Household behavior and individual autonomy: an extended Lindahl mechanism*

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Abstract

We present a comprehensive model of household economic decision covering both full cooperation and noncooperation as well as semi-cooperative cases, varying with income distribution and a parameter vector θ representing degrees of individual autonomy with respect to the public goods. In this model, the concept of "household θ-equilibrium" is introduced through the reformulation of the Lindahl equilibrium for Nash implementation and its extension to semi-cooperation. Existence is proved and some generic properties derived. An example is given to illustrate. An important benefit of this approach is to allow for a compact and unified investigation of the testable (local) restrictions of household demand. A particular decomposition of the pseudo-Slutsky matrix is derived and the testability of the various models discussed.

Keywords: Intra-household allocation · Household financial management · Degree of autonomy · Lindahl prices · Local income pooling · Separate spheres

JEL Classification: D11 · C72 · H41

1 Introduction

Most economic models of household behavior, both theoretical and empirical, have assumed that households act cooperatively, implying that binding marriage agreements under full information and perfect communication are feasible at no (or low) transaction costs. This cooperative approach includes the traditional, but empirically contested, "unitary models" viewing the household as a single decision unit, e.g., Samuelson's consensus model and Becker's altruistic model. It also includes models based on axiomatic bargaining theory, as well as the more recent "collective" models, exploring the restrictions on observable household behavior implied by the assumption of Pareto efficiency without explicitly referring to some decision-making process (Chiappori 1988, 1992; Browning and Chiappori 1998).

However, some theoretical models have formalized the possibility for a household agreement to be noncooperative and in particular to be an equilibrium in a game of voluntary contributions to public goods. In these models (e.g., Ulph 1988; Lundberg and Pollak 1993; Chen and Woolley 2001), the noncooperative equilibrium is introduced as

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the threat (or disagreement) point of a bargaining model[1] and, under some conditions, as a "separate spheres" equilibrium where each spouse is responsible for a distinct set of goods and services within the household, the partition being "defined in terms of traditional gender roles and gender roles expectations" (Lundberg and Pollak 1993, p. 1007).[2] When binding agreements are impossible or too costly to enforce, the noncooperative agreement may even become the acceptable default option in a continuing relationship. In this context, the concept of noncooperative equilibrium for the household has been further analyzed by Lechene and Preston (2005, 2011) and by Browning et al. (2010) who confirm previous results by showing that there are generically only two possible types of noncooperative equilibria, "separate spheres" and "separate spheres up to one public good," the latter meaning that spouses do not contribute jointly to more than one public good.

Such a conclusion, though, seems to be too clear-cut to give account of the variety of household arrangements. In reality, the definition, and hence the division, of tasks and responsibilities to provide some goods and services within the household is ambiguous and may vary over husband–wife pairs. "Child care" or "housekeeping" might be divided into many subtasks that can be differently shared. The perfect partition according to gender roles is a limit case that does not hold for most households in most contemporary societies. Also, arrangements within the household might change along life path (with the number of children, the income level, the geographical location, etc.). Good intentions erode with time, and a genuine cooperative agreement might gradually evolve toward some more traditional division of responsibilities. The way in which the household organizes its finances, in particular whether each spouse has its own account and/or shares some joint account, as well as the type of marriage contract may influence this evolution.[3] As emphasized by Lechene and Preston (2005), "neither the assumption of fully efficient cooperation nor of complete absence of collaboration is likely to be an entirely accurate description of typical household spending behavior and analysis of such extreme cases can be seen as a first step toward understanding of a more adequate model" (p. 19).

Our objective is to follow this route and develop a comprehensive model that includes the cooperative and the noncooperative cases at the extremes, but allows for a continuum of intermediate "semi-cooperative" cases. This comprehensive model is based on a "mechanism design" reformulation of the Lindahl equilibrium for Nash implementation (see Hurwicz 1979; Walker 1981), which we extend to semi-cooperation.[4] In order to implement the household collective objectives while preserving budgetary autonomy and feasibility, a household arrangement (formulated for simplicity in terms of a two-person household) is presented as a mechanism defined by personalized pricing rules for the public goods. For each public good, the pricing rule applying to a spouse will only depend on the strategies of the other spouse. As these strategies are taken as given by the Nash assumption, such a pricing rule is unilaterally non-manipulable. The solution of the resulting game, a household (noncooperative) equilibrium, will vary according to the degree

[1] This is different from earlier models where the disagreement point was taken to be divorce (Manser and Brown 1980; McElroy and Horney 1981).

[2] Indeed, "specialization in the provision of such goods reduces the need for complex patterns of coordination, and traditional gender roles serve as a focal point for tacit division of responsibilities" (Lundberg and Pollak 1993, p. 993).

[3] Household money management systems have been extensively studied both empirically, essentially in the sociological literature (e.g., Pahl 2008; see also the two surveys of the International Social Survey Programme of 1994 and 2002, analyzing representative samples of 38 countries), and experimentally (for recent studies see Ashraf 2009; Schaner 2013).

[4] This extension will be formulated for a 2-person household but can easily be reformulated for more general public good allocation problems.

of autonomy of each spouse. This autonomy results from the possibility for each spouse to go directly to the market and buy individually some portion of the desired quantity of each public good, rather than transferring to the common purse, for collective purchase, the tax determined by the pricing rule.

An important benefit of this comprehensive model is to allow for a compact and unified investigation of the testable (local) restrictions of household demand.[5] These restrictions will include those studied under the assumption of efficiency by Browning and Chiappori (1998), relying on the decomposition of the pseudo-Slutsky matrix as the sum of a matrix with the Slutsky properties and a "deviation matrix" (of rank equal to 1 for a two-person household). They will also include those derived by Lechene and Preston (2011) for the fully noncooperative model, showing that the deviation matrix has generally a larger rank than in the collective model, increasing with the number of public goods. Our paper gives a general derivation of the pseudo-Slutsky matrix allowing to separate different kinds of effects, each effect increasing the maximum possible rank of the deviation matrix. The implementation of such tests becomes more and more demanding in terms of the required number of goods. As we will see, however, the noncooperative and the semi-cooperative models are distinguishable as soon as there is joint contribution to more than one public good.

In Sect. 2, we present a two-person household semi-cooperative decision model, defining the non-manipulable mechanism as well as two related approaches to the concept of "household equilibrium" characterized, either endogenously or exogenously, by the degrees of autonomy of each spouse for each public good. We illustrate via an example the implications of varying the degrees of autonomy. In Sect. 3, we examine the generic local properties of the household equilibrium and derive the pseudo-Slutsky matrix of the household demand, then discussing the testability of the various models. We conclude in Sect. 4.

2 The household decision model

We study a two-adult household, consuming goods that are recognized by both spouses as being either private or public (within the household). Denote by A (the wife) and B (the husband) the two household members, and let $\left(q^A, q^B\right) \in \mathbb{R}^{2n}_+$ be the vector of consumption by the two members of the n private goods and $Q \in \mathbb{R}^m_+$ the consumption vector of the m public goods. The preferences of each spouse J ($J = A, B$) are represented by a utility function $U^J\left(q^J, Q\right)$, which is defined on $\mathbb{R}^{n+m}_+$, increasing and strongly quasi-concave. Each spouse J is supposed to receive an initial income $Y^J \geq 0$, the total household income being $Y = Y^A + Y^B > 0$. Also assumed is an agreed upon mechanism which determines the game played by the household when deciding on its total consumption given the vector of private good prices $p \in \mathbb{R}^n_{++}$ and the vector of public good prices $P \in \mathbb{R}^m_{++}$. The first private good, assumed to be desired in any household environment, is taken as numéraire ($p_1 = 1$).

