Market Power, Coordination Failures and Endogenous Fluctuations*

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December 1994

Abstract

We analyse an overlapping generations economy with Cournotian monopolistic competition in the produced goods markets and perfect competition in the labour market. All prices are perfectly flexible and no adjustments costs are introduced. We show that these features lead to properties qualitatively different from those obtained when all markets are perfectly competitive. First there is a possibility of multiple Pareto-ranked temporary or stationary equilibria. Second, a preliminary analysis of the intertemporal equilibria shows that imperfect competition opens up new possibilities of endogenous fluctuations, and may lead to non-monotonicity of all equilibria in prices and output. These results are essentially based on two features, naturally associated with imperfect competition, increasing returns to scale and the variability of oligopolistic market power as a consequence of the variability of demand elasticity.

1 Introduction

Imperfect competition has become one of the main ingredients of the so-called "New-Keynesian Economics"[1]. Much recent work has been emphasizing Keynesian features of imperfectly competitive macroeconomic models[2], belonging either to the Cournotian tradition, as in Hart (1982), or to one of the monopolistic competition brands, corresponding to the Chamberlinian framework of Dixit and Stiglitz (1977) (Weitzman (1985), Blanchard and Kiyotaki (1987)), or to Hotelling's spatial model (Weitzman (1982)). A somewhat paradoxical aspect of part of this literature is that it actually appears quite unkeynesian, not so much because Keynes himself paid little attention to imperfect competition, but essentially because some of its results, concerning in particular unemployment or the

*We wish to thank Jean-Pascal Bénassy, Huw Dixon and Henri Sneessens for helpful comments, and the Banco de Portugal for its hospitality. Support from the European Commission HCM program is also aknowledged.

This text presents research results of the Belgian programme on Interuniversity Poles of Attraction initiated by the Belgian State, Prime Minister's Office, Science Policy Programming.

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[1]See, for example, Gordon (1990) or Mankiw and Romer (1991).

[2]See the recent surveys of Dixon and Rankin (1991) and Silvestre (1993), (1994).

response to aggregate demand shocks, depend either on union power[3], or on small adjustment costs ("menu costs")[4] and "nearly rational" behaviour of price-setting agents[5], and thus in a sense come back to "classical" explanations, in terms of wage and price rigidities or frictional phenomena.

We do not want to deny the relevance of such explanations. We only propose to explore in this paper some consequences of firms' monopoly power, in a context of full rationality, complete price flexibility, without adjustment costs and in the absence of exogenous shocks. Since we conform to the Keynesian priority given to the product market relatively to the labour market, and since it is the output market power that we want to emphasize, we shall (unrealistically) assume perfect competition in a homogeneous labour market. And, as an attempt to unify the theoretical framework, facilitating comparisons, we shall use a model of "Cournotian monopolistic competition"[6], which generalizes and extends both the Cournotian and the Chamberlinian approaches to imperfect competition. The model involves several goods, each produced by several firms, and includes pure Cournot oligopoly and pure monopolistic competition as particular cases. When making a decision, the producer of a good is supposed to take as given the total quantity of that good produced by other firms and the prices of the other goods, while knowing the "objective" demand, that is, acting on the basis of rational conjectures about consumers' behaviour.

The model further extends early imperfectly competitive models by assuming overlapping generations of consumers. This is of course a convenient step towards the dynamic analysis of intertemporal equilibria, involving in particular the study of the conditions in which endogenous fluctuations occur. Besides, coexistence of two generations of young and old consumers allows to incorporate not only individual income and substitution effects on the demand elasticity, hence on the firms' market power (and the case where the former dominates the latter, the complementarity case, will turn out to be interesting), but also redistributional effects, between the young and the old. Such effects may account for a significant variability of such market power, induced by changes in demand elasticity as the output price varies.

It is this variability, more than the existence of a strong market power in itself, that drives interesting effects[7]. Indeed, alone or in conjunction with decreasing marginal cost[8] (compatible with equilibrium because of imperfect competition), markup variability may be responsible for a reversal of the slope of the adequately defined "oligopolistic labour demand" curve. This reversal, by having that curve cut the labour supply curve more than once, can naturally lead to a multiplicity of temporary equilibria in this model where, on the contrary, a unique equilibrium would have resulted from non-increasing returns together with weak markup variability (as under nearly competitive conditions or under substitutability). Multiple temporary equilibria, corresponding to different employment

[3]This is already the case with Hart (1982), where the presence of an oligopolistic labour market is a necessary condition for the occurrence of unemployment. But the same kind of remark applies for instance to the more recent work of Jacobsen and Schultz (1991) or Schultz (1992), transposing Hart's framework to overlapping generations models.

[4]Examples are Blanchard and Kiyotaki (1987) and Ball and Romer (1991).

[5]See Akerlof and Yellen (1995).

[6]For a presentation of this concept, see d'Aspremont, Dos Santos Ferreira and Gérard-Varet (1991(b)).

[7]The importance of changes in markups, induced in particular by changes in aggregate demand, as a source of endogenous fluctuations and other macroeconomic phenomena, has been stressed by Rotemberg and Woodford (1991), (1992) and (1993). See also Portier (1994).

[8]We do not rely on fixed costs to incorporate increasing returns in the model, as much of the related literature (of which Weitzman (1982), Blanchard and Kiyotaki (1987), Rotemberg and Woodford (1992) are but a few examples). Neither do we consider free entry equilibria, as this literature in general does.

levels, may even be Pareto-ranked, giving rise to a situation of "coordination failure"[9] which cannot occur under perfect competition (as we well know from the first fundamental theorem of welfare economics).

The same factors, increasing returns and variability of market power, that account for a multiplicity of temporary equilibria, also favour, through their influence on the form and movements of the oligopolistic labour demand curve, emergence of endogenous fluctuations. Given enough complementarity, such fluctuations can occur in a perfectly competitive world[10]. But we will show that if, in an extreme form of market power variability, marginal revenue becomes negative[11] when prices are not expected to change much in the next period (a condition which excludes the existence of a stationary equilibrium), and if returns are increasing (a condition that further excludes in our model the existence of a monotonic equilibrium), then *all* equilibria are characterized by persistent endogenous fluctuations in prices and output.

The paper is organized as follows. In Section 2 we describe the model of Cournotian monopolistic competition and define the equilibrium concepts that we will use. Section 3 exhibits a multiplicity of Pareto-ranked (temporary or stationary) equilibria, under increasing returns or under significantly variable market power. More complex intertemporal equilibria, generating endogenous fluctuations, are analysed in section 4. Then we conclude.

2 The model

We use an overlapping generations model in discrete time. In each period $t = 0, 1, 2, \ldots$, there are $m + 2$ commodities: m produced goods ($m \geq 2$), labour and money, which is both the unit of account and the only storable good. The economy is endowed with a constant stock of money M.

We consider two types of agents: consumers and firms. Each consumer lives for two periods, works and earns income when young, but consumes the produced goods both when old and when young. At each period and for every good $k = 1, \ldots, m$, there are n firms producing good k. The labour market is assumed competitive. However, in the produced good markets we assume *Cournotian monopolistic competition*, in the sense that each producer of good k maximises his profit, facing the "true" demand for good k, net of the total quantity produced by the others and given the prices of the other produced goods[12]. In this section we shall first derive the demand functions from consumers' behaviour, then present the firm's problem and introduce both a temporary and

[9] See, for instance, Cooper and John (1988). Multiplicity of equilibria related to demand conditions has already been considered by Heller (1986). Multiplicity due to increasing returns to scale of production or market participation is present, for example, in Kiyotaki (1988), Chatterjee and Cooper (1989), and Manning (1990).

[10] See Grandmont (1985) and Guesnerie and Woodford (1992).

[11] The influence on unemployment of a negative marginal revenue has been analysed, in the context of different models, by d'Aspremont, Dos Santos Ferreira and Gérard-Varet (1984), (1989(a)), (1989(b)) and (1990), Dehez (1985), and Silvestre (1990). The main results concern "cooperation", not "coordination" problems, if we adopt the classification introduced by Silvestre (1993). Both problems appear however in d'Aspremont, Dos Santos Ferreira and Gérard-Varet (1990)). The corresponding results in an overlapping generations model, which may be associated with endogenous fluctuations, are given in d'Aspremont, Dos Santos Ferreira and Gérard-Varet (1991(a)).

[12] See d'Aspremont, Dos Santos Ferreira and Gérard-Varet (1991b) and (1992). An alternative assumption, leading to the so-called *Cournot-Walras equilibrium* (Gabszewicz and Vial (1972)), would make the producer conjecture the quantities produced by all other firms, whether in the same sector or not, implying that the whole system of sectoral demand functions is used to compute the inverse demand function.

an intertemporal concept of Cournotian monopolistic competition equilibrium. Finally we compare our model to others in the literature.

2.1 Consumers' behaviour and demand functions

In each period $t = -1, 0, 1, \ldots$, a continuum of consumers of mass L is born. Each individual lives for two periods and receives, while young, an income (profit or wage). Preferences of young consumers, identical for all individuals, are homothetic and separable, represented by the function

$$U[u(c), u(\hat{c})] - vl, \tag{1}$$

where $c \in \mathbb{R}^m_+$ and $\hat{c} \in \mathbb{R}^m_+$ denote respectively present and future consumption baskets, $l \in \{0, 1\}$ is an (indivisible) labour supply and v is the (non-negative) disutility of work. The function U is twice-differentiable, strongly quasi-concave, increasing, homogeneous of degree one, and such that the indifference curves do not cut the axes. The function u is a symmetric CES function. We assume that the preferences of old consumers are represented by this same function u.

Consumers know and take as given the current prices $p \in \mathbb{R}^m_{++}$ and, when young and hence in the labour market, the current wage $w \in \mathbb{R}_+$. Also, young consumers are assumed to have identical rigid point expectations about future prices $\hat{p} \in \mathbb{R}^m_{++}$.

Separability of preferences entails the following convenient procedure for utility maximization of the young consumer:

1. Optimal consumption baskets are first chosen for each period, assuming given expenditure levels. Let

$$u(c) = m \left(\frac{1}{m} \sum_{k=1}^{m} c_k^{(s-1)/s} \right)^{s/(s-1)} \equiv C, \tag{2}$$

where $s \in \mathbb{R}_{++} \setminus \{1\}$ is the *elasticity of intersectoral substitution*[13]. Notice that the C.E.S. function u can be seen as defining an index for total consumption C, since the expression for C/m in (2) is just a mean of the quantities in the consumption basket. If the *mean price* is defined by the formula

$$P \equiv P(p) \equiv \left(\frac{1}{m} \sum_{k=1}^{m} p_k^{1-s} \right)^{1/(1-s)}, \tag{3}$$

then c^*, the k-th element of the optimal consumption basket for an expenditure x in the first period of consumer's life, can be expressed as

$$c_k^* = \frac{1}{m} \left(\frac{p_k}{P} \right)^{-s} \frac{x}{P}, \tag{4}$$

so that $\sum_k p_k c_k^* = PC = x$. Clearly, similar expressions hold for the optimal consumption of an old consumer as well as for $\hat{c}_k^*$, the future consumption of a young consumer (as a function of the future price $\hat{p}_k$ and the future mean price $\widehat{P}$).

2. An optimal intertemporal consumption decision is then taken, given a total income i. By homogeneity of the function U, we get

$$C^* = h(P/\widehat{P})i/\widehat{P} \quad \text{and} \quad \hat{C}^* = [1 - (P/\widehat{P})h(P/\widehat{P})]i/\widehat{P}, \tag{5}$$

[13] The Cobb-Douglas case ($s = 1$) is discussed later in subsection II.D.1.

where h is a twice-differentiable, decreasing function. We shall henceforward denote:

$$\theta \equiv P/\widehat{P} \quad \text{the } \textit{real interest factor}, \text{ and}$$
$$\alpha(\theta) \equiv \theta h(\theta) \quad \text{the } \textit{propensity to consume}.$$

Notice that $0 < \alpha(\theta) < 1$, since the indifference curves of U do not cut the axes. The optimal intertemporal consumption structure

$$\frac{C^*}{\hat{C}^*} = \frac{h(\theta)}{1-\theta h(\theta)} \equiv \frac{\alpha(\theta)}{\theta}\frac{1}{1-\alpha(\theta)} \equiv H(\theta) \tag{6}$$

is a twice-differentiable, decreasing function of the real interest factor. We shall denote:

$$\sigma(\theta) \equiv -H'(\theta)\theta/H(\theta) \quad \text{the } \textit{elasticity of intertemporal substitution}.$$

A simple calculation shows that $\sigma(\theta) = 1 - (\alpha'(\theta)\theta)/(\alpha(\theta)[1-\alpha(\theta)])$, so that α is increasing (*resp.* decreasing) in the case of intertemporal complementarity, *i.e.* if $\sigma < 1$ (*resp.* substitutability, *i.e.* if $\sigma > 1$). The utility derived from optimal consumption

$$U(C^*, \hat{C}^*) = U(\alpha(\theta), \theta[1-\alpha(\theta)])i/P \equiv U^*(\theta)i/P \tag{7}$$

is linear in the real income i/P and increasing in the real interest factor θ. Indeed, using the first order condition $U_1' = U_2'\theta$, and Euler's identity $U = U_1'\alpha + U_2'\theta(1-\alpha)$, we have:

$$U^{*\prime}\theta = (U_1' - U_2'\theta)\alpha'\theta + U_2'\theta(1-\alpha) = U^*(1-\alpha) > 0. \tag{8}$$

3. Finally, the consumer chooses to work (*resp.* not to work) if $U^*(\theta)w/P - v > 0$ (*resp.* < 0). Indifference prevails in the case of an equality. Thus,

$$\frac{w}{P} = \frac{v}{U^*(\theta)} \tag{9}$$

is the *real reservation wage*, equal to the ratio of labour disutility to utility of optimal consumption.

Since demand is the same linear function of income for all young consumers, income distribution is immaterial (notice that we do not assume that shares of firms are uniformly distributed), so that aggregate demand by the young is a linear function of aggregate income I. In the same way, the distribution of cash balances among the old is also immaterial, and their aggregate demand is a linear function of the money stock M. More precisely, first notice that the aggregate expenditure X is given by the sum of the money stock, which is completely spent by the old, and of the aggregate income multiplied by the propensity to consume, corresponding to what the young spend:

$$X = \alpha(P/\widehat{P})I + M. \tag{10}$$

Thus, as $I = X$ in equilibrium, we obtain the following expression for the *aggregate demand*:

$$\frac{X}{P} = \frac{1}{1-\alpha(P/\widehat{P})}\frac{M}{P} \equiv D(P, \widehat{P}), \tag{11}$$

which is the familiar product of the *Keynesian multiplier* and the autonomous demand. Since demand has the same structure for all consumers, whether young or old, given by eq. (4), we then get the *sectoral demand* for good k:

$$d(p_k, P, \widehat{P}) = \frac{1}{m}\left(\frac{p_k}{P}\right)^{-s} D(P, \widehat{P}). \tag{12}$$

By symmetry, the function d is the same for all produced goods, so that an index k is not required.

