we hedge by mixing a static portfolio of derivatives with a dynamic super-replication strategy.

- The price is obtained by solving the following (formal) program:

$$\text{Price} = \text{Min}\left\{\text{Value of static hedge} + \text{Max}\left(\text{PV of residual liability}\right)\right\}$$

3.4.3 Super hedging portfolio

Suppose we study an upper hedging price on a particular Swaption (Ω_0, T_0). We can find an approximate solution to the previous problem by solving the following problem:

$$\inf_{\lambda} \left\{ \sum_{k=1}^{M} \lambda_k C_k + \sup_{X \succeq 0} \left(BS(Tr(\Omega_0 X)) - \sum_{k=1}^{M} \lambda_k BS\left(Tr(\Omega_k X)\right) \right) \right\}$$

or its dual:

$$\begin{array}{ll} \text{maximize} & BS_0(Tr(\Omega_0 X)) \\ \text{s.t.} & BS_k\left(Tr(\Omega_k X)\right) = C_k \text{ for } k = 1, \ldots, M \\ & X \succeq 0 \end{array}$$

We can write the KKT optimality conditions on this problem:

$$\begin{cases} Z = \frac{\partial BS_0(\Omega_0 X)}{\partial v}\Omega_0 + \sum_{k=1}^M \lambda_k \frac{\partial BS_k(\Omega_k X)}{\partial v}\Omega_k \\ XZ = 0 \\ BS_k\left(Tr(\Omega_k X)\right) = C_i \text{ for } k = 1, ..., M \\ 0 \preceq X, Z \end{cases}$$

hence if y_i^* solves the dual S.D.P:

$$\begin{array}{ll} \text{minimize} & \sum_{k=1}^M y_k \sigma_k^2 T_k \\ \text{s.t.} & 0 \preceq \sum_{k=1}^M y_k \Omega_k - C \end{array}$$

then

$$\lambda_k^* = -y_i^* \frac{\partial BS_0\left(Tr(\Omega_0 X)\right)/\partial v}{\partial BS_k\left(Tr(\Omega_k X)\right)/\partial v}$$

will be the coefficients of a super replicating portfolio in the Swaptions (Ω_k, T_k).