2.1 The Lindahl mechanism for cooperative household decisions

We start by supposing that there is initially enough cooperation so that the two spouses can agree on some mechanism to share the expenses for financing the public goods. The

[5] Another (global) approach to testability is the revealed preference approach. This approach has been introduced for the collective model by Cherchye et al. (2007) and Cherchye et al. (online 2012) and for the noncooperative model by Cherchye et al. (2011).

input of the mechanism is the announced voluntary contributions g_k^J and the desired aggregate quantities Q_k^J by each spouse J for each public good k. The mechanism will be based on the concept of Lindahl equilibrium, which is the best-known "decentralized" allocation mechanism[6] to allocate efficiently the cost of public goods within a group. The Lindahl scheme consists in supposing that there exists a pair of personalized (Lindahl) price vectors $\left(P^A, P^B\right) \in \mathbb{R}_+^{2m}$, satisfying $P^A + P^B = P$, which are posted within the household. However, the version we give of the concept is not the standard one. It is a "mechanism design" version *à la* Hurwicz (1979) and Walker (1981). In the present collective decision problem involving a husband-wife pair, we cannot use the personalized pricing rules proposed by Walker (1981) since they require at least three agents.[7] The mechanism we propose consists in posting for each spouse J ($J = A, B$) a *personalized pricing rule* $\widehat{P}^J$ for the public goods as a function of the voluntary contributions $g^{-J} \in \mathbb{R}_+^m$ and of the desired aggregate quantities $Q^{-J} \in \mathbb{R}_+^m$ chosen by the other spouse, as well as of the market prices of the public goods:

$$\widehat{P}_k^J\left(g_k^{-J}, Q_k^{-J}, P_k\right) \equiv \frac{Q_k^{-J} - g_k^{-J}}{Q_k^{-J}} P_k \text{ , for } J = A, B \text{and any public good } k. \tag{1}$$

Since the proportion $(Q_k^{-J} - g_k^{-J})/Q_k^{-J}$ depends exclusively upon the other spouse's strategies, which are taken as given by the Nash assumption, the personalized price P_k^J cannot be unilaterally manipulated by J. Given this mechanism, we can define a generalized[8] game where the payoffs are the spouses' utility functions. For each spouse J, the strategies $\left(q^J, g^J, Q^J\right)$, where $q^J \in \mathbb{R}_+^n$ is the vector of quantities of private goods, have to respect two constraints. First, there is a budget constraint on the spouse's expenses on private goods (at the market prices p) and on public goods (at the personalized prices P^J). Second, there is a feasibility constraint, whereby the desired quantities Q^J should be equal to the aggregate contributions $g^J + g^{-J}$. A Lindahl equilibrium for the household is then defined as the following social equilibrium of the generalized game defined by the personalized pricing rules $\left(\widehat{P}^A, \widehat{P}^B\right)$:

Definition 1. A vector $\left(q^A, g^A, Q^A, q^B, g^B, Q^B\right) \in \mathbb{R}_+^{2(n+2m)}$ is a *Lindahl household equilibrium* if the vector $\left(q^J, g^J, Q^J\right)$ solves, for each $J = A, B$, the program

$$\begin{aligned} &\max_{(\widetilde{q}^J, \widetilde{g}^J, \widetilde{Q}^J) \in \mathbb{R}_+^{n+2m}} U^J\left(\widetilde{q}^J, \widetilde{Q}^J\right) \\ \text{s.t.} \quad &p\widetilde{q}^J + \sum_{k=1}^m \frac{Q_k^{-J} - g_k^{-J}}{Q_k^{-J}} P_k \widetilde{Q}_k^J \leq Y^J \\ \text{and} \quad &\widetilde{Q}^J = \widetilde{g}^J + g^{-J}. \end{aligned} \tag{2}$$

This equilibrium is clearly an instance of Debreu (1952) social equilibrium.[9] Notice

[6] Introduced by Lindahl (1919) and popularized by Samuelson (1954). Lindahl prices to analyze household decisions are also used in Cherchye et al. (2007) and Cherchye et al. (online 2012). The data consistency of the collective model with public goods requires the existence of Lindahl prices such that individual behavior satisfies the generalized axiom of revealed preference.

[7] Notice that even if we want to extend our model to larger groups, there are more direct ways to do it than to use Walker (1981) formulation, which is based on a strategy space differing from the one we use.

[8] The game is "generalized" in the sense of Debreu (1952), since the choice of each spouse will be restricted to a subset of $\mathbb{R}_+^{n+2m}$ which depends upon the other spouse's choice. See also Yannelis (2009).

[9] Indeed, the payoff functions may be viewed as defined on the whole strategy space $\mathbb{R}_+^{2(n+2m)}$ even, out of equilibrium, at points where the strategies violate one or more of the four constraints.

further that we have $Q^A = g^A + g^B = Q^B$ at equilibrium, implying

$$P_k^J = \widehat{P}_k^J\left(g_k^{-J}, Q_k^{-J}, P_k\right) = \frac{g_k^J}{g_k^J + g_k^{-J}} P_k, \quad \text{for } J = A, B \text{ and any public good } k. \tag{3}$$

Hence, at a Lindahl household equilibrium, for any public good k, $P_k^A + P_k^B = P_k$, and g_k^A and g_k^B are either both positive or both nil.[10] This property ensures the equivalence of the present definition of the Lindahl household equilibrium and the standard definition of a Lindahl equilibrium, where the individualized contribution g_k^J of each spouse J is not introduced, his/her choice being limited to a desired total consumption Q_k^J of each public good k.

For the sake of later comparisons, the first-order conditions for a Lindahl household equilibrium (for $J = A, B$) can be easily derived (we can eliminate Q^J since $Q^J = g^J + g^{-J}$):

$$\begin{gathered}
\frac{1}{\partial_{q_1} U^J\left(q^J, g^J + g^{-J}\right)} \partial_q U^J\left(q^J, g^J + g^{-J}\right) \leq p, \\
\frac{1}{\partial_{q_1} U^J\left(q^J, g^J + g^{-J}\right)} \partial_Q U^J\left(q^J, g^J + g^{-J}\right) \equiv \tau^J\left(q^J, g^J + g^{-J}\right) \leq P^J, \\
pq^J + P^J\left(g^J + g^{-J}\right) = Y^J, \\
\text{with } P_k^J = \frac{g_k^J}{g_k^J + g_k^{-J}} P_k, \quad \text{for every public good } k,
\end{gathered} \tag{4}$$

an inequality becoming an equality for any private good i s.t. $q_i^J > 0$ or any public good k s.t. $g_k^J + g_k^{-J} > 0$. Under the last strict inequality, the Bowen–Lindahl–Samuelson condition for Pareto efficiency

$$\tau_k^A\left(q^A, g^A + g^B\right) + \tau_k^B\left(q^B, g^A + g^B\right) = P_k \tag{5}$$

holds for every public good k.

2.2 Taking into account individual autonomy

From the point of view of Nash implementation, the mechanism we have defined to implement the Lindahl equilibrium has good properties, since it involves non-manipulable personalized pricing rules and it ensures Pareto efficiency. Also, at equilibrium, we have $P^J Q^J = P g^J$; hence, $pq^J + P g^J = Y^J$ for each spouse J, implying that the chosen contributions could actually be bought autonomously by each spouse instead of being paid through the Lindahl transfers to the common purse. Of course, these contributions are also feasible for the household total budget. However, this autonomous feasibility may not hold out of equilibrium, where aggregate contributions can be unfeasible even for the household total budget.