If we now refer to the elasticity of aggregate demand $D(\cdot, \widehat{P})$:

$$-\frac{\partial D}{\partial P}\frac{P}{D} = \alpha\left(\frac{P}{\widehat{P}}\right)\sigma\left(\frac{P}{\widehat{P}}\right) + 1 - \alpha\left(\frac{P}{\widehat{P}}\right) \equiv \Delta\left(\frac{P}{\widehat{P}}\right), \tag{13}$$

and using:

$$\frac{\partial P}{\partial p_k}\frac{p_k}{P} = \frac{1}{m}\left(\frac{p_k}{P}\right)^{1-s}, \tag{14}$$

we can easily compute the elasticity of sectoral demand for good k:

$$\begin{gathered} -\left(\frac{d}{dp_k} d(p_k, P, \widehat{P})\right)\left(\frac{p_k}{d(p_k, P, \widehat{P})}\right) = \\ \left[1 - \frac{1}{m}\left(\frac{p_k}{P}\right)^{1-s}\right] s + \frac{1}{m}\left(\frac{p_k}{P}\right)^{1-s} \Delta\left(\frac{P}{\widehat{P}}\right) \equiv \delta\left(\frac{p_k}{P}, \frac{P}{\widehat{P}}\right). \end{gathered} \tag{15}$$

Hence, the sectoral demand elasticity δ appears as a weighted average of the elasticity of intersectoral substitution s and of the aggregate demand elasticity Δ, itself a weighted average of the elasticity of intertemporal substitution σ and of unity (corresponding to the spending behaviour of the young and the old, respectively). It is necessarily positive.

2.2 Producers' behaviour

Every good $k = 1, \ldots, m$ is produced by the same number n of firms ($n \geq 1$), with identical increasing, but not necessarily convex, isoelastic cost functions: y^β (with $\beta > 0$), giving the labour quantity required by output y.

Each producer j of good k takes as given the wage w in the labour market, while acting as a Cournot oligopolist in the market for the produced good k, on the basis of the "true" residual demand for his output, contingent on the supply of the other firms in the same sector: $d(p_k, P(p_k, p_{-k}), \widehat{P}) - Y^k_{-j}$, with $Y^k_{-j} \equiv \sum_{i \neq j} y^k_i$. We assume that producer j knows the common mean price expectation $\widehat{P}$, and conjectures (correctly at equilibrium) the vector p_{-k} of other commodity prices and the output Y^k_{-j} of good k produced by the other firms. These conjectures are essential for our notion of *Cournotian monopolistic competition*. The function $d(., P(., p_{-k}), \widehat{P})$ being strictly decreasing in p_k, we may introduce the inverse demand function ψ for the output of firm j, which is:

$$p_k = \psi(y_j + Y^k_{-j}, p_{-k}, \widehat{P}) \iff y_j = d(p_k, P(p_k, p_{-k}), \widehat{P}) - Y^k_{-j}. \tag{16}$$

Then, for any given $\overline{Y}, \bar{p}, \widehat{P}$ and w, the typical "producer's problem" is the following profit maximization problem:

$$\max_{y \in \mathbb{R}_+} \pi(y) \equiv \max_{y \in \mathbb{R}_+} \psi(y + \overline{Y}, \bar{p}, \widehat{P})y - wy^\beta \tag{17}$$

(for simplicity of notation, we omit the indices j and k). An optimal positive output y must satisfy the first order condition (with $\psi(y)$ standing for $\psi(y+\overline{Y},\bar{p},\widehat{P})$):

$$\psi(y)\left[1+\frac{\psi'(y)y}{\psi(y)}\right]=w\beta y^{\beta-1}, \tag{18}$$

which is the familiar equality of marginal revenue and marginal cost. It must also entail a non-negative profit: $\psi(y)y \geq wy^{\beta}$ (otherwise it would be dominated by a zero output), so that (using eq. (18)):

$$\epsilon(y) \equiv -\frac{\psi'(y)y}{\psi(y)} \geq 1-\beta. \tag{19}$$

Lemma 1 and Lemma 2 in the Appendix introduce restrictions (on the function ϵ and on the elasticity δ of sectoral demand, respectively), ensuring that the ratio of marginal revenue to marginal cost is decreasing *whenever* condition (19) is verified and the marginal revenue is non-negative. Then conditions (18) and (19) are sufficient for a *global* maximum of the profit function (unique in the positive domain), although that function may fail to be quasi-concave. Based on these lemmata, the following propositions give simple assumptions on s and σ implying that the first order condition together with the non-negative profit condition are necessary and sufficient for profit maximization at a positive level of output. We stick to two typical cases: overall (both intertemporal and intersectoral) complementarity and overall substitutability.

Proposition 1 (Complementarity)**.** *Assume intersectoral complementarity: $s \leq 1$. Also, assume that the elasticity of intertemporal substitution σ satisfies, for any $\theta \in \mathbb{R}_{++}$,*

$$\sigma(\theta)<1 \quad \textit{and} \quad \frac{\sigma'(\theta)\theta}{\sigma(\theta)} \geq -[1-\sigma(\theta)] \tag{20}$$

or, equivalently, that the intertemporal expenditure structure $\theta H(\theta)$ is an increasing and concave function of θ. Then, the first order and non-negative value condition:

$$p\beta \geq p\left[1-\frac{y/(y+\overline{Y})}{\delta(p/P(p,\bar{p}),P(p,\bar{p})/\widehat{P})}\right]=w\beta y^{\beta-1}, \tag{21}$$

where $p=\psi(y+\overline{Y},\bar{p},\widehat{P})$ as defined in (16), is necessary and sufficient for y to be the unique positive solution to the producer's problem (17).

Proof. See Appendix.

Proposition 2 (Substitutability)**.** *Assume intersectoral substitutability:*

$s \geq 1$. Also, assume that the elasticity of intertemporal substitution σ satisfies, for any $\theta \in \mathbb{R}_{++}$,

$$1 \leq \sigma(\theta) \leq \min\{s, 1+\beta\} \quad \text{and} \quad \sigma'(\theta) \geq 0. \tag{22}$$

Then, the first order and non-negative value condition (21) is necessary and sufficient for y to be the unique positive solution to the producer's problem (17) ($\overline{Y}>0$, and hence $n>1$, is required in condition (21) if $s=1$).

Proof. See Appendix.

2.3 Temporary and intertemporal Cournotian monopolistic competition equilibria

Two concepts of equilibrium are successively considered: temporary and intertemporal equilibrium. Temporary equilibrium imposes compatibility of optimal consumers' and producers' decisions in the particular period which is considered, but price expectations remain exogenous and are arbitrarily given:

Definition 1 (Temporary equilibrium)**.** *A temporary equilibrium associated with the expected mean price $\widehat{P}$ is a vector with components $y^*_{kj} \geq 0$ (the outputs chosen by each firm $j = 1, \ldots, n$ in each sector $k = 1, \ldots, m$), $p^*_k > 0$ (the product prices, with $k = 1, \ldots, m$), and $w^* \geq 0$ (the wage), such that:*

1. *for any firm j in any sector k,*

$$y^*_{kj} \in \arg\max_{y \in \mathbb{R}_+} \psi \left(y + \sum_{i \neq j} y^*_{ki}, p^*_{-k}, \widehat{P} \right) y - w^* y^\beta,$$

   with the sectoral inverse demand function ψ as defined in (16),

2. *for any sector k, $\sum_j y^*_{kj} = d(p^*_k, P^*, \widehat{P})$, with $P^* = P(p^*_k, p^*_{-k})$,*

3. *$\sum_k \sum_j y^{*\beta}_{kj} = L$ and $w^*/P^* \geq v/U^*(P^*/\widehat{P})$ (**full employment**) or $\sum_k \sum_j y^{*\beta}_{kj} < L$ and $w^*/P^* = v/U^*(P^*/\widehat{P})$ (**underemployment**).*

The first and second conditions characterize a *Cournotian monopolistic competition* equilibrium in the product markets: every producer maximizes his profit, knowing the "true" consumers' demand function and their expectation of the mean price in the next period, acting as a Cournot oligopolist in his own market, and taking as given the wage and the other products prices. The third condition defines the competitive equilibrium of the labour market by the equality of supply and demand, but taking into account the possibility that only part of the labour force is employed. In this case, since we assume perfect wage flexibility and no rationing, the wage must be equal to its reservation value.

It can be checked that all equilibria are necessarily symmetric in quantities, inside each sector, relatively to active firms (see Lemma 3 in the Appendix). But the possibility that only a number $n^* < n$ of firms be active in equilibrium cannot be excluded when returns are increasing ($\beta < 1$). Also, symmetry is not always warranted in prices, and correspondingly in quantities across sectors, unless we strengthen the assumptions hitherto formulated[14]. For simplicity we shall however simply ignore asymmetric equilibria, and the kind of multiplicity they may introduce.

It will be appropriate to refer to a *symmetric temporary equilibrium* by a triplet (y^*, p^*, w^*) characterized, under the assumptions of either Proposition 1 or Proposition 2, by:

1. the first order and non-negative value condition:

$$p^* \beta \geq p^* \left[1 - \frac{1/n}{\delta(1, p^*/\widehat{P})} \right] = w^* \beta y^{*\beta - 1}, \tag{23}$$

   with δ as given by (15) and (13),

[14] By imposing, for instance, that, for any $\theta \in \mathbb{R}_{++}$, $s \leq \sigma(\theta) < 1$ (in the complementarity case), or $1 \leq \sigma(\theta) < s$ and $(1 - \beta)s < 1$ (in the substitutability case).

2. the product market equilibrium condition:

$$mny^* = \frac{1}{1-\alpha(p^*/\widehat{P})}\frac{M}{p^*}, \tag{24}$$

3. and the labour market equilibrium condition:

$$\begin{aligned} & mny^{*\beta} = L \quad \text{and} \quad w^*/p^* \geq v/U^*(p^*/\widehat{P}) \quad (\textbf{full employment}) \\ \text{or} \quad & mny^{*\beta} < L \quad \text{and} \quad w^*/p^* = v/U^*(p^*/\widehat{P}) \quad (\textbf{underemployment}). \end{aligned} \tag{25}$$

The characterization of symmetric temporary equilibria can be further simplified by referring to just the equilibrium price p^*, which may be determined using workers' and firms' reservation wages. Let us first define the *symmetric marginal revenue function*

$$\rho(p;\widehat{P}) \equiv p\left[1 - \frac{1/n}{\delta(1,p/\widehat{P})}\right], \tag{26}$$

such that $1-\rho(p;\widehat{P})/p = 1/n\delta(1,p/\widehat{P})$ is Lerner's *index of monopoly power*. Then the *firms' reservation wage*[15] is constructed from the first order condition (23) and is equal to the product of the marginal productivity by the marginal revenue. In a symmetric product market equilibrium, using condition (24), this is simply the following function of the price p (conditional on a given price expectation $\widehat{P}$):

$$\omega_f(p;\widehat{P}) = \frac{1}{\beta}\left(\frac{D(p,\widehat{P})}{mn}\right)^{1-\beta}\rho(p;\widehat{P}) \tag{27}$$

The function ω_f has elasticity:

$$\frac{\omega_f'}{\omega_f}p = (\beta-1)\Delta(p/\widehat{P}) + \frac{\rho'(p;\widehat{P})}{\rho(p;\widehat{P})}p, \tag{28}$$

Similarly the *workers'* (nominal) *reservation wage* is constructed from (25) as a function of the same variables:

$$\omega_l(p;\widehat{P}) = \frac{v}{U^*(p/\widehat{P})}p, \tag{29}$$

with elasticity (for $v>0$, and using (8)):

$$\frac{\omega_l'}{\omega_l}p = \alpha(p/\widehat{P}). \tag{30}$$

The following proposition summarizes all the simplifications coming from symmetry:

Proposition 3 (Symmetric temporary equilibria)**.** *An **underemployment symmetric equilibrium** is determined by equalizing both reservation wages, and verifying that a solution p^* to the equation $\omega_f(p;\widehat{P}) = \omega_l(p;\widehat{P})$ satisfies:*

$$L > (mn)^{1-\beta}D(p^*,\widehat{P})^\beta \quad \textit{and} \quad \beta p^* \geq \rho(p^*;\widehat{P}). \tag{31}$$

*A **full employment symmetric equilibrium** is given by the solution p^{**} to the equation $L = (mn)^{1-\beta}D(p^{**},\widehat{P})^\beta$, whenever this solution satisfies:*

$$\omega_f(p^{**};\widehat{P}) \geq \omega_l(p^{**};\widehat{P}) \quad \textit{and} \quad \beta p^{**} \geq \rho(p^{**};\widehat{P}). \tag{32}$$

[15] This is an avowedly improper term, meaning in fact the wage that sustains p^* as an output market equilibrium. It appears however that transactions in the labour market proceed, as we should expect, until the firms' reservation wage (the buyers' reservation price) becomes smaller than the workers' reservation wage (the sellers' reservation price), or until labour endowment is exhausted (see Proposition 3).

We finally give the definition of an intertemporal equilibrium which, in addition to the compatibility of optimal consumers' and producers' decisions in every period, imposes the fulfilment of price expectations.

Definition 2 (Intertemporal equilibrium)**.** *An intertemporal equilibrium is a sequence of temporary equilibria, one for each period $t = 0, 1, 2, \ldots$, associated with an expected mean price $\widehat{P}_t$, characterized by a mean price P_t^* and such that, for any $t > 0$, $P_t^* = \widehat{P}_{t-1}$.*

Clearly, as we restrict ourselves to symmetric equilibria, an intertemporal equilibrium is completely described by a sequence of triplets $(y_t^*, p_t^*, w_t^*)_{t\in\mathbb{N}}$. As suggested by the simplified characterization of symmetric temporary equilibrium given in Proposition 3, a symmetric intertemporal equilibrium can also be simply identified to a sequence of positive prices $(p_t^*)_{t\in\mathbb{N}}$.

2.4 Comparison with the literature

It is useful at this stage to compare our model, and its associate temporary equilibrium concept, with the two main classes of imperfectly competitive macroeconomic models: the one introduced by Hart (1982) and relying on a Cournotian approach, and the one following the Dixit and Stiglitz (1977) Chamberlinian framework, and popularized by Weitzman (1985) and Blanchard and Kiyotaki (1987)[16]. Some important differences immediately appear. Contrary to ours, these models attach importance to labour market imperfections[17], keep intertemporal decisions implicit[18], and (with the exception of Weitzman (1985)) neglect the so-called Ford effects[19] that lead producers to refer to a demand function incorporating a multiplier, such as the one given by (11). Also they do not exhibit an overlapping generations structure, and hence ignore in particular the related distributional effects on demand[20]. It is however easy to capture the meaningful features of these two classes of models, at least from the point of view of product markets, by associating them with two special cases of our own model, assuming a unitary elasticity of, respectively, intersectoral and intertemporal substitution.

2.4.1 Unitary elasticity of intersectoral substitution or pure Cournot oligopoly

Hart (1982) uses a single produced good and assumes that consumers only spend in one sector. The significant point is however that cross price effects do not play any role in his

[16] There are other early and independent references in the second stream, for instance Mankiw (1985) or Svensson (1986).

[17] However, Mankiw (1985) considers monopolistic behaviour in the output markets, with a competitive labour market. Subsequent models of Ball and Romer (1990) or Dixon (1987) also stress, at least in a first stage, product markets imperfections (with suppliers setting prices of differentiated goods, or playing Cournot on the basis of conjectural variations, respectively).

[18] A temporary monetary equilibrium framework with explicit intertemporal decisions and expectations formation is used, in the context of Hart (1982) model, by d'Aspremont, Dos Santos Ferreira and Gérard-Varet (1984), (1989(a)) and (1989(b)) and Rankin (1992) and (1994) and, in the line of Weitzman (1985) model, by d'Aspremont, Dos Santos Ferreira and Gérard-Varet (1990).

[19] See d'Aspremont, Dos Santos Ferreira and Gérard-Varet (1989(a)), (1989(b)), (1990) and (1991(a)). Lack of separability or homotheticity in the consumers' preferences, just as inexistence of a representative consumer, should not prevent us from taking Ford effects into account, even if the construction of an objective demand appears more complex in that case: see for instance Bénassy (1987).