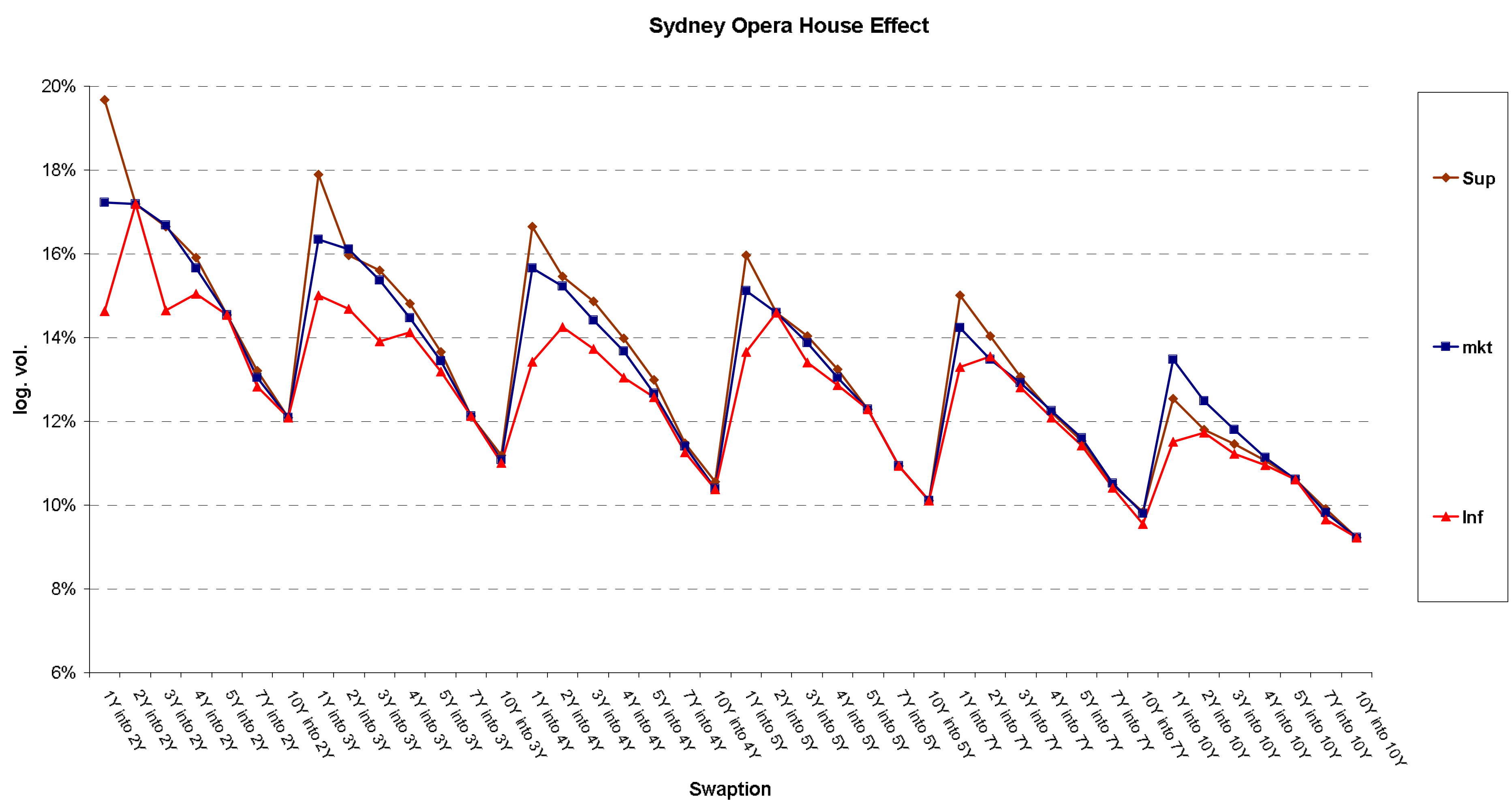

Figure 7: Upper and lower bounds for various Swaption (EUR, 11/6/2000)

3.5 Low rank solution

There is no way to efficiently guarantee that the solution will be of given rank. But there are some excellent heuristical methods. For example, as in Boyd, Fazel & Hindi (2000), we can use another semidefinite positive matrix in the objective to get a low rank solution.