If the spouses attach utmost importance to their individual autonomy and to budgetary feasibility, they may turn out playing a fully noncooperative game. This is the so-called *game with voluntary contributions to public goods* which has been analyzed for household behavior by Lundberg and Pollak (1993), Lechene and Preston (2005, 2011) and

[10] The case where spouse J would prefer to diminish the consumption of public good k at a Lindahl household equilibrium but cannot, because of a binding nonnegative constraint on g_k^J, is eliminated. Indeed, $g_k^J = 0$ and $g_k^{-J} > 0$ imply $P_k^J = 0$, and hence, a contradiction since $g_k^J = 0$ could not be optimal with $P_k^J = 0$, as $U^J\left(q^J, g^J + g^{-J}\right)$ is increasing in g_k^J.

Browning et al. (2010). In this game, each spouse J chooses a strategy $\left(q^J, g^J\right) \in \mathbb{R}_+^{n+m}$ (q^J denoting J's private consumptions and g^J his/her contributions to public goods) in order to solve the program:

$$\begin{aligned} &\max_{(q^J,g^J)\in\mathbb{R}_+^{n+m}} \; U^J\left(q^J, g^J + g^{-J}\right) \\ &\text{s.t.} \quad pq^J + Pg^J \leq Y^J. \end{aligned} \tag{6}$$

A Nash equilibrium of this game can be characterized by the first-order conditions (for $J = A, B$):

$$\begin{aligned} \frac{1}{\partial_{q_1} U^J\left(q^J, g^J + g^{-J}\right)} \partial_q U^J\left(q^J, g^A + g^B\right) &\leq p \\ \tau^J\left(q^J, g^A + g^B\right) &\leq P \\ pq^J + Pg^J &= Y^J, \end{aligned} \tag{7}$$

with an equality for any private good i s.t. $q_i^J > 0$ or any public good k s.t. $g_k^J > 0$. Clearly, the Bowen–Lindahl–Samuelson conditions are now generally violated, so that Pareto efficiency is lost. By contrast, individual autonomy is fully satisfied: Whatever the strategic choices of the two spouses, the two individual budget constraints, hence the household total budget constraint, are always satisfied.

Our objective is now to consider a more general model, where individual autonomy is preserved while some form of cooperation is admitted through the introduction of the same personalized pricing rules $(\widehat{P}^A, \widehat{P}^B)$. Each spouse J, when choosing for each public good k a voluntary contribution g_k^J together with a desired public consumption Q_k^J, should keep the budgetary freedom either to pay the transfer $P_k^J Q_k^J$ to the common purse or to spend instead $P_k g_k^J$ directly in the market.[11] In the new generalized game, we want to consider that these requirements will be incorporated into the individual budget constraints. The social equilibrium of this generalized game, called a *household equilibrium*, can be defined as follows:

Definition 2. A vector $\left(q^A, g^A, Q^A, q^B, g^B, Q^B\right) \in \mathbb{R}_+^{2(n+2m)}$ is a *household equilibrium* if, for $J = A, B$, the strategy $\left(q^J, g^J, Q^J\right)$ solves the program:

$$\max_{(\widetilde{q}^J,\widetilde{g}^J,\widetilde{Q}^J)\in\mathbb{R}_+^{n+2m}} \; U^J\left(\widetilde{q}^J, \widetilde{g}^J + g^{-J}\right) \tag{8}$$

$$\begin{aligned} &\text{s.t.} \quad p\widetilde{q}^J + \sum_{k=1}^{m} \max\left\{\frac{Q_k^{-J} - g_k^{-J}}{Q_k^{-J}} P_k \widetilde{Q}_k^J,\, P_k \widetilde{g}_k^J\right\} \leq Y^J, \\ &\text{and} \quad \widetilde{Q}^J = \widetilde{g}^J + g^{-J}. \end{aligned} \tag{9}$$

This modification of the game has considerable consequences. First, since we are again dealing with a generalized game, the constraints are not necessarily satisfied out of equilibrium. However, autonomous budgetary feasibility $pq^J + Pg^J \leq Y^J$ for $J = A, B$ is satisfied, in and out of equilibrium, for any admissible deviation g^{-J}, implying that $g^J + g^{-J}$ is budgetary feasible for the household. Second, as will be shown in the

[11] Some money management systems could facilitate this type of individual autonomy. Each spouse having its own bank account and the household having one joint account is such a system (called 'partial pool' by Vogler et al. 2006). Of course this is neither necessary nor sufficient to ensure individual autonomy.

following, there is now a continuum of household equilibria (that can be meaningfully parameterized), as opposed to the generic finiteness of the Lindahl equilibria. Third, it will follow that the set of equilibria includes the Lindahl equilibria at one extreme (the only efficient ones) and the noncooperative equilibria of the game with voluntary contributions to public goods at the other extreme.

2.3 Characterization of a household equilibrium

The concept of household equilibrium can be more easily characterized by using an alternative formulation. In the definition of a household equilibrium, each spouse J faces a non-smooth optimization program (because of the max function in the budget constraint). This program can be replaced by an equivalent smooth program with 2^m budget constraints indexed by $\delta^J \in \Delta^J \equiv \{0,1\}^m$:

$$p\widetilde{q}^J + \sum_{k=1}^{m} \left(\delta_k^J P_k \widetilde{g}_k^J + \left(1 - \delta_k^J\right) \frac{Q_k^{-J} - g_k^{-J}}{Q_k^{-J}} P_k \widetilde{Q}_k^J \right) \leq Y^J. \tag{10}$$

Each $\delta^J \in \Delta^J$ describes a configuration specifying the public goods for which spouse J goes to the market ($\delta_k^J = 1$) and those for which J accepts to contribute to the common purse ($\delta_k^J = 0$). All such configurations have to be budgetary feasible for spouse J. We then have the following result:

Lemma 1. *A household equilibrium is characterized by the following system of inequalities and equalities for $J = A, B$:*

$$\frac{1}{\partial_{q_1} U^J \left(q^J, g^J + g^{-J}\right)} \partial_q U^J \left(q^J, g^J + g^{-J}\right) \leq p \tag{11}$$

and, for all k and some $\theta_k^J \in [0,1]$,

$$\tau_k^J \left(q^J, g^J + g^{-J}\right) \leq \theta_k^J P_k + \left(1 - \theta_k^J\right) P_k^J, \quad \textit{with } P_k^J = \frac{g_k^J}{g_k^J + g_k^{-J}} P_k, \quad \textit{and}$$

$$pq^J + \sum_{k=1}^{m} \left(\theta_k^J P_k g_k^J + \left(1 - \theta_k^J\right) P_k^J \left(g_k^J + g_k^{-J}\right) \right) = Y^J, \tag{12}$$

an inequality becoming an equality for any private good i s.t. $q_i^J > 0$ or any public good k s.t. $Q_k^J > 0$.