[20] However, some recent macroeconomic overlapping generations models introduce imperfect competition in output markets, whether in a Cournotian approach, as Schultz (1992), or under monopolistic competition, as Bénassy (1991(a)) and (1991(b)), or both, as d'Aspremont, Dos Santos Ferreira and Gérard-Varet (1991(a)).

model. Such is also the case when consumers spend a constant sum in each sector, as a result of a Cobb-Douglas intersectoral subutility function u, with unit elasticity. Indeed, as one can easily check from (15), with $s = 1$ the sectoral demand elasticity has the real interest factor $\theta = P/\widehat{P}$ as its sole argument, and becomes independent of the relative price p/P:

$$\delta = 1 - \frac{1}{m} + \frac{1}{m}\Delta(\theta) = 1 + \frac{1}{m}\alpha(\theta)[\sigma(\theta) - 1]. \tag{33}$$

Notice that $\delta > 1 - 1/m \geq 1/n$ (for $n > 1$, and recalling that $m \geq 2$, by assumption), so that the marginal revenue is necessarily positive, as *assumed* by Hart. As it already appears from the analysis in Heller (1986), this is a crucial restriction, making in particular less likely the emergence of multiple Pareto-ranked equilibria (as we shall see in Section III). Such restriction also excludes the cases of economies in which *all* intertemporal equilibria are inflationary and converge to the autarkical state (d'Aspremont, Dos Santos Ferreira and Gérard-Varet (1991(a))) or are non-monotonic in prices and output (Section IV.D.2).

2.4.2 Unitary elasticity of intertemporal substitution and pure monopolistic competition

Comparison with pure monopolistic competition models, in the line of Dixit and Stiglitz (1977), supposes a single firm in each sector: $n = 1$. Also, neglecting, as these authors and their followers systematically do, any indirect individual price effects (through the mean price), amounts to assume an infinite number of sectors ($m = \infty$), leading to a sectoral demand elasticity equal to the elasticity of intersectoral substitution: $\delta = s$. Indeed, we know that the mean price P is a function of p_k, the individual price of good k, given by (3). The elasticity of this function, given by (14), becomes negligible if m is very large[21], which is precisely the reason put forward by Dixit and Stiglitz for taking P as (approximately) constant when determining the demand elasticity for good k (see d'Aspremont, Dos Santos Ferreira and Gérard-Varet (1994)). If we look beyond the simplifying role of such an assumption, we can see that it leads to the important consequence that the sectoral demand elasticity is constant and larger than one (otherwise no equilibrium would exist), thus excluding again the possibility of a negative marginal revenue and, more fundamentally, ruling out any change in market power.

This limitation remains, even without resorting to an infinite number of sectors (and to a sole producer of each good), as long as we assume, with Weitzman (1985) and Blanchard and Kiyotaki (1987), but contrary to Dixit and Stiglitz (1977), that the intertemporal utility function U is restricted to be of the Cobb-Douglas type, *i.e.* that the corresponding elasticity of substitution is equal to one. Indeed, with $\sigma(\theta) \equiv 1$, the sectoral demand elasticity becomes a weighted average of s and 1:

$$\delta = \left[1 - \frac{1}{m}\left(\frac{p}{P}\right)^{1-s}\right] s + \frac{1}{m}\left(\frac{p}{P}\right)^{1-s}. \tag{34}$$

With symmetry in prices, this elasticity is equal to $(1 - 1/m)s + 1/m$, a constant, necessarily larger than $1/n$ for an equilibrium to exist. As a consequence, the symmetric marginal revenue is always positive and, more significantly, market power (or markup

[21] As already mentioned in the introduction, we do not impose the number of products to be very large, as we should in a model of pure monopolistic competition. By considering Cournotian competition instead, close substitutes may be subsumed in the same sector, and the number of sectors may accordingly be kept relatively small.

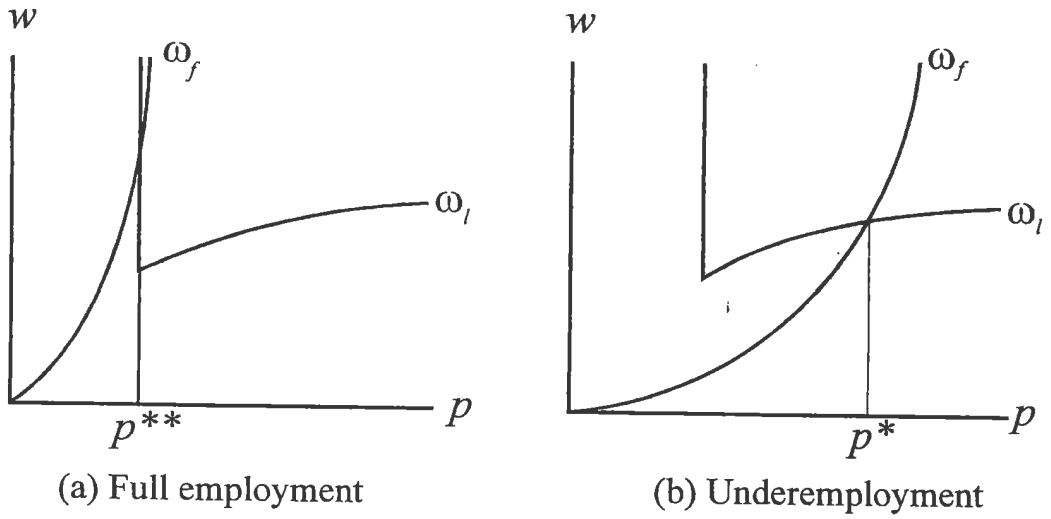


Figure 1

over marginal cost) is constant, so that both multiple temporary equilibria (Section III) and endogenous fluctuations (Section IV) are less likely to occur.

3 Multiplicity of Pareto-ranked equilibria

We first examine the static properties of our model and study how imperfect competition may contribute to the emergence of multiple Pareto-ranked equilibria. These static properties will be useful both to analyse temporary equilibria and, under stationarity, intertemporal equilibria.

3.1 Temporary equilibria

We have seen that underemployment equilibria are determined by equalizing the firms' and the workers' reservation wage functions, ω_f and ω_l respectively, and that a full employment equilibrium requires the value of ω_f at the full employment price to be at least as large as the value of ω_l. This is illustrated in Figure 1. Thus, existence of at most one equilibrium is clearly ensured if the graph of ω_f can only cut the graph of ω_l from below.

That condition is satisfied if (using (28) and (30))

$$\frac{\omega_f'}{\omega_f}p = (\beta - 1)\Delta + \frac{\rho'}{\rho}p > \alpha = \frac{\omega_l'}{\omega_l}p \tag{35}$$

for any price higher than the full employment one. In particular this is afforded by the assumption of non-increasing returns ($\beta \geq 1$), together with the restriction: $\rho' p/\rho > \alpha$, for any admissible p. Thus, multiple temporary equilibria can never arise in our model when competition is perfect (requiring both non-increasing returns and equality of price and marginal revenue, so that $\rho' p/\rho = 1 > \alpha$). Multiple equilibria are also impossible in the models of Weitzman (1985) and Blanchard and Kiyotaki (1987), once we exclude fixed costs and "menu" costs, since the demand elasticity is constant, entailing as in the pure competition case a unitary elasticity of ρ, the marginal revenue function. More

generally, non-increasing returns rule out equilibrium multiplicity in the substitutability case[22], where (as stated in Lemma 4 in the Appendix) $\rho' p/\rho$ is always larger than α:

Proposition 4 (Uniqueness)**.** *In the substitutability case, with non-increasing returns, and under the assumptions of Proposition 2, there is at most one symmetric temporary equilibrium.*

There are however cases of multiple equilibria. The proposition suggests to consider the case of increasing returns and the case of complementarity.

3.1.1 Increasing returns

That increasing returns are a possible source of multiplicity of symmetric temporary equilibria should be obvious from the preceding discussion. In order to focus on the specific role of increasing returns, let us neglect changes in market power, by resorting to the quite simple case of a Cobb-Douglas intertemporal utility function (leading to $\Delta = 1$ and $\rho' p/\rho = 1$). We assume that the (constant) propensity to consume α is larger than the elasticity of the cost function β, so that

$$\frac{\omega_f'}{\omega_f} p = \beta < \alpha = \frac{\omega_l'}{\omega_l} p. \tag{36}$$

Thus, the equality of the firms' and workers' reservation wages $\omega_f = Bp^\beta = Ap^\alpha = \omega_l$ (where A and B denote constant factors[23]) is satisfied at a unique price p^*, characterizing an underemployment equilibrium if conditions (31) are satisfied, that is:

$$\begin{gathered} p^* = \left(\frac{B}{A}\right)^{1/(\alpha-\beta)} > (mn)^{1/\beta-1} \frac{M/L^{1/\beta}}{1-\alpha} = p^{**} \\ \text{and} \quad \beta \geq 1 - \frac{m/n}{(m-1)s+1}. \end{gathered} \tag{37}$$

In fact as shown in Figure 2, when these conditions hold, there is, besides the underemployment equilibrium, a full employment equilibrium corresponding to a price p^{**} (we have $Bp^{**\beta} > Ap^{**\alpha}$, so that conditions (32) are also verified)[24]. Otherwise, except in the singular case $p^* = p^{**}$, or in the case $v = 0$ (implying $A = 0$), where the full employment equilibrium is unique, no equilibrium exists.

Multiplicity of equilibria does not in general imply that the equilibria are Pareto-ranked. An old consumer obviously prefers the lowest price, here the full employment price p^{**}. A young consumer, as a worker, is unable to choose among underemployment equilibria: in any such equilibrium, the utility he derives from working is $U^*(p/\widehat{P})w/p - v = 0$, which is the utility he gets when not working and not receiving any non-wage income. On the other hand, he definitely prefers, as a worker, a full employment equilibrium, if the wage is higher than its reservation value. The difficulty comes from his position as a

[22] Substitutability is of course not the only way to ensure uniqueness. Hart (1982) uses (together with non-increasing returns) the assumption of an increasing marginal revenue function ρ and the absence of labour disutility, so that $\omega_l' p/\omega_l = 0 < (\beta - 1)\Delta + \rho' p/\rho$.

[23] As one can easily check, the constants in the reservation wage functions, are:

$B = \frac{1}{\beta}\left[\frac{M}{(1-\alpha)mn}\right]^{1-\beta}\left[1 - \frac{m/n}{(m-1)s+1}\right]$ and $A = \frac{v\widehat{P}^{1-\alpha}}{\alpha^\alpha(1-\alpha)^{1-\alpha}}$.

[24] The graphs in Figure 2 have been plotted for the parameter values: $\alpha = .75$, $\beta = .25$, $A = 1$ (entailed for instance by $\widehat{P} = .75$ and $v \simeq .612$), $B = 1$ (corresponding for instance to $n = 1$, $m = 20$, $s = 4/3$ and $M \simeq 5.265$), and $p^{**} = .5$ (resulting, given the other parameter values, from $L \simeq 24.093$)

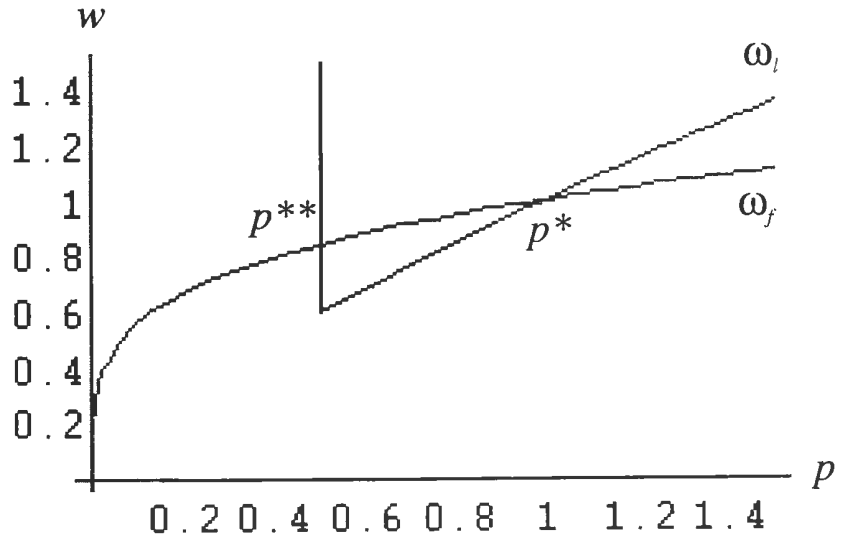


Figure 2

shareholder, receiving a given fraction of total profits Π. So, in order to assert that p^{**} Pareto-dominates p^*, one has to check whether

$$U^*\left(\frac{p^*}{\widehat{P}}\right)\frac{\Pi^*}{p^*} \leq U^*\left(\frac{p^{**}}{\widehat{P}}\right)\frac{\Pi^{**}}{p^{**}}. \tag{38}$$

Since $U^*(p/\widehat{P})/p$ is decreasing (by (29) and (30), it has elasticity $-\alpha$), this inequality holds if Π is a non-increasing function of p, which is true in this example, since $pD(p,\widehat{P}) = M/(1-\alpha)$, so that $\Pi = pD(p,\widehat{P}) - (mn)^{1-\beta}Bp^{\beta}D(p,\widehat{P})^{\beta}$ is in fact constant.

Multiple equilibria are thus Pareto-ranked in this simple example with increasing returns and constant market power, giving rise to a *coordination failure*[25].

3.1.2 Complementarity and significant variability of market power

We will now consider the complementarity case, putting aside increasing returns. For simplicity, we stick again to a straightforward class of examples[26], assuming that the symmetric marginal revenue function ρ is positive over some price range but, as the price increases, eventually becomes negative since[27]:

$$\begin{aligned}\lim_{\theta\to\infty}\delta(1,\theta) &= (1-1/m)s + (1/m)\lim_{\theta\to\infty}\sigma(\theta) < 1/n \\ &< (1-1/m)s + 1/m = \lim_{\theta\to 0}\delta(1,\theta).\end{aligned} \tag{39}$$

An example is given by a C.E.S. function U, with $\sigma < m/n - (m-1)s < 1$.

[25] By applying the same type of argument, based on an elasticity of the workers' reservation wage function larger than the elasticity of the firms' reservation wage function, at least for arbitrarily high prices, the previous result may be extended, under supplementary assumptions, to the complementarity case.

[26] We are using the same type of assumption as Heller (1986).

[27] In the two equalities, we use the fact that, for $\sigma < 1$, α increases with θ from 0 to 1, so that Δ tends to 1 when the price tends to 0, to the same limit as σ when the price tends to infinity. Only if σ converges to 1 for a price tending to zero (infinity) can the limit of α be larger than zero (smaller than 1), but Δ will still have the stated limit, 1 (the same as σ), in that case.