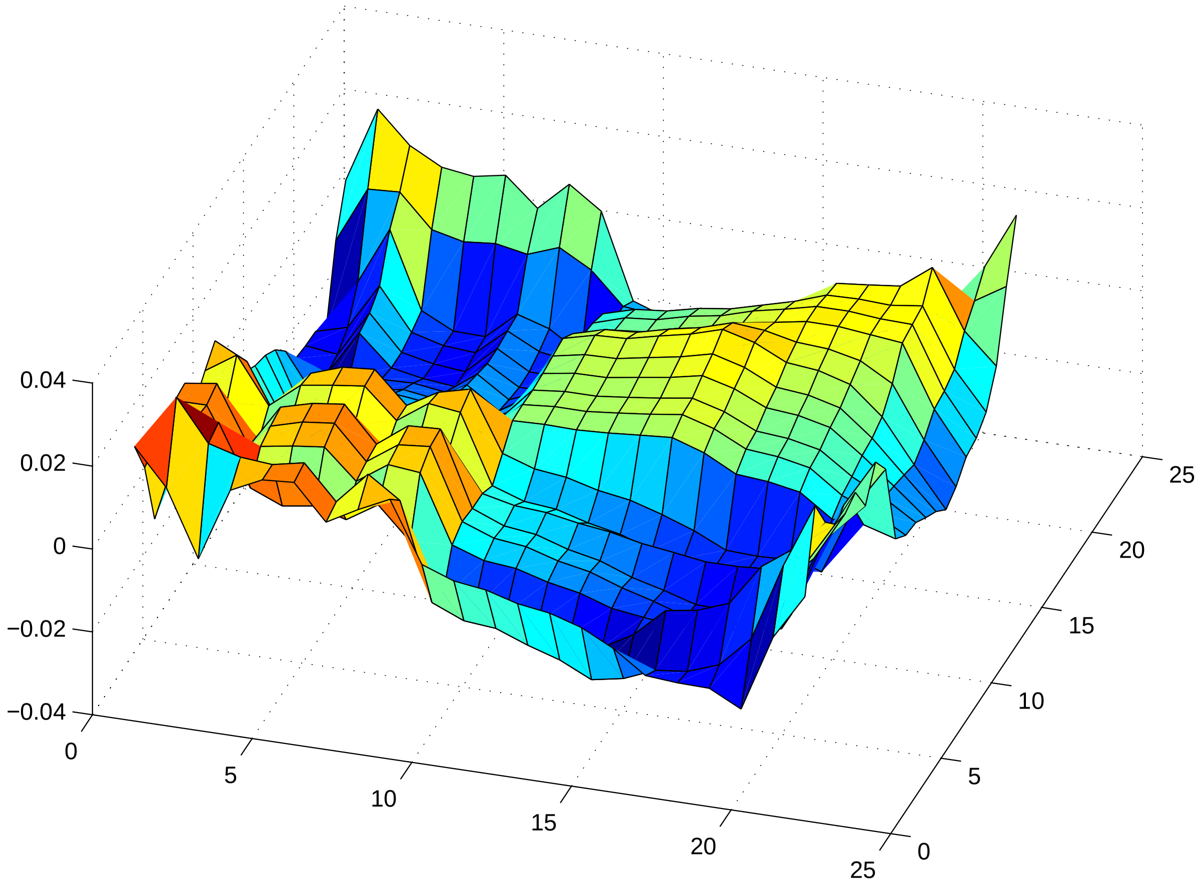

Figure 8: Low rank solution

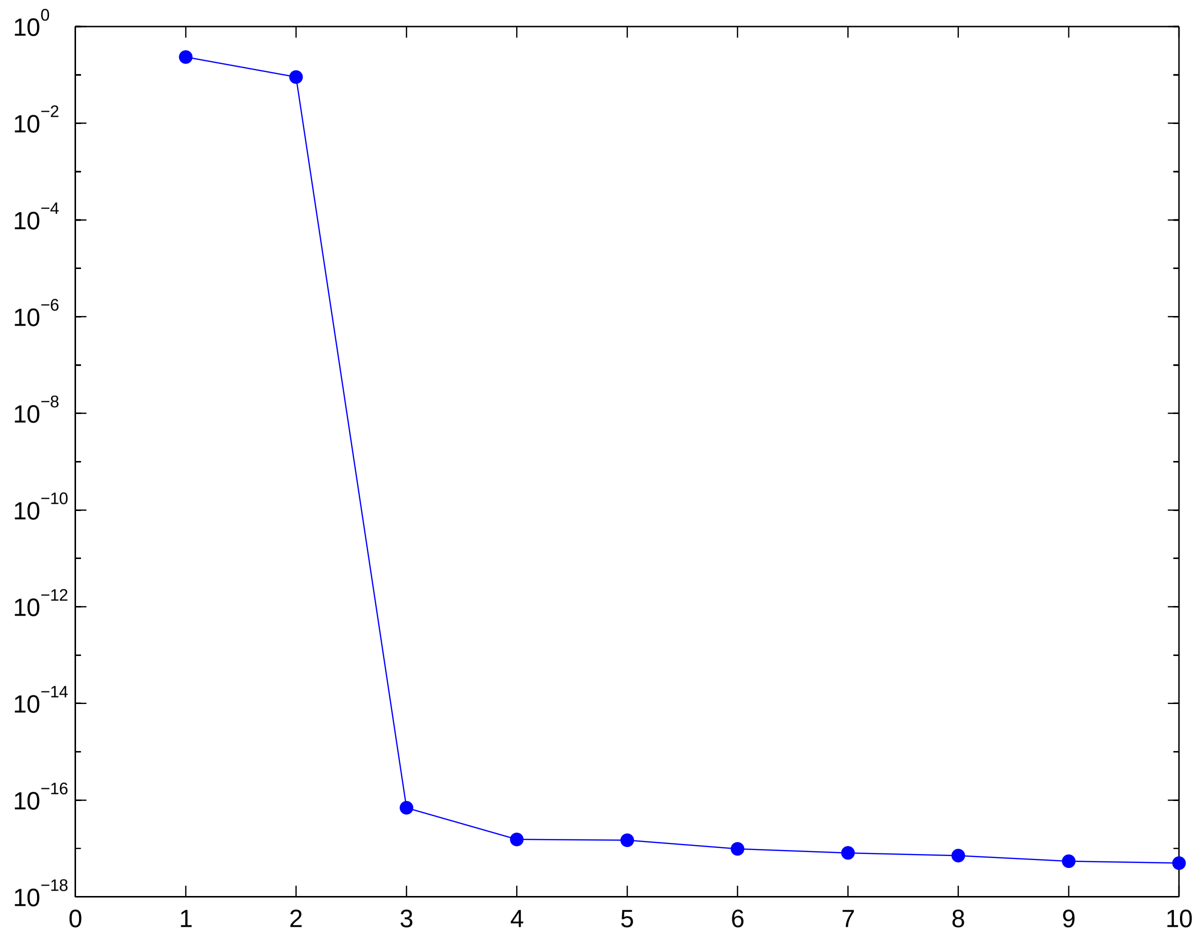

Figure 9: Eigenvalues of the low rank solution (semilog).