Proof. Suppose that, in the definition of a household equilibrium, we replace the budget constraint in program (8) by the set of constraints (10). The program of each spouse consists then in maximizing a strongly quasi-concave utility function under linear constraints. Taking first-order conditions, we immediately get the wanted inequalities for the private goods and, for any public good k and spouse J, for some vector of Lagrange multipliers $\lambda_{\delta^J}^J \in \mathbb{R}_+^{2^m}$,

$$\begin{aligned}
&\frac{1}{\partial_{q_1} U^J(q^J, g^J + g^{-J})} \partial_{Q_k} U^J(q^J, g^J + g^{-J}) \\
&\quad \leq \frac{1}{\sum\limits_{\delta^J \in \Delta^J} \lambda_{\delta^J}^J} \left(\sum_{\delta^j \in \{\Delta^J : \delta_k^J = 1\}} \lambda_{\delta^J}^J P_k + \sum_{\widehat{\delta}^J \in \{\Delta^J : \widehat{\delta}_k^J = 0\}} \lambda_{\widehat{\delta}^J}^J \widehat{P}_k^J \left(g_k^{-J}, Q_k^{-J}, P_k\right) \right) \\
&\quad \equiv \theta_k^J P_k + \left(1 - \theta_k^J\right) P_k^J \text{ (with } \theta_k^J \text{ the normalized Lagrange multiplier),}
\end{aligned}$$

so that we get the wanted inequality for public good k using the fact that $\widehat{P}_k^J(g_k^{-J}, Q_k^{-J}, P_k) = P_k^J$ at equilibrium. Finally, because $P_k^J\left(g_k^J + g_k^{-J}\right) = P_k g_k^J$ for every k, we get the equality

$$pq^J + \sum_{k=1}^{m}\left(\theta_k^J P_k g_k^J + \left(1-\theta_k^J\right)P_k^J\left(g_k^J + g_k^{-J}\right)\right) = Y^J,$$

and the result follows. □

Comparing the characterization of a household equilibrium with the one of a Lindahl household equilibrium, we immediately see that the two coincide for $\theta_k^J = 0$, for all k and $J = A, B$. Similarly, the household equilibrium coincides with the Nash equilibrium of the game with voluntary contributions to public goods for $\theta_k^J = 1$, for all k and $J = A, B$. But there are many such equilibria, at least one for each $\theta = \left(\theta^A, \theta^B\right) \in [0,1]^{2m}$, as it will be shown in the next Subsection. Considering the "smooth" definition of a household equilibrium with 2^m budget constraints (10) for $J = A, B$, we see that each θ_k^J is the (normalized) sum of the shadow prices associated with those J's budget constraints corresponding to an autonomous purchase of public good k in the market ($\delta_k^J = 1$). Each θ_k^J can thus be interpreted as the price J would be ready to pay to relax anyone of the constraints. In that sense, θ_k^J measures the *degree of autonomy* of spouse J with respect to public good k. This interpretation is reinforced by the way the budget constraint is written in the characterization of a household equilibrium given in the lemma. The expenses of spouse J to finance its contribution to public good k are divided into two parts, one autonomous valued at market price P_k, and the other collective valued at the personalized price P_k^J. This suggests an alternative approach to the household game.

2.4 The game with given degrees of autonomy

In our analysis, up to now, the degrees of autonomy of both spouses are fixed endogenously, ex post, as characteristics of a specific equilibrium. We may, however, invert the approach and take the parameters as preliminarily and conventionally fixed ex ante within the household. The idea is that the expenses of each spouse on each public good can be divided into two parts, one autonomously fixed and the other collective according to the mechanism. This mechanism remains the same, as well as the strategies and the payoffs. This leads to the following definition of a social equilibrium relative to fixed degrees of autonomy $(\theta^A, \theta^B) \in [0,1]^{2m}$ and based on the personalized pricing rules $(\widehat{P}^A, \widehat{P}^B)$:

Definition 3. A vector $\left(q^A, g^A, Q^A, q^B, g^B, Q^B\right) \in \mathbb{R}_+^{2(n+2m)}$ is a *household θ-equilibrium* with degrees of autonomy $\left(\theta^A, \theta^B\right) \in [0,1]^{2m}$ if, for $J = A, B$, the strategy $\left(q^J, g^J, Q^J\right)$ solves the program:

$$\begin{aligned}
&\max_{(\widetilde{q}^J, \widetilde{g}^J, \widetilde{Q}^J) \in \mathbb{R}_+^{n+2m}} U^J\left(\widetilde{q}^J, \widetilde{Q}^J\right)\\
\text{s.t.}\quad & p\widetilde{q}^J + \sum_{k=1}^{m}\left(\theta_k^J P_k \widetilde{g}_k^J + \left(1-\theta_k^J\right)\frac{Q_k^{-J} - g_k^{-J}}{Q_k^{-J}} P_k \widetilde{Q}_k^J\right) \leq Y^J,\\
\text{and}\quad & \widetilde{Q}^J = \widetilde{g}^J + g^{-J}.
\end{aligned} \tag{13}$$

In this concept, which appears more appropriate in view of empirical testing, the division into two parts of the expenses of each spouse on each public good is made explicit through the budget constraint, according to the preestablished degrees of autonomy for

that public good. The following proposition clarifies the relationship between the two concepts, that of household equilibrium on the one hand and of household θ-equilibrium on the other.

Proposition 1. *A vector* $\left(q^A, g^A, Q^A, q^B, g^B, Q^B\right) \in \mathbb{R}_+^{2(n+2m)}$ *with associated normalized Lagrange multipliers* $\theta \in [0,1]^{2m}$ *is a household equilibrium for the personalized pricing rules* $\left(\widehat{P}^A, \widehat{P}^B\right)$ *if and only if this vector is a household* θ*-equilibrium for the same* θ*. For each* $\theta \in [0,1]^{2m}$*, a* θ*-equilibrium exists. When* $\theta_k^J = 0$ *for any* J *and any* k*, the household* θ*-equilibrium is a Lindahl household equilibrium. When* $\theta_k^J = 1$ *for any* J *and any* k*, the household* θ*-equilibrium is a Nash equilibrium of the game with voluntary contributions to public goods.*

Proof. In both cases, the program of each spouse consists in maximizing a strongly quasi-concave utility function under linear constraints. Taking the first-order conditions for a household θ-equilibrium with degrees of autonomy $\left(\theta^A, \theta^B\right)$, we see that they coincide with those for a household equilibrium with associated normalized Lagrange multipliers $\left(\theta^A, \theta^B\right)$, as given by Lemma 1. These conditions also coincide with the first-order conditions for a Lindahl household equilibrium, as given by (4), when $\theta_k^J = 0$ for any J and k, and with the first-order conditions for a Nash equilibrium of the game with voluntary contributions to public goods, as given by (7), when $\theta_k^J = 1$ for any J and k.

As to the existence of a household θ-equilibrium for every $\theta \in [0,1]^{2m}$, the strategies $\left(q^J, g^J, Q^J\right)$ of J are constrained to be in a compact, convex, nonempty set S^J defined by $0 \leq q_i^J \leq Y^J/p_i$ for $i = 1, \ldots, n$, and $0 \leq g_k^J \leq Q_k^J \leq Y/P_k$ for $k = 1, \ldots, m$. We further define for each spouse J a choice correspondence associating with each strategy $\left(q^{-J}, g^{-J}, Q^{-J}\right) \in S^{-J}$, the subset of S^J satisfying $Q^J = g^J + g^{-J}$ and

$$pq^J + \sum_{k=1}^{m} \left(\theta_k^J P_k g_k^J + \left(1 - \theta_k^J\right) P_k \frac{Q_k^{-J} - g_k^{-J}}{Q_k^{-J}} \left(g_k^J + g_k^{-J}\right) \right) \leq Y^J.$$

This correspondence is continuous, and nonempty convex valued. Since the payoff functions U^J are continuous and quasi-concave, Debreu's (1952) social equilibrium existence theorem applies. □

Notice that, whenever it exists, a household θ-equilibrium with *separate spheres*, namely with $g_k^A g_k^B = 0$ for all k, coincides with an equilibrium of the game with voluntary contributions to public goods. To see this, suppose a separate sphere equilibrium with $g_k^{-J} = 0$ whereas $g_k^J > 0$, implying $P_k^{-J} = 0$ and $P_k^J = P_k$. At such an equilibrium, the unit expenditure on public good k for a deviation by spouse J would be $\theta_k^J P_k + \left(1 - \theta_k^J\right) P_k = P_k$, the same unit expenditure for a deviation in the game with voluntary contributions to public goods. As to spouse $-J$, who decides not to contribute to public good k when the unit expenditure on this good would be $\theta_k^{-J} P_k$, he/she would a fortiori not want to deviate in the game with voluntary contributions to public goods where this unit expenditure would be $P_k \geq \theta_k^{-J} P_k$. This proves the following proposition.