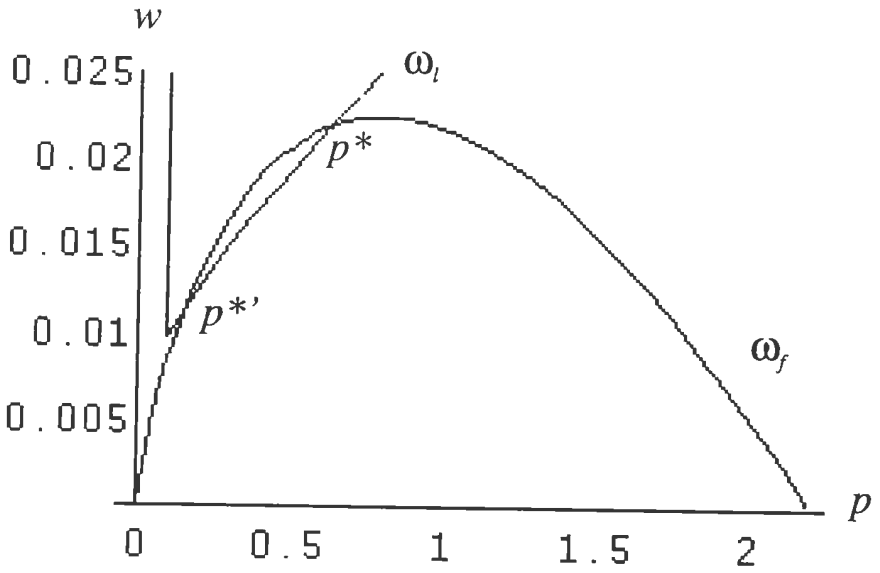


Figure 3

In this class of examples, because of intertemporal complementarity (dominance of the income effect over the substitution effect), a higher price leads to a higher expenditure by the young consumers, thus increasing the weight of the generation with the lowest demand elasticity. So market power increases with price, and the firms' reservation wage correspondingly decreases (if marginal productivity does not increase too sharply with diminishing output). Since the firms' reservation wage function eventually becomes negative, it must indeed be decreasing for high prices. But it is increasing for low prices since, by (27), it converges to zero, as the price tends to zero. As a matter of fact, except in singular cases, it is this property, and not the fact that the function eventually takes negative values, that rules out equilibrium uniqueness. As shown in Figure 3, if the marginal revenue is positive at the full employment price, there will be, for a labour disutility v sufficiently weak to ensure existence of an equilibrium, and by continuity of the reservation wage functions, at least two intersections of their graphs, and hence two equilibria. Notice that, contrary to the simple case of a unitary elasticity of intertemporal substitution with increasing returns, this case may entail, and indeed does entail in the example of Figure 3, two underemployment equilibria[28]. Also, multiplicity of equilibria is compatible in the present case with the absence of labour disutility.

Are such equilibria Pareto-ranked? From the discussion about the increasing returns case, in order to establish that lower price equilibria Pareto-dominate higher price ones, it is sufficient to show that the utility derived from the profit income $U^*\Pi/p$ is non-increasing in price in the relevant range. As the elasticity of this function is equal, by (8), to $-\alpha$ plus the elasticity of the function

$$\begin{aligned}\Pi(p;\widehat{P}) &\equiv pD(p,\widehat{P}) - \omega_f(p;\widehat{P})(mn)^{1-\beta}D(p,\widehat{P})^{\beta} \\ &= pD(p,\widehat{P})(1/\beta)[\beta - 1 + 1/n\delta(1,p/\widehat{P})],\end{aligned} \tag{40}$$

[28]The graphs in Figure 3 correspond to the case of a C.E.S. utility function with $\sigma = .5$ and a discount factor for future consumption $\gamma = .81$. Other parameter values are $s = .15$, $v \simeq .0108$, $m = 10$, $n = 5$, $\beta = 1$, $\widehat{P} = .5$ and $p^{**} = .1$ (resulting from $M/L \simeq .0668$).

this amounts to show that:

$$\frac{\partial \Pi}{\partial p}\frac{p}{\Pi} = 1 - \Delta - \frac{\delta_2' \theta/\delta}{(\beta-1)n\delta + 1} \leq \alpha. \tag{41}$$

By referring to (13) and (15), and using $\Delta'\theta \geq -\Delta(1-\Delta)$ (implied by the assumption of Proposition 1)[29], together with the fact that $n\delta > 1$ in the relevant range, we indeed obtain:

$$\begin{aligned} \frac{\partial \Pi}{\partial p}\frac{p}{\Pi} &\leq (1-\Delta)\left[1 + \frac{1}{(\beta-1)n\delta+1}\frac{\Delta}{\Delta + (m-1)s}\right] \\ &< \alpha(1-\sigma)\left[1 + \frac{1}{\beta(1+(m-1)s)}\right] \leq \alpha. \end{aligned} \tag{42}$$

The last inequality is satisfied if we assume: $\sigma(\theta) \geq 1/[1+\beta(1+(m-1)s)]$ for the relevant values of θ (those which lead to $\delta(1,\theta) > 1/n$). A simple example of existence of multiple Pareto-ranked temporary equilibria based on the C.E.S. utility function (as in the example of Figure 3) is thus obtained for values of σ such that:

$$\frac{1}{1+\beta(1+(m-1)s)} \leq \sigma < m/n - (m-1)s < 1. \tag{43}$$

3.2 Stationary equilibria

Under temporary equilibria, expectations are exogenous and arbitrary, and one may always wonder whether the multiplicity of equilibria obtained should not be traced back to the degree of freedom thus allowed to the model maker. Considering intertemporal equilibria, such degree of freedom is lost since expectations, even though rigid and devoid of cognitive foundations, are required to be fulfilled.

The study of intertemporal equilibria is often limited to stationary equilibria, allowing to remove the arbitrariness of expectations without giving up the simplicity of the static analysis. Imposing stationarity may however be misleading. Indeed, in the symmetric temporary equilibrium context, it shares with the assumption of a unitary elasticity of intertemporal substitution the property that demand elasticity, hence market power, is constant (since $\theta_t = 1$ for all t). As a result, we obtain, like in perfect competition, a unitary elasticity of the symmetric marginal revenue function. Thus, we should not be surprised to find that non-decreasing returns *per se* exclude (generically) the possibility of multiple stationary equilibria, independently of the characteristics of intertemporal substitution. By contrast, given existence, increasing returns necessarily entail such a multiplicity (except in a singular case).

The analysis of stationary equilibria is straightforward. Reservation wage functions take quite simple forms when the real interest factor is fixed at unity:

$$\omega_l(p;p) = Ap \quad \text{and} \quad \omega_f(p;p) = Bp^{\beta}, \tag{44}$$

with $A = v/U^*(1)$ and $B = [1 - 1/n\delta(1,1)](1/\beta)[M/mn(1-\alpha(1))]^{1-\beta}$. If $A > 0$ and $B > 0$, and except in the case of constant returns ($\beta = 1$), there is a unique price, say p^*, at which the two reservation wages are equal. Under decreasing returns ($\beta > 1$), if this price is larger than the full employment price p^{**}, it sustains an underemployment stationary equilibrium. Otherwise the only stationary equilibrium has full employment. Under increasing returns ($\beta < 1$), both p^* and p^{**} are stationary equilibrium prices if $p^* \geq p^{**}$ and if the non-negative profit condition is satisfied. Otherwise no stationary

[29] See (84) in the Appendix (proof of Proposition 1).

equilibrium exists. Equality of p^* and p^{**} corresponds to a degree $\bar{v}$ of labour disutility satisfying:

$$\bar{v} \equiv U^*(1)Bp^{**\beta-1} = U^*(1)[1 - 1/n\delta(1,1)](1/\beta)(L/mn)^{1/\beta-1}. \tag{45}$$

We can use $\bar{v}$ to summarize the preceding remarks in a proposition.

Proposition 5 (Stationary equilibria)**.** *Under the assumptions of either Proposition 1 or Proposition 2, and referring to $\bar{v}$ as defined in (45),*

- *if returns are* decreasing *($\beta > 1$), there exists a* unique *stationary equilibrium, with full employment if $0 \leq v \leq \bar{v}$ (one inequality strict), or with underemployment if $0 < \bar{v} < v$; in the* singular *case in which $v = \bar{v} = 0$, there exist multiple stationary equilibria, one with full employment and a continuum with underemployment, corresponding to any price lower than the full employment price; and, if $\bar{v} \leq 0 < v$, no stationary equilibrium exists;*

- *if returns are* constant *($\beta = 1$), there exists a* unique *stationary equilibrium with full employment if $v < \bar{v}$; there exist multiple stationary equilibria, one with full employment and a continuum with underemployment, corresponding to any price lower than the full employment price, in the* singular *case in which $v = \bar{v}$; and no stationary equilibrium exists if $\bar{v} < v$;*

- *if returns are* increasing *($\beta < 1$) and if $\beta \geq 1 - 1/n\delta(1,1)$, there exist* two *stationary equilibria, one of each type, if $0 < v < \bar{v}$, and only one stationary full employment equilibrium in the* singular *cases in which $0 < v = \bar{v}$ or $0 = v < \bar{v}$; there exist multiple stationary equilibria, one with full employment and a continuum with underemployment, corresponding to any price lower than the full employment price, in the* singular *case in which $0 = v = \bar{v}$; and no stationary equilibrium exists if $\bar{v} < v$ or if $\beta < 1 - 1/n\delta(1,1)$.*

Are multiple stationary equilibria Pareto-ranked? Because of stationarity and the associated constancy of both expenditure and market power, the answer is based on the same argument as in the context of temporary equilibria when the elasticity of intertemporal substitution is equal to one. Lower price equilibria are obviously preferred by all generations when old, and are at least no less desired by young consumers as wage earners (the utility derived from their worker's status is zero in any underemployment equilibrium and may be greater at full employment) and also as profit earners (with a constant real interest factor, the utility derived from the shareholder's status is proportional to real profits, and nominal profits are constant in this case).

4 Endogenous fluctuations

By its very simplicity, the study of stationary equilibria is both appealing and delusive. Indeed, stationary equilibria are but a special type of intertemporal equilibria, and should be viewed as such in a genuine dynamic analysis. Non-stationary equilibria may be quite complex, and it is in particular possible to find, along an equilibrium sequence, both under- and full-employment phases. For the sake of simplicity, we shall however emphasize the two "pure" dynamic regimes, corresponding to the full employment and the underemployment regimes. These two regimes will be successively examined in the next subsection. Then, in the following subsection, monotonicity of all equilibrium trajectories

will be established for the substitutability case with non-increasing returns. In the two remaining subsections, we will then show how imperfect competition may contribute to the emergence of endogenous fluctuations, by allowing increasing returns on the one hand, and through variability of market power on the other hand.

4.1 Two regimes of dynamics

4.1.1 Full employment

The first significant feature of full employment intertemporal equilibria, whether stationary or not, is that stationarity prevails at least in output, given symmetry and the constancy of labour supply and productivity. Thus, the demand must satisfy, for any $t = 0, 1, 2, \ldots$,

$$D(p_t, p_{t+1}) = D(p_{t+1}, p_{t+2}). \tag{46}$$

The second important feature of full employment intertemporal equilibria is that they are determined quite independently of market structure, according to a dynamic system which is the same, whether competition is perfect or imperfect. It is only through the two admissibility conditions, of non-negative profits and proper ranking of the reservation wages, that market power may have an influence, by ruling out inadmissible trajectories.

Because of homogeneity (of degree -1) of the demand function, the difference equation of order 2 given by (46) can be transformed into a difference equation of order 1 in the variable $\theta_t = p_t/p_{t+1}$:

$$D(\theta_t, 1) = \theta_{t+1} D(\theta_{t+1}, 1), \tag{47}$$

or equivalently, since $[1 - \alpha(\theta_t)]\theta_t = M/D(\theta_t, 1)$ by (11):

$$[1 - \alpha(\theta_t)]\theta_t = 1 - \alpha(\theta_{t+1}). \tag{48}$$

The left-hand side of this equation is an increasing function of θ_t (with elasticity $\Delta(\theta_t)$), so that equation (48) always has, for any θ_{t+1} in some interval of $\mathbb{R}_{++}$ (in general $\mathbb{R}_{++}$ itself), a unique solution in θ_t, and thus characterizes well-defined *backward* dynamics in the real interest factor.

In the simple cases that we are considering, the substitutability and the complementarity cases, α is a monotonic function. So it becomes possible to consider also well-defined *forward* dynamics: $\theta_{t+1} = \Phi_{FE}(\theta_t)$ on some interval of $\mathbb{R}_{++}$. As $\alpha(\theta) < 1$ for any positive θ, the only fixed point of Φ_{FE} is clearly 1. Moreover, since Φ_{FE} is increasing in the case of intertemporal substitutability (α decreasing), decreasing in the case of intertemporal complementarity (α increasing), the dynamics are monotonic (both in θ_t and in p_t) in the first case, alternating (in both variables) in the second case[30] (see Figure 4). Stability of the stationary equilibrium can be assessed by referring to the absolute value of Φ'_{FE} at $\theta_t = 1$, with

$$\Phi'_{FE}(1) = \left.\frac{d\theta_{t+1}}{d\theta_t}\right|_{\theta_{t+1}=\theta_t=1} = \frac{\Delta(1)}{\Delta(1) - 1}. \tag{49}$$

When investigating existence of non-stationary intertemporal equilibria, one can of course use the dynamic system described by Φ_{FE}, keeping in mind that any sequence $(\theta_t)_{t\in\mathbb{N}}$ derived from an initial value θ_0 by successive iterations of Φ_{FE}, characterizes an

[30] The price sequence (p_t) is monotonic in the substitutability case because θ_t cannot switch from $]0, 1[$ to $]1, \infty[$ or vice versa. On the contrary in the complementarity case θ_t, when different from 1, takes alternatively values in each one of the two intervals, so that prices are alternatively increasing and decreasing.

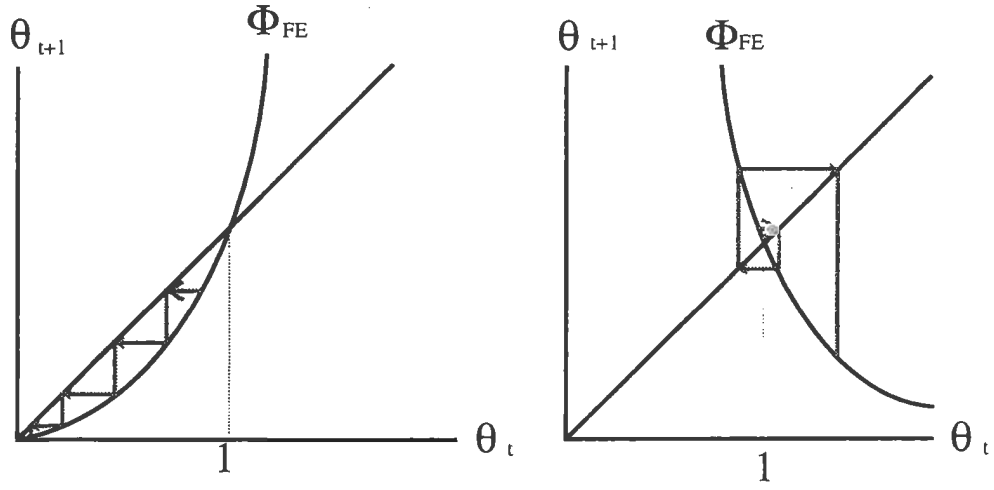


Figure 4

equilibrium only if profits are non-negative and reservation wages are properly ranked in all periods, *i.e.* if, for any t,

$$\beta\left(\frac{L}{mn}\right)^{1-1/\beta} v/U^*(\theta_t) \leq 1 - 1/n\delta(1,\theta_t) \leq \beta. \tag{50}$$

4.1.2 Underemployment

Underemployment equilibria must satisfy, for any $t = 0, 1, 2, \ldots$, the equality of the two reservation wages:

$$\omega_f(p_t, p_{t+1}) = \omega_l(p_t, p_{t+1}), \tag{51}$$

i.e., using homogeneity (of degree β for ω_f and 1 for ω_l):

$$\omega_f(\theta_t, 1)p_{t+1}^{\beta-1} = \omega_l(\theta_t, 1). \tag{52}$$

As can be seen from this equation, the dynamics in θ_t degenerate if the returns are constant ($\beta = 1$), or if labour has no disutility (i.e. if $v = 0$, leading to $\omega_l \equiv 0$). Any underemployment equilibrium is then constant in the real interest factor, hence quasi-stationary. Deflationary equilibria are excluded, since demand would be unbounded along the equilibrium path. Thus, in each one of those two situations (except in the singular case in which 1 is a solution to equation (52)), any equilibrium must be inflationary and declining: prices and the nominal wage increase at the same constant rate $1/\theta - 1$, with outputs decreasing at rate $1 - \theta$.