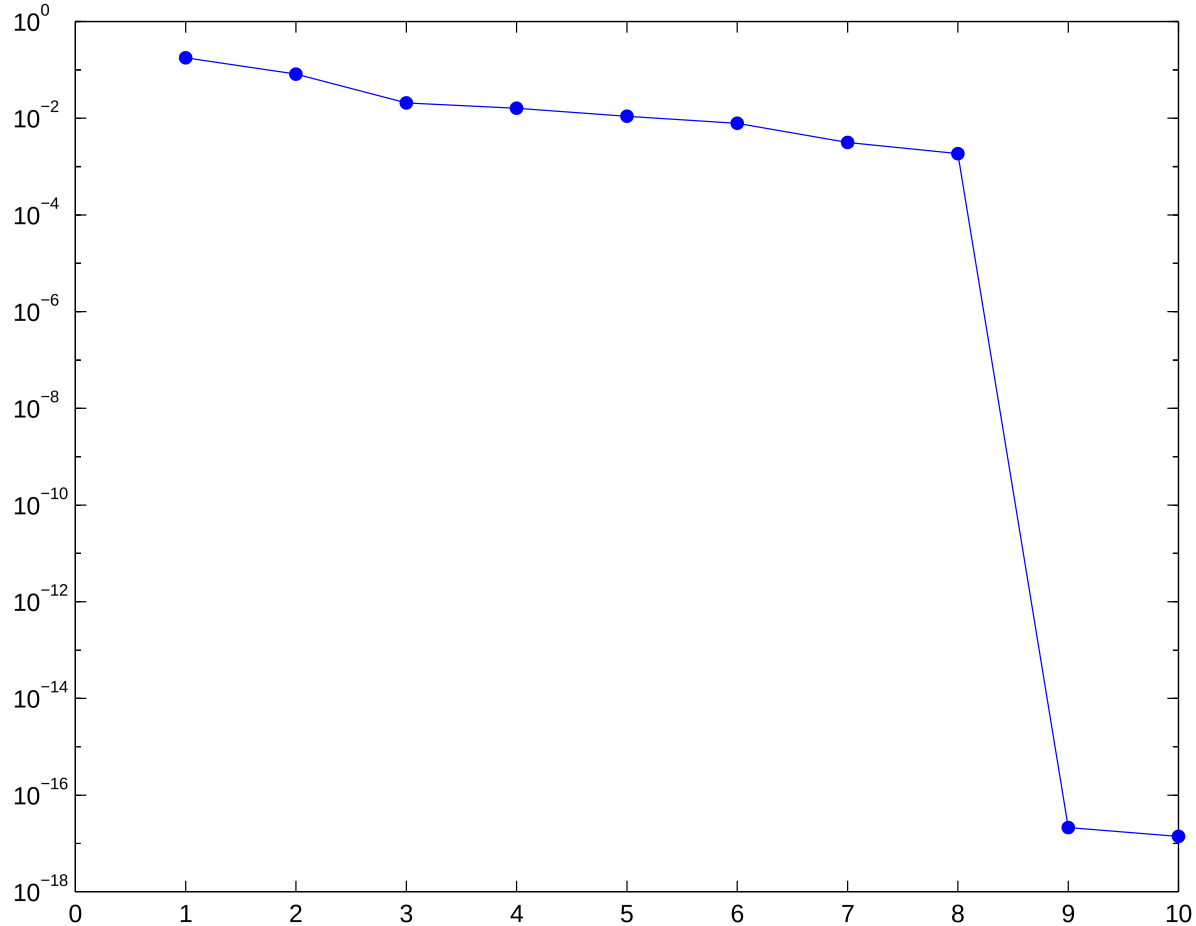

Figure 10: Eigenvalues of the smooth solution (semilog).

4 Conclusion

- Semidefinite programming provides a fast, reliable calibration method for the LMM model.

- The improvement in the solution's stability should reduce unnecessary hedging costs.

- The dual solution provides all the essential local sensitivity results.

- The final trade-off in the calibration problem is "low rank" vs. "stability".

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