Proposition 2. *Let* $0 < \theta_k^J \leq 1$ *for any* J *and* k*. Suppose* $(q^A, g^A, Q^A, q^B, g^B, Q^B) \in \mathbb{R}_+^{2(n+2m)}$ *is a household* θ*-equilibrium such that* $g_k^A g_k^B = 0$ *for all* k *(separate spheres). Then,* $\left(q^A, g^A, q^B, g^B\right)$ *is a Nash equilibrium of the game with voluntary contributions to public goods.*

A last remark in this subsection concerns Pareto efficiency. Take the first-order conditions relative to the public good k for both spouses' programs, as given by Lemma 1. For efficiency, the Bowen–Lindahl–Samuelson condition requires that the sum $\tau_k^A + \tau_k^B$ of the two marginal willingnesses to pay be equal to the corresponding market price $P_k = P_k^A + P_k^B$. This condition is generally violated as soon as cooperation is less than full. Indeed, the sum of the two marginal willingnesses to pay is equal, if both spouses contribute to public good k, to $P_k + \theta_k^A P_k^B + \theta_k^B P_k^A$, larger than P_k outside the case $\theta_k^A = \theta_k^B = 0$, and the more so the higher the degrees of autonomy of the two spouses. Also, if a spouse, say the wife, contributes alone to public good k, $\tau_k^A = P_k$, so that $P_k < \tau_k^A + \tau_k^B$, leading to a similar conclusion.

2.5 An example

In order to discuss the conditions leading to the regimes of separate spheres and of joint contribution to one or more public goods, we shall refer to the example used by Browning et al. (2010) in the context of full noncooperation. In this example, the spouses have Cobb–Douglas preferences over one private and two public goods. We denote by x and z the private consumptions of spouses A and B, respectively, and by X and Z the quantities of the two public goods. The utility functions are given by:

$$U^A(x, X, Z) = xX^a Z^\alpha \text{ and } U^B(z, X, Z) = zX^b Z^\beta, \tag{14}$$

with positive parameter values a, α, b and β. The wife A is supposed to care more about the first public good and the husband B about the second, so that

$$s \equiv \frac{\alpha/a}{\beta/b} < 1, \tag{15}$$

where the term on the LHS can be taken as the *degree of symmetry* of the spouses' preferences for the two public goods. We use the normalization

$$p_x = p_z = P_X = P_Z = Y = 1, \tag{16}$$

with an income distribution given by $Y^A = \rho$ and $Y^B = 1 - \rho$. The degrees of autonomy are assumed to be symmetric with respect to the two public goods, so that they will be simply denoted θ^A and θ^B.

Browning et al. (2010) show the existence of three kinds of regimes. Two kinds correspond to separate spheres, prevailing both for extremely unequal income distributions, where the spouse with the higher income contributes alone to both public goods, and for relatively equal income distributions, where each spouse contributes to her/his preferred public good. The third kind appears in intermediate cases of income distribution and is characterized by separate spheres up to one public good (and local income pooling): The spouse with the higher income contributes to both public goods, while the other spouse contributes solely to her/his preferred public good. A fourth kind of regime, with both spouses contributing to both public goods, is generically excluded under full noncooperation, but not in the semi-cooperative case, as we shall show.

– Consider the regime of *separate spheres* where both spouses contribute, A to her preferred public good (X) and B to his (Z). Hence, $P_Z^A = P_X^B = 0$ and $P_X^A = P_Z^B = 1$. By the first-order conditions for public goods,

$$\begin{aligned} ax/X = \theta^A + \left(1 - \theta^A\right) P_X^A = 1,\ \alpha x/Z < \theta^A + \left(1 - \theta^A\right) P_Z^A = \theta^A, \\ \beta z/Z = \theta^B + \left(1 - \theta^B\right) P_Z^B = 1,\ bz/X < \theta^B + \left(1 - \theta^B\right) P_X^B = \theta^B. \end{aligned} \tag{17}$$

Using the equilibrium budget equations

$$x + X = \rho,\ z + Z = 1 - \rho, \tag{18}$$

we easily obtain the solution

$$x = \frac{\rho}{1+a},\ X = \frac{a\rho}{1+a},\ z = \frac{1-\rho}{1+\beta},\ Z = \frac{\beta\,(1-\rho)}{1+\beta}. \tag{19}$$

This solution is constrained by the two first-order conditions on the noncontributed goods, expressed as inequalities:

$$\frac{\alpha}{\beta}\,\frac{1+\beta}{1+a}\,\frac{\rho}{1-\rho} < \theta^A,\ \frac{b}{a}\,\frac{1+a}{1+\beta}\,\frac{1-\rho}{\rho} < \theta^B. \tag{20}$$

Clearly, one of these two conditions will be violated for small enough or large enough values of ρ, so that separate spheres with both spouses contributing to public consumption can prevail only if the income distribution between the two spouses is not too unequal. Also, by multiplying both sides of the first inequality by the corresponding sides of the second, we obtain

$$0 < s < \theta^A\theta^B \leq 1. \tag{21}$$

Hence, the existence of this regime requires a relatively high average degree of autonomy of the two spouses, the higher this degree of autonomy, the larger the degree of symmetry of their preferences for the two public goods. The fully noncooperative case, where $\theta^A\theta^B = 1$, always satisfies this condition, provided there is no full symmetry in the spouses' preferences.

– Now, consider the regime of *joint contribution to both public goods*, which is generically excluded under full autonomy of the two spouses. By first-order conditions, the marginal willingnesses to pay for the public goods are as follows:

$$\begin{aligned} ax/X &= \theta^A + \left(1-\theta^A\right) P_X^A,\ \alpha x/Z = \theta^A + \left(1-\theta^A\right) P_Z^A, \\ \beta z/Z &= \theta^B + \left(1-\theta^B\right) P_Z^B,\ bz/X = \theta^B + \left(1-\theta^B\right) P_X^B. \end{aligned} \tag{22}$$

Division of both sides of the second and third equations by the corresponding sides of the first and fourth, respectively, leads to

$$\frac{a}{\alpha}\theta^A < \frac{a}{\alpha}\,\frac{\theta^A + (1-\theta^A)\,P_Z^A}{\theta^A + (1-\theta^A)\,P_X^A} = \frac{X}{Z} = \frac{b}{\beta}\,\frac{\theta^B + (1-\theta^B)\,P_Z^B}{\theta^B + (1-\theta^B)\,P_X^B} < \frac{b}{\beta}\,\frac{1}{\theta^B}, \tag{23}$$

the two inequalities being easily checked to be true (by taking the extreme values $P_X^A = 1$ and $P_Z^A = 0$). We thus obtain

$$0 \leq \theta^A\theta^B < s < 1, \tag{24}$$

an existence condition just opposite to the one we found for separate spheres. For both spouses to contribute to both public goods, their average degree of autonomy must be small enough, the smaller this average degree of autonomy, the more asymmetric their preferences for the public goods.