If $\beta \neq 1$ and $v \neq 0$, one can divide both sides of equation (52) by the corresponding sides of the same equation, applying to the next period, to get a difference equation of order 1 in the variable θ_t:

$$f(\theta_t) \equiv \frac{\omega_f(\theta_t, 1)}{\omega_l(\theta_t, 1)} = \frac{\omega_f(\theta_{t+1}, 1)}{\omega_l(\theta_{t+1}, 1)}\theta_{t+1}^{1-\beta} \equiv g(\theta_{t+1}). \tag{53}$$

Any solution to this equation characterizes an underemployment intertemporal equilibrium, provided the non-negativity condition on profits and the labour availability constraint are satisfied in each period t. Since the output derived from the equality of the

two reservation wage functions is given by:

$$y_t = \left[\frac{1}{\beta v} U^*(\theta_t)(1 - 1/n\delta(1,\theta_t))\right]^{\frac{1}{\beta-1}}, \tag{54}$$

the two admissibility conditions are respectively:

$$\beta \geq 1 - 1/n\delta(1,\theta_t) \quad \text{and} \quad \frac{L}{mn} > \left[\frac{1}{\beta v} U^*(\theta_t)(1 - 1/n\delta(1,\theta_t))\right]^{\frac{\beta}{\beta-1}}. \tag{55}$$

Using equations (28) and (30), elasticities of the left-hand and of the right-hand sides of equation (53) with respect to θ_t and θ_{t+1} are respectively:

$$\frac{f'}{f}\theta_t = \frac{\omega_f'}{\omega_f}\theta_t - \frac{\omega_l'}{\omega_l}\theta_t = (\beta - 1)\Delta + \frac{\rho'}{\rho}\theta_t - \alpha \tag{56}$$

and

$$\frac{g'}{g}\theta_{t+1} = \frac{\omega_f'}{\omega_f}\theta_{t+1} - \frac{\omega_l'}{\omega_l}\theta_{t+1} + 1 - \beta = (\beta - 1)(\Delta - 1) + \frac{\rho'}{\rho}\theta_{t+1} - \alpha. \tag{57}$$

Neither of these elasticities has a well determined sign, the same for all possible cases, or even, in some cases, the same for all values of θ_t or θ_{t+1}. Thus, contrary to the full employment difference equation (48), equation (53) does not necessarily lead to well-defined dynamics, either backwards or forwards. However, putting together equality (56) and inequality (35), one sees that the condition entailing uniqueness of temporary equilibrium (that the elasticity of the firms' reservation wage be larger than the elasticity of the workers' reservation wage) naturally leads to well-defined *backward* underemployment dynamics (since f is an increasing function in that case). As we have seen, that condition is in particular satisfied when competition is perfect, so that increasing returns are excluded (i.e. $\beta \geq 1$) and marginal revenue is equal to price (hence $\rho'\theta/\rho = 1$).

Also, in the substitutability case, we know from Lemma 4 in the Appendix that the elasticity of the marginal revenue function ρ is always larger than the one of the propensity to consume function α. We can then conclude that, for $\beta \geq 1$, both sides of equation (53) are increasing. Thus, both the forward and the backward dynamics are well-defined in this case. We can in particular consider a function Φ_{UE}, defined on some interval of $\mathbb{R}_{++}$, and describing the *forward* dynamics characterized by equation (53): $\theta_{t+1} = \Phi_{UE}(\theta_t)$. The function Φ_{UE} is increasing, with a derivative larger than 1 at 1, its unique fixed point:

$$\Phi'_{UE}(1) = \left.\frac{d\theta_{t+1}}{d\theta_t}\right|_{\theta_{t+1}=\theta_t=1} = \frac{(\beta - 1)\Delta(1) + \rho'(1)/\rho(1) - \alpha(1)}{(\beta - 1)(\Delta(1) - 1) + \rho'(1)/\rho(1) - \alpha(1)}. \tag{58}$$

Before we show how imperfect competition can contribute to endogenous fluctuations, let us consider this particularly simple case, in which fluctuations are precisely excluded.

4.2 The substitutability case with non-increasing returns

Assume first decreasing returns ($\beta > 1$). The two dynamics are then essentially the same: they can both be defined forwards by explicit functions Φ_{FE} and Φ_{UE} (in full employment and underemployment, respectively), which are increasing, have 1 as their unique fixed point, with derivatives larger than 1 at this point (see equations (49) and (58)). So, the stationary equilibrium is always unstable. If $\theta_0 > 1$, θ_t would indefinitely increase,

implying that the price would converge to zero. But this is impossible, since demand would then tend to infinity whereas production is bounded. Thus, besides the stationary equilibrium, only equilibria with explosive inflation (such that θ_t converges to 0) can exist. As a matter of fact, there generally exists a continuum of them, for any $\theta_0 < 1$, as we are going to see.

In order to assert existence of such equilibria, we must take the admissibility conditions (50) and (55) into account. Let us define, as a function of the degree of labour disutility, the value $\tilde{\theta}(v)$ of the real interest factor that leads to equality of firms' and workers' reservation wages at full employment:

$$(1/\beta)(L/mn)^{1/\beta-1}[1 - 1/n\delta(1, \tilde{\theta}(v))] = v/U^*(\tilde{\theta}(v)). \tag{59}$$

As $\rho'\theta/\rho > \alpha$ when substitutability prevails, the elasticity with respect to θ of the left-hand side of this equation is larger than the one of right-hand side, so that a solution $\tilde{\theta}(v)$ is uniquely determined. A positive solution exists if $v > \underline{v}$, with

$$\underline{v} \equiv (1/\beta)(L/mn)^{1/\beta-1} \lim_{\theta \to 0} \{[1 - 1/n\delta(1, \theta)]U^*(\theta)\}. \tag{60}$$

Also, notice that $\tilde{\theta}$ is increasing and that, taking $\bar{v}$ as defined in (45), we obtain $\tilde{\theta}(\bar{v}) = 1$. Finally, since by equation (54) output y_t increases with the real interest factor θ_t, any value of θ_t will be associated with underemployment if $\theta_t < \tilde{\theta}(v)$ and compatible with full employment if $\theta_t \geq \tilde{\theta}(v)$.

Two cases must be considered now. If $v \leq \underline{v}$, full employment results from any positive value of the real interest factor, so that a sequence starting with any $\theta_0 \in]0, 1[$ and determined by successive iterations of the function Φ_{FE} characterizes a full employment equilibrium (see Figure 4a), provided those iterations can go on forever, that is provided $\lim_{\theta \to 0} \alpha(\theta) = 1$ (a condition implied by $\lim_{\theta \to 0} \sigma(\theta) > 1$)[31]. This also applies to the constant returns case ($\beta = 1$), since full employment dynamics are not affected by technological characteristics.

If $v > \underline{v}$, we can associate with any $\theta_0 \in]0, \tilde{\theta}(v)[$ a sequence determined by successive iterations of the function Φ_{UE} (we show in the Appendix that this function is indeed defined on the whole interval $]0, \tilde{\theta}(v)[$), a sequence characterizing an underemployment equilibrium (see Figure 5). But, if $v < \bar{v}$ and hence $\tilde{\theta}(v) < 1$, we can also take a sequence starting at $\theta_0 \in [\tilde{\theta}(v), 1[$, which will characterize an equilibrium with full employment in a finite number of periods and underemployment thereafter. As long as θ_t belongs to the interval $[\Phi_{FE}^{-1}(\tilde{\theta}(v)), 1[$, dynamics are, as we know, described by the function Φ_{FE}. However, if $\theta_t \in [\tilde{\theta}(v), \Phi_{FE}^{-1}(\tilde{\theta}(v))[$ equilibrium involves a transition from the full employment to the underemployment regime. We show in the proof of Proposition 6, in the Appendix, that such transition is described by an increasing function Φ_T which continuously extends both Φ_{UE} and Φ_{FE}, so that the dynamics are essentially the same in this case as in the pure regimes.

In the constant returns case, there is a single possible real interest factor $\tilde{\theta}(v)$, but there still exists a continuum of underemployment equilibria, associated with any initial price larger than the full employment price (the price corresponding to $\tilde{\theta}(v)$), as soon as $\tilde{\theta}(v) < 1$, *i.e.* $v < \bar{v}$. Of course, no deflationary equilibrium can exist, so that $\tilde{\theta}(v) > 1$ leads to the inexistence of an intertemporal equilibrium. We thus obtain the following proposition:

[31] Indeed, using the equality $\alpha'\theta = \alpha(1 - \alpha)(1 - \sigma)$, it is easy to calculate: $\alpha(\theta) = [1 + e(\theta)]^{-1}$, with $e(\theta) = \theta^{-1} \exp \int (\sigma(\theta)/\theta)\, d\theta$. Hence, from $\lim_{\theta \to 0} \alpha(\theta) \in]0, 1[$, so that (using L'Hospital's rule) $\lim_{\theta \to 0} e(\theta) = \lim_{\theta \to 0} [e(\theta)\sigma(\theta)] \in]0, \infty[$, we obtain $\lim_{\theta \to 0} \sigma(\theta) = 1$.

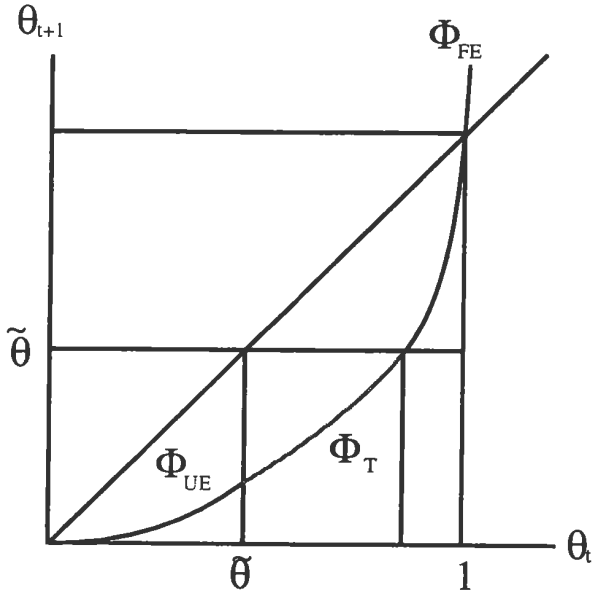


Figure 5

Proposition 6 (Monotonicity)**.** *Consider the substitutability case, as characterized by Proposition 2, and assume non-increasing returns. Refer to* $\underline{v}$*,* $\bar{v}$ *and* $\tilde{\theta}(v)$ *as defined by (60), (45) and (59), respectively.*

- *If* $0 \leq v \leq \underline{v}$*, there only exist full employment intertemporal equilibria, the stationary one and, provided* $\lim_{\theta \to 0} \alpha(\theta) = 1$*, a continuum of equilibria with explosive inflation associated with any* $\theta_0 \in]0, 1[$*.*

- *If* $\underline{v} < v \leq \bar{v}$*, there is a stationary full employment equilibrium and a continuum of inflationary equilibria with eventually declining output. If* $\beta > 1$*, those equilibria, associated with any* $\theta_0 \in]0, 1[$*, are characterized by explosive inflation. If* $\beta = 1$ *and* $v < \bar{v}$*, those equilibria have constant inflation at the rate* $1/\tilde{\theta}(v) - 1$*, and are associated with any initial price at least as large as the full employment price corresponding to* $\tilde{\theta}(v)$*.*

- *If* $\bar{v} < v$ *and* $\beta > 1$*, there only exist underemployment intertemporal equilibria, the stationary one and, associated with any* $\theta_0 \in]0, 1[$*, a continuum of equilibria with explosive inflation and declining output. No equilibrium exists if* $\bar{v} < v$ *and* $\beta = 1$*.*

Proof. See Appendix.

It is again quite easy to establish that monotonic equilibria are Pareto-ranked, at least if the aggregate demand elasticity Δ is a decreasing function (for instance, if U is C.E.S.). Indeed, real wage costs per unit of output are then a decreasing function of the real interest factor, since market power is positively related to this variable. Besides, U^* and, if returns are decreasing, underemployment equilibrium output are increasing functions of the real interest factor θ. Thus young consumers, as shareholders, unambiguously prefer in both regimes higher values of θ. As we know, they are indifferent, as workers, among underemployment equilibria, and prefer in full employment higher values of θ, since the favourable effect of an increase of the real interest factor on U^* (given by the elasticity $1 - \alpha$) dominates, when substitutability prevails, the negative effect through market

power of that increase on the real wage (given by the elasticity $\rho'\theta/\rho - 1$). Finally, old consumers would object to lower values of θ. Indeed, the Keynesian multiplier is in this case a decreasing function of θ, so that higher prices must compensate for lower values of the real interest factor and higher associated values of the multiplier, if demand is to be kept at its full employment level. As for underemployment equilibria, the same relation prevails between prices and values of θ, as can be seen from equation (52). Thus, and because of monotonicity, equilibria in the decreasing returns case are positively related (according to the Pareto criterion) to the initial values of the real interest factor. The preceding argument also applies to the constant returns case, except that underemployment equilibria are negatively related to initial prices.

As we see, the multiplicity of Pareto-ranked intertemporal equilibria may occur even in the substitutability case under non-increasing returns, a situation which excludes the multiplicity of temporary equilibria. This conclusion is however by no means related to market power, and is quite compatible with perfect competition. It is the overlapping generations structure together with an infinite horizon of the economy, and not the imperfectly competitive nature of its markets, that explains the equilibrium indeterminacy, through the indeterminacy of potentially harmful yet self-fulfilling expectations, leading to Pareto-inferior intertemporal equilibria.

4.3 Increasing returns

Substitutability and non-increasing returns together lead to monotonicity of intertemporal equilibria. A continuum of inflationary and declining equilibria may then exist, but fluctuations are excluded. This is no longer the case under increasing returns. Of course, as soon as multiple equilibria are non-monotonic, they will in general not be Pareto-ranked. In any case, it is the possibility of occurrence of endogenous fluctuations in itself that will hold, from now on, our attention.

In order to focus on the role of increasing returns as a source of endogenous fluctuations, we shall first stick to the substitutability case (the combined influence of increasing returns and complementarity will be considered in the next subsection). No difference arises from the assumption of increasing returns in full employment dynamics, which are independent of technological specifications, because of stationarity in output. As for underemployment dynamics, by (56) and (57), the elasticities of both the left- and right-hand sides of the difference equation (53) will in general take negative values for θ close to 0, and positive values for θ large enough. Indeed, if we exclude the singular case in which $\lim_{\theta\to 0}\sigma(\theta) = 1$ (so that $\lim_{\theta\to 0}\alpha(\theta) = 1$) and, taking into account the fact that, by (A20) and (A29) in the Appendix, $\lim_{\theta\to 0}\rho'(\theta)\theta/\rho(\theta) = 1$, we obtain:

$$\lim \frac{f'(\theta)}{f(\theta)}\theta = (\beta - 1)\lim_{\theta\to 0}\sigma(\theta) < 0,$$

$$\lim_{\theta\to 0}\frac{g'(\theta)}{g(\theta)}\theta = (\beta - 1)\left(\lim_{\theta\to 0}\sigma(\theta) - 1\right) < 0,$$

and, as $\lim_{\theta\to\infty}\alpha(\theta) = 0$ and $\lim_{\theta\to\infty}\rho'(\theta)\theta/\rho(\theta) = 1$,

$$\lim_{\theta\to\infty}\frac{f'(\theta)}{f(\theta)}\theta = \beta > 0, \quad \lim_{\theta\to\infty}\frac{g'(\theta)}{g(\theta)}\theta = 1 > 0. \tag{61}$$

Thus, the functions f and g are non-monotonic, and the dynamics generated by equation (53) are not uniquely defined, either forwards or backwards.

To keep things straightforward, let us concentrate on the existence of a 2-cycle, the simplest form of persistent endogenous fluctuations. We look for a pair (θ^*, θ^{**}) of two different positive values of the real interest factor verifying the difference equation (53), with $\theta_t = \theta_{t+2} = \theta^*$ and $\theta_{t+1} = \theta^{**}$. Since this equation has the form

$$f(\theta_t) = f(\theta_{t+1})\theta_{t+1}^{1-\beta}, \tag{62}$$

we obtain:

$$f(\theta^*) = f(\theta^{**})\theta^{**1-\beta} = \left[f(\theta^*)\theta^{*1-\beta}\right]\theta^{**1-\beta}$$

and, since f takes only positive values, we get: $\theta^{**} = 1/\theta^* > 1$ (without loss of generality). Thus, the difference equation (62) admits a 2-cycle as a solution if and only if there exists $\theta^* \in]0, 1[$ verifying the equation:

$$F(\theta) \equiv \frac{f(\theta)}{f(1/\theta)}\theta^{1-\beta} = 1, \tag{63}$$

where F has elasticity:

$$\frac{F'(\theta)}{F(\theta)}\theta = \frac{f'(\theta)}{f(\theta)}\theta + \frac{f'(1/\theta)}{f(1/\theta)}\frac{1}{\theta} + 1 - \beta. \tag{64}$$

As $F(1) = 1$, a 2-cycle can only exist if F is non-monotonic, hence (for $\beta < 1$) if f is somewhere decreasing, which is precisely the *condition required for multiple temporary equilibria*. By continuity of F, a sufficient condition for the existence of a solution $\theta^* \in]0, 1[$ to the equation (63) is that $F'(1) < 0$ together with $\lim_{\theta\to 0} F(\theta) = 0$. Let us assume that $\lim_{\theta\to\infty} \alpha(\theta) = 0$ (which results simply from excluding the case of a Cobb-Douglas intertemporal utility function). Then, by (64), (56), and Lemma 4 in the Appendix, and recalling that $\lim_{\theta\to 0} \sigma(\theta) \leq 1 + \beta$, by the assumption of Proposition 2, we get:

$$\lim_{\theta\to 0} \frac{F'(\theta)}{F(\theta)}\theta \geq (\beta - 1) \lim_{\theta\to 0} \sigma(\theta) + 1 \geq \beta^2 > 0,$$

and hence $\lim_{\theta\to 0} F(\theta) = 0$. Let us now consider the sign of $F'(1)$. Using again (64) and (56), we obtain:

$$F'(1) = 2\frac{f'(1)}{f(1)} + 1 - \beta = -2\left[(1-\beta)\left(\Delta(1) - \frac{1}{2}\right) + \alpha(1) - \frac{\rho'(1)}{\rho(1)}\right]. \tag{65}$$

Referring to this equation and to equation (58), one straightforwardly checks that $F'(1) < 0$ if and only if the stationary equilibrium is unstable in the (forward) underemployment dynamics, i.e. if $|\Phi'_{UE}(1)| > 1$ (where Φ_{UE} is now the implicit function defined by the difference equation (53) in a neighbourhood of the stationary state).

From (65) we see that favourable circumstances for the existence of a 2-cycle are: (i) a high degree of impatience (think of a C.E.S. intertemporal utility function with a small discount factor for future consumption) leading to a high value $\alpha(1)$ of the propensity to consume when young and leading also to a large elasticity $\Delta(1)$ of aggregate demand (indirectly and in conjunction with a high elasticity of intertemporal substitution); (ii) significantly increasing returns (small β); (iii) some variability of market power (expressed, when Δ is a decreasing function, as in the case of constant elasticity of intertemporal substitution, by a small value of $\rho'(1)/\rho(1)$).

Of course, in order that the 2-cycle $(\theta^*, 1/\theta^*)$ (with $0 < \theta^* < 1$) satisfying the difference equation (62) be an intertemporal equilibrium, the usual admissibility conditions must

be verified, both the non-negative profits condition: $\beta \geq 1 - 1/n\delta(1, \theta^*)$ (for a decreasing Δ, the condition prevails also for $1/\theta^*$), and the labour availability condition: either $\theta^* \geq \tilde{\theta}(v)$ (with $\tilde{\theta}(v)$ as defined in (59) and taking into account the fact that y_t, as given by (54), is now a decreasing function of θ_t) or else $v \leq \underline{v}$ (with $\underline{v}$ as defined by (60)). The first condition is verified, even for θ^* arbitrarily small, if $\beta \geq 1 - 1/ns$ (since $\delta < s$ if $\sigma \leq s$). As for the second condition, if v is larger than $\underline{v}$ but small enough, so that $0 < \tilde{\theta}(v) \leq \theta^*$, the inflationary full employment equilibria cease to exist (see Proposition 6), but the existence of the 2-cycle as an underemployment equilibrium is preserved.

4.4 Complementarity and strong variability of market power

We will first show in this subsection that complementarity may contribute to the occurrence of 2-cycles, and then that, by leading to negative values of the marginal revenue for some range of the real interest factor (an extreme form of variability of market power), it may also generate persistent endogenous fluctuations in *all* intertemporal equilibria.

4.4.1 Existence of cycles of period two when the stationary equilibrium is unstable

The argument used in the former subsection to establish existence of a 2-cycle in the underemployment dynamics also applies to the complementarity case. By (64), (56), and (84) and (97) in the Appendix, and assuming that $\lim_{\theta \to 0} \alpha(\theta) = 0$ (which is implied by $\lim_{\theta \to 0} \sigma(\theta) < 1$), and that $(1 - 1/m)s + (1/m) \inf_{\theta \in]0,\infty[} \Delta(\theta) > 1/n$, we get:

$$\lim_{\theta \to 0} \frac{F'(\theta)}{F(\theta)}\theta \geq 1 + (\beta - 1) \lim_{\theta \to \infty} \sigma(\theta) \qquad \begin{array}{l} \geq 1 \text{ if } \beta > 1 \\ \geq \beta \text{ if } \beta < 1. \end{array}$$

As the elasticity of F always tends to a positive limit when θ tends to 0, we have: $\lim_{\theta \to 0} F(\theta) = 0$. Thus, a sufficient condition for the existence of a 2-cycle in the underemployment dynamics is again $F'(1) < 0$ or, equivalently, the instability of the stationary state in the forward underemployment dynamics.

A simple inspection of (65) shows that a large variability of market power expressed, when Δ is a decreasing function, by a small value of $\rho'(1)/\rho(1)$, makes that condition easier to satisfy. Also, the lower is the value of β, for $\sigma(1) \geq 1/2$ (and hence $\Delta(1) > 1/2$), the easier is that condition to satisfy. Increasing returns are still favourable to the occurrence of a 2-cycle when complementarity is weak. In that case, the same conclusions apply to the (favourable) influence of a high degree of impatience (leading to a large $\alpha(1)$) and of a not too small elasticity of intertemporal substitution $\sigma(1)$. If returns are decreasing, it is on the contrary a small value of $\sigma(1)$ that favours occurrence of a 2-cycle.

We should point out that, as long as complementarity and variability of market power are both weak ($\sigma(\theta) \geq 1/2$ and $\rho'(\theta)\theta/\rho(\theta) \geq \alpha(\theta)$ for any θ), no 2-cycle can exist under non-increasing returns in either of the two regimes. Indeed, existence of a 2-cycle in underemployment dynamics requires equation (63) to have more than one solution, hence F to be non-monotonic. But, by (64) and (56), we obtain in that case:

$$\begin{aligned} \frac{F'(\theta)}{F(\theta)}\theta = (\beta - 1)\left[\Delta(\theta) + \Delta\left(\frac{1}{\theta}\right) - 1\right] + \frac{\rho'(\theta)}{\rho(\theta)}\theta - \alpha(\theta) \\ + \frac{\rho'(1/\theta)}{\rho(1/\theta)}\frac{1}{\theta} - \alpha\left(\frac{1}{\theta}\right) \geq 0. \end{aligned}$$

Exactly the same argument applies to full employment dynamics, once $\Delta(\theta)+\Delta(1/\theta)-1 \geq 0$.

If complementarity is strong enough, more precisely if the elasticity of intertemporal substitution is allowed to take values smaller than 1/2, one can however obtain the existence of 2-cycles, either in the full employment regime ($\Delta(1) < 1/2$ is then a sufficient condition) or in the underemployment regime, and even with decreasing returns and a constant, in fact nil, market power, i.e. even under perfect competition. In the latter, given high degrees of impatience and a large intertemporal complementarity, more complex dynamics, with cycles of higher periods, are possible. Thus, strong complementarity is as such, and quite independently of its effects through market power, a potential source of persistent endogenous fluctuations. We are however interested in effects that are specific to imperfect competition. Such effects work through the variability of market power, an extreme form of which is the emergence of negative values of the marginal revenue. That possibility not only favours occurrence of endogenous fluctuations, it can in fact threaten existence of stationary or, in conjunction with increasing returns, even monotonic intertemporal equilibria[32].

4.4.2 Non-stationarity and non-monotonicity of all intertemporal equilibria

It is not difficult to find conditions implying that any intertemporal equilibrium is non-stationary, even non-monotonic. But a related question has to be considered: do such non-stationary (or non-monotonic) equilibria exist, or on the contrary do those conditions preclude existence of intertemporal equilibria in general? Indeed existence of a stationary equilibrium, either in full employment or underemployment, is clearly ruled out if the marginal revenue is negative at a real interest factor equal to one: $\rho(1,1) < 0$. If in addition returns are increasing, no monotonic intertemporal equilibrium can exist. We know indeed that deflationary equilibria are impossible, since θ_t cannot converge to 1 without making the marginal revenue negative, so that prices would necessarily converge to zero in such an equilibrium, leading to a violation of the labour availability constraint. On the other hand, inflationary equilibria are also impossible in this case, since the fact that θ_t cannot converge to 1 has the further consequence that the sequence of prices would be unbounded, and hence that output and, under increasing returns, labour productivity would both converge to zero. The firm's reservation wage would then eventually become smaller than the labourers' reservation wage, which is incompatible with an equilibrium. So, if intertemporal equilibria exist, they cannot be monotonic.

Now, in order to get existence in the case of decreasing returns[33], let us assume, together with the conditions of Proposition 1, that[34]

$$\Delta(1) < m/n - (m-1)s < 1 = \lim_{\theta \to 0} \Delta(\theta), \tag{66}$$

or equivalently, by (15) and (26), that the marginal revenue function takes negative values for θ close to 1, and positive values for θ close to 0. The marginal revenue function and consequently the firms' reservation wage function must then switch from positive to negative values at some $\underline{\theta} \in]0,1[$. Assuming regularity ($f'(\underline{\theta}) < 0$), the difference

[32] The first possibility has been shown in d'Aspremont, Dos Santos Ferreira and Gérard-Varet (1991).

[33] Existence is also compatible with constant returns if marginal revenue is positive at $\tilde{\theta}(v)$, as defined by (59).

[34] A simple example, with a decreasing demand elasticity, is given by a C.E.S. intertemporal utility function, with elasticity of substitution $\sigma < m/n - (m-1)s < 1$ and a discount factor for future consumption γ such that $\gamma^\sigma < [m/n - (m-1)s - \sigma]/[1 - m/n + (m-1)s]$.

equation (53), of which $\underline{\theta}$ is clearly a stationary solution, defines in a neighbourhood of $\underline{\theta}$ an implicit function Φ_{UE} with $\underline{\theta}$ as a fixed point and such that $\theta_{t+1} = \Phi_{UE}(\theta_t)$. Also,

$$\begin{aligned} 0 < \Phi'_{UE}(\underline{\theta}) &= \left.\frac{d\theta_{t+1}}{d\theta_t}\right|_{\theta_{t+1}=\theta_t=\underline{\theta}} \\ &= \frac{f'(\underline{\theta})}{f'(\underline{\theta})\underline{\theta}^{1-\beta} + f(\underline{\theta})(1-\beta)\underline{\theta}^{-\beta}} = \underline{\theta}^{\beta-1} < 1, \end{aligned} \tag{67}$$

since $f(\underline{\theta}) = 0$. Thus, the stationary solution $\underline{\theta}$ is stable in the forward dynamics and attracts the trajectories starting with values of the initial real interest factor θ_0 in a neighbourhood of $\underline{\theta}$. Since the marginal revenue remains positive and the labour availability constraint is satisfied for θ_t smaller than, but sufficiently close to, $\underline{\theta}$ (by (54), y_t tends to 0 if θ_t tends to $\underline{\theta}$), we obtain existence of a continuum of equilibria converging to $\underline{\theta}$ and associated with different values of $\theta_0 \in]0, \underline{\theta}[$ close enough to $\underline{\theta}$. These equilibria are inflationary. Also, as θ_t is bounded away from 1 and the sequence of prices is consequently unbounded, whereas the multiplier is bounded, these equilibria must be such that demand, and hence output, converge to zero. We sum up the preceding result in a proposition.

Proposition 7 (Non-stationarity)**.** *With decreasing returns, under the assumptions of Proposition 1, and also assuming that*

$$\Delta(1) < m/n - (m-1)s < 1 = \lim_{\theta \to 0} \Delta(\theta),$$

there exists a continuum of inflationary intertemporal equilibria, such that output converges to zero. No stationary equilibrium exists.

Now observe that, if marginal revenue is negative for any θ larger than 1 (as it will necessarily be the case if the demand elasticity Δ is a decreasing function), then *all* equilibria are of course inflationary and such that output converges to zero. Stationarity is excluded, but so are in that case endogenous fluctuations. Let us therefore assume that the marginal revenue function takes positive values when the real interest factor is either sufficiently smaller or sufficiently larger than 1:

$$\Delta(\theta) > m/n - (m-1)s \quad \text{iff} \quad \theta \in]0, \underline{\theta}[\cup]\bar{\theta}, \infty[, \quad \text{with } \underline{\theta} < 1 < \bar{\theta}. \tag{68}$$

Such behaviour of the function Δ is possible if we assume, for instance, that σ is an increasing function[35]. Existence of fluctuating equilibria is then possible in any one of the two regimes and under both decreasing and increasing returns. Assuming for instance that $\underline{\theta}\bar{\theta} > 1$, we immediately obtain existence of a 2-cycle in the underemployment dynamics. Indeed, using again the function F as defined in (63), recalling that $\lim_{\theta \to 0} F(\theta) = 0$ (if $\lim_{\theta \to 0} \alpha(\theta) = 0$), and noticing that $\lim_{\theta \to (1/\bar{\theta})^-} F(\theta) = +\infty$ (since $1/\bar{\theta} < \underline{\theta}$ and by (68)), we obtain by continuity of F in the interval $]0, 1/\bar{\theta}[$ existence in that interval of a solution θ^* to equation (63). That solution satisfies: $0 < \theta^* < 1/\bar{\theta} < \underline{\theta} < 1 < 1/\underline{\theta} < \bar{\theta} < 1/\theta^*$, so that the cycle $(\theta^*, 1/\theta^*)$ lies in the admissible domain where a positive marginal revenue prevails. Of course, the non-negative profits condition and the labour availability constraint should also be satisfied. The former requires a large enough value of β; if Δ is quasi-convex, a sufficient condition is clearly:

$$\beta \geq 1 - \frac{1/n}{(1-1/m)s + 1/m}.$$

[35] A simple example is given in d'Aspremont, Dos Santos Ferreira and Gérard-Varet (1991).

A sufficient condition for the latter is a large (small) degree v of labour disutility if returns are decreasing (increasing). Indeed, by (54), the outputs associated with each one of the values θ^* and $1/\theta^*$ (independent of v) of the real interest factor along the cycle tend to 0 when v tends to infinity (zero).