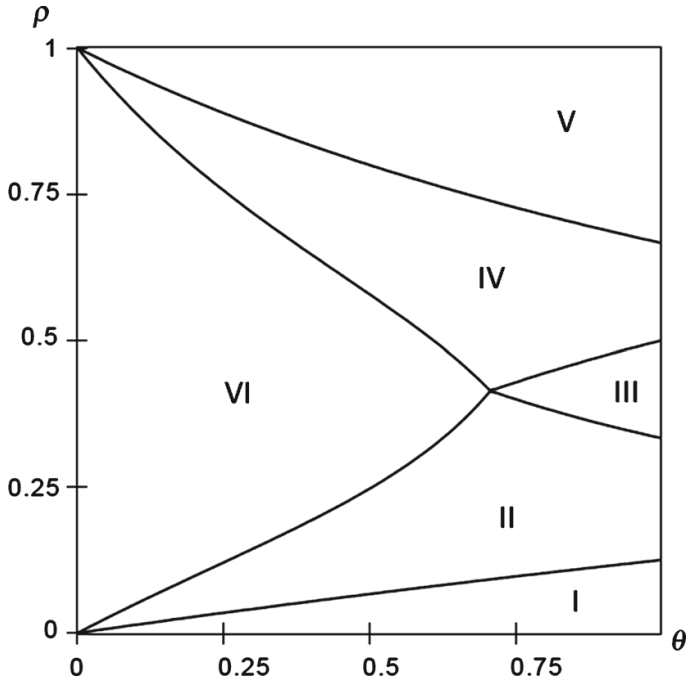


Figure 1: Regime switching values of ρ as θ varies

Thus, for not too unequal income distributions, separate spheres appear as a characteristic of high individual autonomy in household decisions. As the spouses become less and less autonomous, the regime prevailing when their incomes are not too different is rather the one where both contribute to both public goods, which is the rule under full cooperation.

More generally, in order to represent the parameter configurations leading to the different regimes, we take the same values as in Browning et al. (2010), namely $a = 5/3$, $\alpha = 8/9$, $b = 15/32$ and $\beta = 1/2$, leading to a degree of symmetry equal to 0.5, and we stick now to equal degrees of autonomy $\theta^A = \theta^B = \theta$. This is given in Fig. 1, with θ and ρ varying from 0 to 1 along the horizontal and the vertical axes, respectively. Six different regimes are possible: (I) where B is the only spouse to contribute to (both) public goods, (II) where A contributes to her preferred public good and B still contributes to both, (III) where each spouse specializes on his/her preferred public good (separate spheres with both spouses contributing), (IV) and (V) symmetric to (II) and (I), respectively (with inverted roles of A and B), and (VI) where both spouses contribute to both public goods.

As already mentioned, we see that the regime (III) of separate spheres exists only for a sufficiently high degree of autonomy (higher than the square root of the degree of symmetry $\sqrt{1/2} = 0.707$) and for relatively equal income shares. The corresponding regime (VI) for a lower degree of autonomy is the one where both spouses contribute to both public goods, allowing for more and more income disparities as we approach full cooperation. By contrast, the regimes (I) and (V) of exclusive contribution to public spending by the richer spouse are compatible with a lower and lower amplitude in income distribution as we approach full cooperation.

3 Local analysis of household θ-equilibria

We shall now successively examine the local properties of the system of equations characterizing a household θ-equilibrium (extending the analysis in Browning et al. 2010), and the local properties of the associated household demand function (extending the analysis in Browning and Chiappori 1998; Lechene and Preston 2011).

3.1 Local determinacy

For an environment (p, P, Y) and an income distribution $\left(Y^A, Y^B\right)$, take a household θ-equilibrium $\left(q^A, g^A, Q^A, q^B, g^B, Q^B\right) \in \mathbb{R}_+^{2(n+2m)}$ and refer to the partition $\left\{M^A, M^B, M^{AB}, M^0\right\}$ of the set M of public goods, where M^A and M^B are the subsets of goods exclusively contributed by spouses A and B, respectively, M^{AB} is the subset of goods to which both spouses contribute, and M^0 is the subset of goods that are not consumed by the household at this equilibrium. Denote by m^A, m^B, m^{AB} and m^0 the cardinals of the respective subsets in this partition.

The variables Q^A and Q^B can be ignored in the following, since they are equal to the sum $g^A + g^B$. Also, $m^A + m^B + 2m^0$ variables characterizing the equilibrium are trivially determined, namely $g_k^J = 0$ for $k \in M^{-J} \cup M^0$, $J = A, B$. Thus, we are left with $2n + 2m - \left(m^A + m^B + 2m^0\right)$ unknowns. There are 2 budget equations, $2(n-1)$ equations expressing the first-order conditions for the private goods,[12] and $m^A + m^B + 2m^{AB}$ equations expressing the first-order conditions for public goods. Hence, the number of unknowns is equal to the number of equations, so that an equilibrium is (generically) locally determinate.

Now, consider the consequences of full autonomy for both spouses relative to some subset of jointly contributed public goods $M^\theta \subset M^{AB}$ (with cardinal m^θ): For any $k \in M^\theta$, $\theta_k^A = \theta_k^B = 1$. Observe that the contributions to public consumption of $k \in M^\theta$ appear in the first-order conditions only through the sums $g_k^A + g_k^B$. Then, the equation system reduced to the $2(n-1) + m^A + m^B + 2m^{AB}$ first-order conditions has only $2n + m^A + m^B + 2m^{AB} - m^\theta$ unknowns, namely q_i^J (for $J = A, B$ and $i = 1, \ldots, n$), g_k^J (for $J = A, B$ and $k \in M^J \cup (M^{AB} \setminus M^\theta)$) and $g_k^A + g_k^B$ (for $k \in M^\theta$). Hence, there is an excess of the number of equations over the number of unknowns equal to $m^\theta - 2$, so that there is generically *over*determinacy if $m^\theta > 2$, or even if $m^\theta = 2$, since the household consumption is entirely determined in this case by the sole first-order conditions, independently of any budget constraint. Determinacy thus implies either $m^\theta = 0$ or $m^\theta = 1$: There is at most one public good for which the autonomy of the two spouses is full, which is jointly contributed. If there is one, say good k, we may ignore the decomposition of Q_k into g_k^A and g_k^B and completely determine the system by just adding the *household* budget equation. In other words, there is *local income pooling*.

This analysis can be straightforwardly applied to the game with voluntary contributions to public goods (where $m^\theta = m^{AB}$). Generically, either $m^{AB} = 0$ (*separate spheres*), with the two individual budget equations making the system determinate, or $m^{AB} = 1$ (*separate spheres up to one public good*), with *local income pooling*, as shown by Browning et al. (2010).

[12] We are assuming for simplicity that all the n private goods are consumed by both spouses at the equilibrium we refer to.

3.2 Foundations of the spouses' demand functions

In order to pursue our local analysis, let us establish, for exogenous degrees of autonomy, the foundations of the spouses' demand functions to be aggregated into the household demand function. The Marshallian demand function of spouse $J \in \{A, B\}$, conditional to a given choice $g^{-J} \in \mathbb{R}_+^m$ of the other spouse, can be straightforwardly derived from his/her utility maximization program:

$$\begin{aligned} x^J\left(p, \mathcal{P}^J, \mathcal{Y}^J, g^{-J}\right) \equiv \arg \max_{(q^J, g^J) \in \mathbb{R}_+^{n+m}} & U^J\left(q^J, g^J + g^{-J}\right) \\ & pq^J + \mathcal{P}^J g^J \leq \mathcal{Y}^J, \end{aligned} \tag{25}$$

with $\mathcal{P}_k^J \equiv \theta_k^J P_k + (1 - \theta_k^J) P_k^J = P_k - (1 - \theta_k^J) P_k^{-J}$ for $k = 1, \ldots, m$, and $\mathcal{Y}^J \equiv Y^J - \sum_k (1 - \theta_k^J) P_k^J g_k^{-J}$, equal at a household θ-equilibrium to $Y^J - \sum_k (1 - \theta_k^J) P_k^{-J} g_k^J$.

We shall limit our local analysis to an open set $\Omega \subset \mathbb{R}_+^{n+m+1}$ of environment values $\omega \equiv (p, P, Y)$. We assume that the sharing rule $\rho \in (0, 1)$, defined by $\left(Y^A, Y^B\right) \equiv \left(\rho^A, \rho^B\right) Y \equiv (\rho, 1 - \rho) Y$, and the degrees of autonomy $\left(\theta^A, \theta^B\right) \in (0, 1)^{2m}$ of the two spouses are differentiable functions of the environment, which are defined on Ω. We further assume equilibrium *uniqueness*,[13] as well as *no regime switching* over Ω:[14] The private goods purchased by each spouse and the public goods to which she/he *actually* contributes (corresponding to the nonzero elements of equilibrium vectors (q^A, g^A) and (q^B, g^B)) are the same for each element of Ω. This allows us to refer to differentiable functions $g^J : \Omega \to \mathbb{R}_+^m$ $(J = A, B)$ associating with each environment value ω the equilibrium values of the individual contributions $g^J(\omega)$. Because of no regime switching, we may admit without loss of generality (by simple redefinition of the utility function) that all private and public goods are consumed at equilibrium. We keep the notation of the preceding subsection: M^J and M^{AB} for the sets of public goods contributed by spouse J and by both spouses, respectively, m^J and m^{AB} for the corresponding cardinals (with $m^A + m^B + m^{AB} = m$).

By differentiability of the functions ρ, θ^J and g^J (implying that of the functions P^J with $P_k^J(\omega) = P_k g_k^J(\omega) / (g_k^J(\omega) + g_k^{-J}(\omega))$), we obtain from the differentiable Marshallian conditional demand functions x^J, as defined by (25), differentiable functions $\xi^J : \Omega \to \mathbb{R}_+^{n+m}$, depending directly upon the environment. We take their sum $\xi = \xi^A + \xi^B$ as the household demand function.

3.3 The pseudo-Slutsky matrix of household demand at a household θ-equilibrium

At any equilibrium, we can straightforwardly compute the pseudo-Slutsky matrix $\Psi = \left[\partial_{(p,P)} \xi\right] + \left[\partial_Y \xi\right] \left[{}^T\left(x^A + x^B\right)\right]$ of the household demand function as the sum of the two corresponding matrices $\Psi^J = \left[\partial_{(p,P)} \xi^J\right] + \left[\partial_Y \xi^J\right] \left[{}^T\left(x^A + x^B\right)\right]$ of the individual demand functions ξ^J, for $J = A, B$. By using the notations $\widetilde{P}_k^{-J} = \left(1 - \theta_k^J\right) P_k^{-J}$, so that $\mathcal{P}^J = P - \widetilde{P}^{-J}$ and $\mathcal{Y}^J = \rho^J Y - \widetilde{P}^{-J} g^J$, $\mathbf{\Pi}^J \equiv \left[\partial_{(p,P)} \widetilde{P}^J\right] + \left[\partial_Y \widetilde{P}^J\right] \left[{}^T\left(x^A + x^B\right)\right]$, $\mathbf{\Gamma}^J \equiv \left[\partial_{(p,P)} g^J\right] + \left[\partial_Y g^J\right] \left[{}^T\left(x^A + x^B\right)\right]$ and $\mathbf{R} = \left[\partial_{(p,P)} \rho\right] + (\partial_Y \rho) \left[{}^T\left(x^A + x^B\right)\right]$, we

[13] Lechene and Preston (2011), studying the fully noncooperative case, also rely on the uniqueness assumption.

[14] Notice that no regime switching holds generically for Ω small enough.

obtain:

$$\begin{aligned}\Psi = &\overbrace{\left[\partial_{(p,\mathcal{P})}x^A\right] + \left[\partial_{\mathcal{Y}}x^A\right]\left[{}^T x^A\right]}^{\Sigma^A} + \overbrace{\left[\partial_{(p,\mathcal{P})}x^B\right] + \left[\partial_{\mathcal{Y}}x^B\right]\left[{}^T x^B\right]}^{\Sigma^B} \\ &- \overbrace{\left(\left[\partial_{\mathcal{Y}}x^A\right] - \left[\partial_{\mathcal{Y}}x^B\right]\right)\left[{}^T\left(\rho^B x^A - \rho^A x^B\right) - Y\mathbf{R}\right]}^{\Delta} \\ &+ \overbrace{\begin{pmatrix}\left(\left[\partial_g x^A\right] - \left[\partial_{\mathcal{Y}}x^B\right]\left[\widetilde{P}^A\right]\right)\mathbf{\Gamma}^B + \\ \left(\left[\partial_g x^B\right] - \left[\partial_{\mathcal{Y}}x^A\right]\left[\widetilde{P}^B\right]\right)\mathbf{\Gamma}^A\end{pmatrix}}^{\Xi} \\ &- \overbrace{\begin{pmatrix}\left(\left[\partial_{\mathcal{P}}x^A\right] + \left[\partial_{\mathcal{Y}}x^A\right]\left[{}^T g^A\right]\right)\mathbf{\Pi}^B + \\ \left(\left[\partial_{\mathcal{P}}x^B\right] + \left[\partial_{\mathcal{Y}}x^B\right]\left[{}^T g^B\right]\right)\mathbf{\Pi}^A\end{pmatrix}}^{\Theta}.\end{aligned} \tag{26}$$

The two matrices Σ^A and Σ^B are the Slutsky matrices of the individual demand functions ξ^A and ξ^B. Their sum Σ has all the properties of a Slutsky matrix. However, the pseudo-Slutsky matrix Ψ differs from Σ by a deviation matrix $\Sigma - \Psi = \Delta - \Xi + \Theta$, which is the sum of three matrices. The first one, Δ, the only element of $\Sigma - \Psi$ in the fully cooperative case, is an outer product, with rank no larger than 1, as shown by Browning and Chiappori (1998). It expresses an *aggregation effect*, working in the general case where there is no "representative consumer", independently of the existence of public goods. The second matrix, $-\Xi$, expresses *externality effects*, working through the presence of the contributions of spouse $-J$ in the second argument of the utility function of spouse J (leading to the products $\left[\partial_g x^J\right]\mathbf{\Gamma}^{-J}$, $J = A, B$). These effects appear, in addition to the aggregation effect, in the fully noncooperative case and have been analyzed by Lechene and Preston (2011). They extend in our framework, through the personalized prices, to the net income $\mathcal{Y}^J$ (adding $-\left[\partial_{\mathcal{Y}}x^J\right]\left[\widetilde{P}^{-J}\right]\mathbf{\Gamma}^J$, $J = A, B$, to the former product). The third matrix, Θ, expresses the sum of the *substitution and income effects* of changes in the transformed personalized prices $\widetilde{P}^J$ (expressed by $\left[\partial_{\mathcal{P}}x^J\right]\mathbf{\Pi}^{-J}$ and $\left[\partial_{\mathcal{Y}}x^J\right]\left[{}^T g^J\right]\mathbf{\Pi}^{-J}$, respectively, with $J = A, B$).