Now, persistent endogenous fluctuations, taking for instance, as in the preceding example, the form of a 2-cycle, or converging to it, or taking possibly more complex forms, will characterize *any* intertemporal equilibrium if returns are increasing, a condition that excludes monotonicity of the equilibrium trajectories. Also, if intertemporal complementarity is not too strong, those fluctuations will involve output as well as prices. Indeed, if $\Delta(\theta) > 1/2$ for any θ, no 2-cycle exists in the full employment regime, as we have seen. In fact, no full employment intertemporal equilibrium can then exist[36]. We sum up these conclusions in the following proposition.

Proposition 8 (Non-monotonicity)**.** *Under the assumptions of Proposition 1, and also assuming that* $\lim_{\theta\to 0}\alpha(\theta) = 0$ *and that, for some* $\underline{\theta}$ *and* $\bar{\theta}$ *such that* $0 < \underline{\theta} < 1 < \bar{\theta}$ *and* $\underline{\theta}\bar{\theta} > 1$*,*

$$\Delta(\theta) > m/n - (m-1)s \quad \textit{iff} \quad \theta \in]0, \underline{\theta}[\cup]\bar{\theta}, \infty[,$$

there exists a cyclical underemployment equilibrium of period 2, either under decreasing returns and for a sufficiently high degree v *of labour disutility or under increasing returns, large enough values of the elasticity* β *of the cost function and sufficiently low values of* v*. Also, under increasing returns,* any *intertemporal equilibrium is characterized by persistent fluctuations in prices and in the real interest factor, and, if* Δ *takes only values larger than* $1/2$*, also in output.*

5 Conclusion

We have analysed an overlapping generations economy with "Cournotian monopolistic competition" in the produced goods markets. This generalizes and extends previous macroeconomic models with imperfect competition, either of the Cournotian or of the Chamberlinian variety. All prices have been assumed perfectly flexible and no adjustment costs were introduced. The labour market has been supposed perfectly competitive, in order to emphasize the effects of firms' oligopolistic market power, as opposed to the effects of union power.

Such effects work in our model through two main channels: increasing returns to employment, more precisely decreasing *marginal* costs, and variability of markups over marginal costs, induced by changes in demand elasticity as prices vary. Both channels of influence have often been neglected in previous models, particularly the ones using the Dixit and Stiglitz monopolistic competition framework, by assuming both a constant (at least non-decreasing) marginal cost and a constant markup over marginal cost. This practice cancels the effects we wanted to stress, making output market power somewhat unimportant.

Increasing returns and significant variability of market power, isolated or in conjunction, have been shown to be responsible for multiple Pareto-ranked temporary or stationary equilibria on the one hand, and for the occurrence of endogenous equilibrium

[36]Indeed, the stationary state is unstable in the full employment dynamics if $\Delta(1) > 1/2$ (see (49)), and any sequence of real interest factors generated by equation (48) would be unbounded, since it cannot converge to a cycle. But equation (48) does not have a solution in θ_{t+1} for a too large θ_t since $\lim_{\theta\to\infty}[\theta(1-\alpha(\theta))]$ belonging to $]0,1]$ implies $\lim_{\theta\to\infty}\sigma(\theta) = 0$ (as can be seen using L'Hospital's rule and the equality $\alpha'\theta = \alpha(1-\alpha)(1-\sigma)$).

fluctuations, in an economy which is not subject to shocks, on the other hand. Perfect competitive economies can certainly experience such endogenous fluctuations, related in particular to intertemporal complementarity, that is, to the dominance of income effects over substitution effects. But output market power, through the two channels of influence we have emphasized, not only makes endogenous fluctuations more plausible, but can actually exclude any equilibrium devoid of persistent fluctuations. This is the consequence of possible negative marginal revenue making the stationary state inadmissible, and of sharply increasing marginal costs as output diminishes along inflationary trajectories, excluding monotonicity. Variability of market power can however in this case entail positive marginal revenue along fluctuating equilibria, and the occurrence of such equilibria is clearly a possibility.

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Appendix

Lemma 1. *Consider the problem*

$$\max_{y\in\mathbb{R}_+} \pi(y) \equiv \max_{y\in\mathbb{R}_+}\{\psi(y)y - wy^{\beta}\}, \tag{69}$$

where $w \in \mathbb{R}_+$, $\beta \in \mathbb{R}_{++}$, *and* $\psi : \mathbb{R}_{++} \to \mathbb{R}_{++}$ *is a* C^2 *decreasing function (continuously extended to the domain* $\mathbb{R}_+$*), such that (denoting* $\epsilon(y) \equiv -\frac{\psi'(y)y}{\psi(y)}$*) for any* $y \in \mathbb{R}_{++}$,

$$1-\beta \leq \epsilon(y) \leq 1 \;\Rightarrow\; \epsilon'(y)y > -[1-\epsilon(y)][\epsilon(y)-(1-\beta)]. \tag{70}$$

Then, the following condition on $y^* \in \mathbb{R}_{++}$*:*

$$\psi(y^*)\beta \geq \psi(y^*)[1-\epsilon(y^*)] = w\beta y^{*\beta-1} \tag{71}$$

is necessary and sufficient for y^* *to be a solution to the problem. If it exists, this solution is the unique positive solution.*

Proof. Necessity: Suppose the problem has a positive solution y^*. Then y^* is a critical point of π:

$$\pi'(y^*) = \psi(y^*)[1-\epsilon(y^*)] - w\beta y^{*\beta-1} = 0, \tag{72}$$

and has a non-negative value:

$$\pi(y^*) = \psi(y^*)y^* - wy^{*\beta} \geq \lim_{y\to 0}\pi(y) \geq 0. \tag{73}$$

Sufficiency: Suppose some positive y^* satisfies the first order and non-negative value condition (71). Then, $1-\beta \leq \epsilon(y^*) \leq 1$, so that

$$\pi''(y^*) = -\frac{\psi(y^*)}{y^*}\{\epsilon'(y^*)y^* + [1-\epsilon(y^*)][\epsilon(y^*)-(1-\beta)]\} < 0, \tag{74}$$

by assumption. Hence, $\pi(y^*)$ is a strict local maximum. Suppose π has another local maximum $\bar{y}$ in $[a,b]$, with a and b arbitrarily close to 0 and ∞, respectively. As π is C^2, the two local maxima must be separated by an interior minimum at say $\underline{y}$, so that $\pi'(\underline{y}) = 0$ (hence $\epsilon(\underline{y}) \leq 1$) and $\pi''(\underline{y}) \geq 0$. Given assumption (70) on ϵ, it must be the

case that $\epsilon(\underline{y}) < 1-\beta$. But that implies in fact: $\forall y \in]0, \underline{y}]$, $\epsilon(y) < 1-\beta$, since by the same assumption $\epsilon'(y) > 0$ whenever $\epsilon(y) = 1-\beta$. So, we must admit that $a \leq \bar{y} < \underline{y} < y^*$, with $\pi'(\bar{y}) \leq 0$. Then (as $\beta < 1 - \epsilon(\bar{y})$) $\pi(\bar{y}) = (\bar{y}/\beta)[\psi(\bar{y})\beta - w\beta\bar{y}^{\beta-1}] < (\bar{y}/\beta)\pi'(\bar{y}) \leq 0 \leq \pi(y^*)$, contradicting the hypothesis that $\bar{y}$ is a maximum. We conclude that $\pi(y^*)$ is a global maximum and that y^* is the unique positive solution to the problem. □

Lemma 2. *Consider the problem*

$$\max_{y \in \mathbb{R}_+} \pi(y; \overline{Y}) \equiv \max_{y \in \mathbb{R}_+} \{d^{-1}(y + \overline{Y})y - wy^{\beta}\}, \tag{75}$$

where $w \in \mathbb{R}_+$, $\beta \in \mathbb{R}_{++}$, $\overline{Y} \in \mathbb{R}_+$, and $d : \mathbb{R}_{++} \to \mathbb{R}_{++}$ is a C^2 decreasing function such that (denoting $\delta(p) \equiv -\frac{d'(p)p}{d(p)}$) for any $p \in \mathbb{R}_{++}$,

$$\delta(p) \leq 1 \;\Rightarrow\; \frac{\delta'(p)p}{\delta(p)} > -[1 - \delta(p)] \tag{76}$$

$$\begin{gathered} 1 \leq \delta(p) \quad \textit{and either } \beta \geq 1 \textit{ or } \delta(p) \leq \frac{1}{1-\beta} \;\Rightarrow \\ \frac{\delta'(p)p}{\delta(p)} > -[\delta(p) - 1][1 - (1-\beta)\delta(p)]. \end{gathered} \tag{77}$$

Then, the following condition on $y^* \in \mathbb{R}_{++}$, with $p^* = d^{-1}(y^* + \overline{Y})$:

$$p^*\beta \geq p^* \left[1 - \frac{y^*/(y^* + \overline{Y})}{\delta(p^*)}\right] = w\beta y^{*\beta-1} \tag{78}$$

is necessary and sufficient for y^* to be a solution to the problem. If it exists, this solution is the unique positive solution.

Proof. First, observe that if we identify $\psi(y) \equiv d^{-1}(y + \overline{Y})$, we obtain:

$$\epsilon(y) = \frac{\eta}{\delta(p)}, \quad \text{with } \eta = \frac{y}{y + \overline{Y}} \text{ and } p = d^{-1}(y + \overline{Y}). \tag{79}$$

Thus, the condition on y^* is just the transposition of the first order and non-negative value condition of Lemma 1. Hence, it suffices to apply that lemma to prove the present one, if only we can show that the assumption of the former is implied by the assumption of the latter. By a simple calculation we get:

$$\frac{\epsilon'(y)y}{\epsilon(y)} = 1 - \eta + \epsilon(y)\frac{\delta'(p)p}{\delta(p)}. \tag{80}$$

Consider first the case: $\delta(p) \leq 1$. By assumption,

$$\begin{aligned} \epsilon'(y)y &> \epsilon(y)\{1 - \eta - \epsilon(y)[1 - \delta(p)]\} \\ &= \epsilon(y)[1 - \epsilon(y)] \geq 0 \quad \text{if } \epsilon(y) \leq 1. \end{aligned} \tag{81}$$

Thus, the assumption of Lemma 1 is fulfilled in this case. Now, consider the case: $\delta(p) \geq 1$. By assumption, if either $\beta \geq 1$ or $\delta(p) \leq \frac{1}{1-\beta}$ (otherwise, $\epsilon(y) < 1 - \beta$, and ϵ is unconstrained),

$$\begin{aligned} \epsilon'(y)y &> \epsilon(y)\{1 - \eta - \epsilon(y)[\delta(p) - 1][1 - (1-\beta)\delta(p)]\} \\ &= \epsilon(y)[1 - \eta] - [\eta - \epsilon(y)][\epsilon(y) - (1-\beta)\eta] \\ &= (1 - \eta)[2\epsilon(y) - (1-\beta)(1 + \eta - \epsilon(y))] \\ &\quad - [1 - \epsilon(y)][\epsilon(y) - (1-\beta)] \\ &\geq -[1 - \epsilon(y)][\epsilon(y) - (1-\beta)] \quad \text{if } 1 - \beta \leq \epsilon(y) \leq 1. \end{aligned} \tag{82}$$

Again, we see that the assumption of Lemma 1 is fulfilled. □

Proposition 1 (Complementarity)**.** *Assume intersectoral (non-strict) complementarity:* $s \leq 1$*. Also, assume that the elasticity of intertemporal substitution* σ *(a* C^1 *function) satisfies, for any* $\theta \in \mathbb{R}_{++}$*,*

$$0 < \sigma(\theta) < 1 \quad \textit{and} \quad \frac{\sigma'(\theta)\theta}{\sigma(\theta)} \geq -[1 - \sigma(\theta)]. \tag{83}$$

Then, the first order and non-negative value condition (78) is necessary and sufficient for y^* *to be the unique positive solution to the producer's problem (75), with* d *as defined in Section II (equations (11) and (12)).*

Proof. By Lemma 2, it suffices to show that assumption (83) implies assumption (76). Denoting $q \equiv \frac{\partial P}{\partial p}\frac{p}{P} = \frac{1}{m}\left(\frac{p}{P}\right)^{1-s} \in]0,1[$, we have $\delta = (1-q)s + q\Delta$ (see (15), Section II), with $\Delta = \alpha\sigma + 1 - \alpha$ (see (13), Section II), so that $\delta < 1$ for any values of p/P and θ, and hence δ should always satisfy $\frac{\delta' p}{\delta} > \delta - 1$. First notice that

$$\begin{aligned} \frac{\Delta'\theta}{\Delta} &= \frac{\alpha\sigma'\theta + \alpha'\theta(\sigma-1)}{\Delta} = \frac{\alpha\sigma'\theta - \alpha(1-\alpha)(1-\sigma)^2}{\Delta} \\ &\geq \frac{-\alpha(1-\sigma)[\sigma + (1-\alpha)(1-\sigma)]}{\Delta} = -(1-\Delta). \end{aligned} \tag{84}$$

Also, with elasticity $\frac{dq}{dp}\frac{p}{q} = (1-s)(1-q)$, we obtain:

$$\begin{aligned} \frac{\delta' p}{\delta} &= \frac{q(1-q)(1-s)(\Delta - s) + q^2\Delta'\theta}{\delta} \\ &\geq \frac{q(1-q)(1-s)(\Delta - s) - q^2\Delta(1-\Delta)}{\delta} > -(1-\delta) \end{aligned} \tag{85}$$

if

$$\begin{aligned} &q(1-q)(1-s)(\Delta - s) - q^2\Delta(1-\Delta) + \delta(1-\delta) \\ &\quad = [-\Delta(2-3s) + s(1-2s)]q^2 + (\Delta - s)(2-3s)q + s(1-s) \\ &\quad \equiv f(q) > 0. \end{aligned} \tag{86}$$

That is precisely the case, since f is quadratic, $f(0) = s(1-s) \geq 0$, $f(1) = 0$, and $f'(1) = -\Delta(2-3s) - s^2 < 0$. □

Proposition 2 (Substitutability)**.** *Assume intersectoral (non-strict) substitutability:* $s \geq 1$*. Also, assume that the elasticity of intertemporal substitution* σ *satisfies, for any* $\theta \in \mathbb{R}_{++}$*,*

$$1 \leq \sigma(\theta) \leq \min\{s, 1+\beta\} \quad \textit{and} \quad \sigma'(\theta) \geq 0. \tag{87}$$

Then, the first order and non-negative value condition (78) is necessary and sufficient for y^* *to be the unique positive solution to the producer's problem (*$\overline{Y} > 0$*, and hence* $n > 1$*, is required in condition (78) if* $s = 1$*).*

Proof. Again, by Lemma 2, it suffices to show that assumption (87) implies assumption (77). We have, in this case:

$$\begin{aligned} \Delta'\theta &= \alpha\sigma'\theta - \alpha(1-\alpha)(\sigma-1)^2 \\ &\geq -\alpha(1-\alpha)(\sigma-1)^2 = -(\Delta - 1)(\sigma - \Delta), \end{aligned} \tag{88}$$

and hence

$$\begin{aligned}\frac{\delta' p}{\delta} &= \frac{q(1-q)(s-1)(s-\Delta)+q^2\Delta'\theta}{\delta}\\ &\geq \frac{q}{\delta}[(1-q)(s-1)(s-\Delta)-q(\Delta-1)(\sigma-\Delta)]\\ &> -(\delta-1)[1-(1-\beta)\delta]\end{aligned} \tag{89}$$

for $(1-\beta)\delta \leq 1$, if (since $q/\delta < 1$)

$$\begin{aligned}&(1-q)(s-1)(s-\Delta)-q(\Delta-1)(\sigma-\Delta)+(\delta-1)[1-(1-\beta)\delta]\\ &\quad = -(1-\beta)(s-\Delta)^2q^2\\ &\qquad +[(s-\Delta)(s-1+(1-2s)\beta)-(\Delta-1)(\sigma-\Delta)]q\\ &\qquad +(s-1)(1+\beta s-\Delta)\equiv g(q)>0.\end{aligned} \tag{90}$$

This inequality is satisfied. Indeed, g is quadratic and, using the assumption $\sigma \leq 1+\beta$, we have for $s > 1$,

$$\begin{aligned}g(0) &= (s-1)(1+\beta s-\Delta) \geq \beta(s-1)^2 > 0,\\ g(1) &= (\Delta-1)(1+\beta\Delta-\sigma) \geq \beta(\Delta-1)^2 \geq 0.\end{aligned} \tag{91}$$

If $\beta \leq 1$, g is concave, so that $g(q) > 0$ for $q \in]0,1[$. If $\beta > 1$, using the assumption $\sigma \leq s$,

$$\begin{aligned}g'(1) &= -(s-\Delta)[(\beta-1)(2\Delta-1)+s]-(\Delta-1)(\sigma-\Delta)\\ &\leq -(s-\Delta)(\beta-1+s)-(\Delta-1)(\sigma-\Delta)<0,\end{aligned} \tag{92}$$

and we get the same result. In the case $s = 1$, since $1 \leq \sigma \leq s$, so that $\Delta(\theta) \equiv 1$, we have, by (89), $\delta'(\theta) \equiv 0$ and, by (79) and (80), $\epsilon'(y)y = (1-\eta)\epsilon(y) > 0$ (for $\overline{Y} > 0$, and hence $\eta < 1$). Thus, condition (70) is always satisfied and we can directly apply Lemma 1 in order to complete the proof. □

Lemma 3 (Symmetry in quantities)**.** *Any temporary equilibrium is symmetric in quantities, inside each sector $k = 1,\ldots,m$, relatively to the active firms: $y^*_{ki} = y^*_{kj}$ for any i and j in the same sector k, such that $y^*_{ki} > 0$ and $y^*_{kj} > 0$.*

Proof. Consider the first order necessary condition (78), and notice that it must be verified in equilibrium by any active firm for the same price p^* and with $y^* + \overline{Y} = d(p^*)$. The left-hand side of the equation is decreasing in y^* and the right-hand side is non-decreasing in the same variable if $\beta \geq 1$. Thus, there is at most one value of y^* satisfying the condition, the same for all active firms. The argument does not however apply to the case $\beta < 1$. Suppose in this case that the first order and non-negative value condition (78) is verified by two outputs y and y', such that $0 < y < y'$. Dividing both sides of the equality which expresses for y the first order condition by the corresponding sides of the y' equality, we obtain:

$$\frac{1-y/\delta^* d^*}{1-y'/\delta^* d^*} = \left(\frac{y}{y'}\right)^{\beta-1}, \tag{93}$$

with $\delta^* = \delta(p^*)$ and $d^* = d(p^*)$. Non-negativity of the profit resulting from y imposes:

$$1-\beta \leq \frac{y}{\delta^* d^*} = \frac{(y'/y)^{1-\beta}-1}{(y'/y)^{2-\beta}-1} \tag{94}$$

or

$$1-\beta-(y/y')+\beta(y/y')^{2-\beta} \leq 0. \tag{95}$$

The derivative with respect to y/y' of the expression on the first-hand side of this inequality, $-1+\beta(2-\beta)(y/y')^{1-\beta}$, is negative since y/y' and $\beta(2-\beta)$ are both less than 1. Thus, as $y/y'<1$, that expression is always positive, and we get a contradiction. □

Lemma 4. *In the substitutability case, under the assumptions of Proposition 2, the elasticity of the symmetric marginal revenue function $\rho' p/\rho$ is larger than the propensity to consume α. As a consequence, in this case and with non-increasing returns, the elasticity of the firms' reservation wage function $\omega_f' p/\omega_f$ is always larger than the elasticity of the labourers' reservation wage function $\omega_l' p/\omega_l$.*

Proof. From (28) and (30) in Section II,

$$\frac{\omega_f'}{\omega_f}p=(\beta-1)\Delta+\frac{\rho'}{\rho}p>\alpha=\frac{\omega_l'}{\omega_l}p, \tag{96}$$

if $\beta\geq 1$ and $\rho' p/\rho>\alpha$. But, using (15) and (26) in Section II, and (88) in the Appendix,

$$\frac{\rho'}{\rho}p=1+\frac{\Delta'\theta}{m\delta(n\delta-1)}\geq 1-\frac{m\alpha(1-\alpha)(\sigma-1)^2}{[(m-1)s+\Delta][n(m-1)s+n\Delta-m]}. \tag{97}$$

Now, observe that the denominator of the last term in this expression is equal to

$$\begin{aligned}[(m-1)(s-1)+(\Delta-1)+m][n(m-1)(s-1)+n(\Delta-1)+(n-1)m]\\ >m(\Delta-1)(s-1).\end{aligned} \tag{98}$$

Indeed, all terms in the left-hand side are non-negative, so that the inequality results from the fact that the coefficient of the term with $(\Delta-1)(s-1)$, $2n(m-1)$ is not smaller than m (as $m\geq 2$). Strictness is a consequence of either $s>1$ (so that $n(m-1)^2(s-1)^2>0$) or $n>1$ (so that $m^2(n-1)>0$). Since $\Delta-1=\alpha(\sigma-1)$ and $\sigma\leq s$ by assumption, we obtain from (97) and (98):

$$\frac{\rho'}{\rho}p>1-(1-\alpha)\frac{m\alpha(\sigma-1)^2}{m\alpha(\sigma-1)^2}=\alpha \tag{99}$$

if $\sigma>1$, and

$$\frac{\rho'}{\rho}p\geq 1>\alpha \tag{100}$$

if $\sigma=1$. □

Proposition 6 (Monotonicity)**.** *In the substitutability case, as characterized by Proposition 2, and under decreasing returns, there only exist a stationary equilibrium and, possibly, a continuum of equilibria with explosive inflation, associated with any $\theta_0\in]0,1[$. Let $\underline{v}$ and $\bar{v}$ be defined as follows:*

$$\begin{aligned}\underline{v}&\equiv(1/\beta)(L/mn)^{1/\beta-1}\lim_{\theta\to 0}\{[1-1/n\delta(1,\theta)]U^*(\theta)\}\\ \bar{v}&\equiv(1/\beta)(L/mn)^{1/\beta-1}[1-1/n\delta(1,1)]U^*(1).\end{aligned} \tag{101}$$

- *If $0\leq v\leq\underline{v}$, full employment prevails in all intertemporal equilibria. The inflationary equilibria exist if and only if $\lim_{\theta\to 0}\alpha(\theta)=1$.*
- *If $\underline{v}<v\leq\bar{v}$, the stationary equilibrium has full employment. The inflationary equilibria exist and are characterized by eventually declining output.*

- *If $\bar{v} < v$, underemployment prevails in all intertemporal equilibria. The inflationary equilibria exist and are characterized by declining output.*

Under constant returns, the proposition applies to the case: $0 \leq v \leq \underline{v}$. No equilibrium exists if $\bar{v} < v$. If $v = \bar{v}$, there exists a stationary full employment equilibrium and a continuum of stationary underemployment equilibria. If $\underline{v} < v < \bar{v}$, there exists a full employment stationary equilibrium, a continuum of equilibria with constant inflation and declining output, and a countable infinity of inflationary equilibria with increasing inflation and full employment in a finite number of periods, constant inflation and declining output thereafter.

Proof. We have seen in Subsection IV.B that dynamics can be defined forwards, in full employment and, if $\beta > 1$, in underemployment, by explicit functions, Φ_{FE} and Φ_{UE} respectively, which are increasing, have 1 as their unique fixed point, and have at this point derivatives larger than 1. Hence, a sequence $(\theta_t)_{t\in\mathbb{N}}$ associated with any $\theta_0 > 1$ and determined by successive iterations of Φ_{FE} or Φ_{UE} will monotonically diverge, whatever the regime, implying that the corresponding sequence of prices converges to 0. But this is incompatible with the upper bound imposed by labour availability on output. Thus, all non-stationary equilibria start at some $\theta_0 \in]0,1[$ and are necessarily inflationary, with θ_t converging to 0 (since the stationary equilibrium is unstable, and no fixed point exists in $]0,1[$).

Now, consider the different cases associated with the potential values of v. If $v \leq \underline{v}$, the firms' reservation wage is larger than the workers' reservation wage at any θ, since $[1 - 1/n\delta(1,\theta)]U^*(\theta)$ is increasing (by Lemma 4, it has elasticity $\rho'\theta/\rho - \alpha > 0$). Hence, full employment prevails for all θ, so that for $\beta \geq 1$ and for any $\theta_0 \in]0,1[$, the sequence $(\Phi_{FE}^t(\theta_0))_{t\in\mathbb{N}}$ of the images of θ_0 by the successive iterates of Φ_{FE} clearly characterizes an equilibrium if $\lim_{\theta\to 0}\alpha(\theta) = 1$. Otherwise, the function Φ_{FE} is not defined over the whole interval $]0,1[$ (as one can easily check by looking at the difference equation characterizing full employment dynamics), and the stationary equilibrium is the only intertemporal equilibrium.

For $v > \underline{v}$, we can use the value $\tilde{\theta}(v)$ of the real interest factor that equalizes the two reservation wages at full employment:

$$v = (1/\beta)(L/mn)^{1/\beta-1}[1 - 1/n\delta(1,\tilde{\theta}(v))]U^*(\tilde{\theta}(v)), \tag{102}$$

with a function $\tilde{\theta}$ which is increasing and verifies $\tilde{\theta}(\bar{v}) = 1$. If $v > \bar{v}$, underemployment prevails for any $\theta \leq 1$, since the right-hand side of this equation is increasing in θ. We can express, for $\beta > 1$ and using the equality of firms' and workers' reservation wages, the output y_t as an increasing function of θ_t:

$$y_t = \left[\frac{1}{\beta v}[1 - 1/n\delta(1,\theta_t)]U^*(\theta_t)\right]^{\frac{1}{\beta-1}}. \tag{103}$$

If $\beta = 1$, so that $\tilde{\theta}(v) > 1$ is the only value of the real interest factor that is compatible with the equality of the two reservation wages, no equilibrium can exist. But if $\beta > 1$, the sequence $(\Phi_{UE}^t(\theta_0))_{t\in\mathbb{N}}$ of the images of θ_0 by the successive iterates of Φ_{UE} characterizes an equilibrium with explosive inflation and declining output, for any $\theta_0 \in]0,1[$. Contrary to Φ_{FE}, the function Φ_{UE} is indeed always defined on the whole interval $]0,1[$. This results from: $\lim_{\theta\to 0}\omega_f(\theta,1)\theta^{1-\beta}/\omega_l(\theta,1) = 0$ (implying $\lim_{\theta\to 0}\omega_f(\theta,1)/\omega_l(\theta,1) = 0$), or equivalently from: $\lim_{\theta\to 0}[(1-\alpha(\theta))^{\beta-1}U^*(\theta)]$, which is always true, because either $\lim_{\theta\to 0}\alpha(\theta) = 1$ or $\lim_{\theta\to 0}U^*(\theta) = 0$. To see this, suppose that $\lim_{\theta\to 0}[U^{*\prime}(\theta)\theta/U^*(\theta)] =$

$\lim_{\theta\to 0}[1-\alpha(\theta)] \in]0,1[$ (implying $\lim_{\theta\to 0}\sigma(\theta)=1$), with $\lim_{\theta\to 0}U^*(\theta)>0$. Then we get $\lim_{\theta\to 0}U^{*\prime}(\theta)\theta \in]0,\infty[$ and, by applying L'Hospital's rule, $\lim_{\theta\to 0}U^{*\prime}(\theta)\theta = -\lim_{\theta\to 0}U^{*\prime\prime}(\theta)\theta^2$, so that $\lim_{\theta\to 0}U^{*\prime\prime}(\theta)\theta/U^{*\prime}(\theta)=-1$. But $\lim_{\theta\to 0}U^{*\prime\prime}(\theta)\theta/U^{*\prime}(\theta) = \lim_{\theta\to 0}[-\alpha'(\theta)\theta/(1-\alpha(\theta))-\alpha(\theta)] = \lim_{\theta\to 0}[\alpha(\theta)(\sigma(\theta)-2)] = -\lim_{\theta\to 0}\alpha(\theta) > -1$, and we obtain a contradiction.

Finally, if $\underline{v} < v \leq \bar{v}$, any value of θ_t will be associated with full employment in period t if $\theta_t \geq \tilde{\theta}(v)$ and with underemployment if $\theta_t < \tilde{\theta}(v)$. Also, if $\theta_t \leq \tilde{\theta}(v)$, we can apply the underemployment dynamics, since we then get: $\Phi_{UE}(\theta_t) < \theta_t \leq \tilde{\theta}(v)$. Full employment dynamics can however only be used if $\theta_t \geq \Phi_{FE}^{-1}(\tilde{\theta}(v))$, so that if $\tilde{\theta}(v) < \theta_t < \Phi_{FE}^{-1}(\tilde{\theta}(v))$, we must consider the transition from the full employment in period t to the underemployment regime in period $t+1$. From the equalities $L=(mn)^{1-\beta}D(p_t,p_{t+1})^\beta$ and $\omega_f(p_{t+1},p_{t+2})=\omega_l(p_{t+1},p_{t+2})$ (see Proposition 3), we straightforwardly obtain the difference equation:

$$[1-\alpha(\theta_t)]\theta_t = \\ [1-\alpha(\theta_{t+1})]\left[\frac{1}{\beta v}\left(\frac{L}{mn}\right)^{1/\beta-1}\left[1-\frac{1}{n\delta(1,\theta_{t+1})}\right]U^*(\theta_{t+1})\right]^{\frac{1}{\beta-1}}. \tag{104}$$

Both the left- and the right-hand sides of this equation are increasing in the real interest factor, so that forward dynamics are well-defined in the interval $[\tilde{\theta}(v),\Phi_{FE}^{-1}(\tilde{\theta}(v))]$, and can be described by a function Φ_T such that $\Phi_T(\tilde{\theta}(v))=\Phi_{UE}(\tilde{\theta}(v))$ and $\Phi_T\circ\Phi_{FE}^{-1}(\tilde{\theta}(v)) = \tilde{\theta}(v) = \Phi_{FE}\circ\Phi_{FE}^{-1}(\tilde{\theta}(v))$, as can be easily checked from (104) and the corresponding difference equations for the two pure regimes (see also (102)). We thus obtain a continuous increasing function Φ, which coincides with Φ_{UE} in $]0,\tilde{\theta}(v)]$, with Φ_T in $[\tilde{\theta}(v),\Phi_{FE}^{-1}(\tilde{\theta}(v))]$, and with Φ_{FE} in $[\Phi_{FE}^{-1}(\tilde{\theta}(v)),1]$ (see Figure 5). Since this function has no fixed point in $]0,1[$, any sequence starting in this interval and determined by the successive iterations of Φ converges to 0 and characterizes an equilibrium with explosive inflation and declining output.

If $\beta=1$, the only admissible value of θ in the underemployment regime is $\tilde{\theta}(v)$ (implying $v>\underline{v}$). This excludes underemployment equilibria for $v>\bar{v}$ (and hence $\tilde{\theta}(v)>1$). A continuum of equilibria results from the possibility of taking $\theta_0=\tilde{\theta}(v)$ with any p_0 at least as high as the full employment price corresponding to $\tilde{\theta}(v)$. A transition between regimes still arises if $\theta_t=\Phi_{FE}^{-1}(\tilde{\theta}(v))$, so that an equilibrium may start with full employment at any $\theta_0=\Phi_{FE}^{-k}(\tilde{\theta}(v))$, for any $k\in\mathbb{N}$. We obtain inflationary equilibria, with full employment in a finite number $k+1$ of periods, and declining output afterwards. Inflation is constant once the underemployment phase is attained. □