The following proposition establishes the maximum rank of the deviation matrix $\Sigma - \Psi$ at different regimes.

Proposition 3. *At a household θ-equilibrium associated with an environment $\omega = (p, P, Y)$, the household demand function has a pseudo-Slutsky matrix Ψ which deviates from a Slutsky matrix $\Sigma = \Sigma^A + \Sigma^B$: (i) under separate spheres, by a matrix $\Delta - \Xi$ of rank at most equal to*

$$r_{\Delta-\Xi} = m + \min\left\{n - \max\left\{m^A - m^B, 1\right\}, 1\right\},$$

where $m^A - m^B$ is assumed nonnegative WLOG; (ii) under joint contribution to m^{AB} public goods, by a matrix $\Delta - \Xi + \Theta$ of rank at most equal to

$$r_{\Delta-\Xi+\Theta} = m + 3m^{AB} + \min\left\{n - \max\left\{m^A - m^B - 1, 1 + 3m^{AB}\right\}, 1\right\}.$$

Proof. See Appendix. □

As it will become apparent by the reading of the proof, the increase in the maximum rank of the deviation matrix when going from the regime of separate spheres to a regime of joint contribution to m^{AB} public goods is partly due to the substitution and income effects of the changes in personalized prices, expressed by the matrix Θ, of maximum rank $2m^{AB}$. It is also partly due to the utility externality effects, responsible for an increase of m^{AB} in the maximum rank of matrix Ξ, and to the budget externality effects, responsible for a supplementary unit increase.[15]

3.4 Empirical testing

The upper bound imposed upon the rank of the deviation matrix can be used to test the different models of household behavior. Browning and Chiappori (1998) have used this upper bound to discriminate between the unitary model (which predicts that the matrix $\Psi - \left({}^{T}\Psi\right)$ has rank 0 because of the symmetry of $\Psi = \Sigma$) and the collective model (which predicts that $\Psi - \left({}^{T}\Psi\right)$ has rank at most 2, since $\Psi = \Sigma - \Delta$, Δ having a rank at most equal to 1). They have also shown that this test requires at least 5 goods, a requirement that stems from the fact that the rank of $\Psi - \left({}^{T}\Psi\right)$ cannot be higher than 2 if the number $n + m$ of goods is not larger than 4 (given the linear dependence of the columns of Ψ introduced by the homogeneity of degree zero of the demand functions).

Lechene and Preston (2011) have shown that, in order to reject the noncooperative model, one must have $n \geq m + 5$. Their Lemma A.1 shows indeed that if $\Psi - \left({}^{T}\Psi\right)$ has rank at most $n+m-1$, then Ψ can always be expressed as the sum of a symmetric matrix and a matrix of rank not higher than r such that $2r + 1 \geq n + m - 1$. For $r = m + 1$, as in the noncooperative case, the test works only if $2(m + 1) + 1 < n + m - 1$, that is, if $n > m + 5$.

As previously emphasized, there is no possibility of discriminating between full and partial autonomy under separate spheres, since the noncooperative and the semi-cooperative models are then observationally equivalent. Discrimination between the two models becomes possible under joint contribution to at least one public good. In the semi-cooperative case, if we apply Lechene and Preston (2011) Lemma A.1 to the preceding Proposition, with $r = m + 3m^{AB} + 1$, we see that $n > m + 2\left(3m^{AB} + 2\right)$ is needed to discriminate between full and partial autonomy.[16] Moreover, as soon as there is joint contribution to more than one public good,[17] the noncooperative model is generically excluded, and full cooperation is still rejected at a rank of $\Psi - \left({}^{T}\Psi\right)$ larger than 4, leaving the possibility of semi-cooperation.

[15] However, in the case of symmetry of the two spouses' degrees of autonomy with respect to the jointly contributed public goods, there is no increase in the maximum rank of matrix Ξ. This can be shown by supposing a reallocation of the spouses' contributions, accompanied by a compensating income transfer between the spouses, so as to recover separate spheres without changing the household's demand. Hence, the difference in the maximum rank of the deviation matrix relative to its value under separate spheres comes, in this case, solely from the matrix Θ.

[16] If, for instance, we are in the special case of only one public good to which both spouses contribute, at least 12 private goods are required. The maximum possible rank of $\Psi - \left({}^{T}\Psi\right)$, given homogeneity of degree 0 of the demand functions, is then 12. As the observed rank increases from 0 to 10, the test successively rejects the unitary model (at 2), full cooperation (at 4), full autonomy (at 6) and the semi-cooperative model as a whole (at 12).

[17] In a recent empirical application of the noncooperative model to data drawn from the Russia Longitudinal Monitoring Survey, Cherchye et al. (2011) could not reject joint contributions to two or three public goods by some households.

4 Conclusion

Our purpose in this paper has been to develop models of the household that do not impose collective efficiency. This assumption is often contradicted by empirical evidence.[18] Also, noncooperation has not been restricted to the pure voluntary contributions model, the consequences of which seem to be rather special (separate spheres or separate spheres up to one public good). However, collective efficiency and pure noncooperative behavior are not excluded but included as particular models in a more comprehensive approach trying to give account of the large variety of formal and informal contractual arrangements and decision procedures that are used by households all over the world. The concept of household equilibrium is meant to be flexible by allowing the household members to adopt various degrees of autonomy, either determined endogenously as characteristics of a specific equilibrium among many others, or taken as parameters preliminarily and conventionally fixed within the household and reflecting some selection process.

Another objective was to derive testable (local) restrictions at equilibrium in a way that incorporates the similar testable restrictions already derived for the collective and the noncooperative models. The paper provides a unified and compact derivation of the pseudo-Slutsky matrix allowing to separate different effects: an "aggregation effect", the only one working in the collective model, "externality effects" appearing with noncooperation and, finally, "substitution and income effects" due to partial autonomy. Each effect increases the maximum possible rank of the deviation matrix, hence increasing the requirements for empirical testing, although some properties become easier to test as soon as joint contribution to more than one public good can be assessed.

But many difficulties remain. An important one is the identification of a good consumed by the household as being private or public (or semi-public).[19] This is crucial for the testability of the various semi-cooperative or noncooperative models whether using the differentiable local approach we have explored here, or revealed preference global approach adopted elsewhere. The public or private nature of a good is of course linked to some objective characteristics, like the possibility of being nonexclusively consumed or the presence of external effects, but also linked to the recognition of such characteristics by the spouses themselves and their agreement to share the good. Another unsolved related issue is the fixing of the autonomy parameters. The techniques used by the New Empirical Industrial Organization when estimating conduct parameters could possibly be adapted.[20] Further work is obviously required.

Appendix

Proof of Proposition 3

Separate spheres

This is the simpler case, in which semi-cooperation is observationally equivalent to noncooperation. We first show that the matrix Θ expressing the substitution and income effects of changes in prices $\widetilde{P}^A$ and $\widetilde{P}^B$ is null (as it certainly is when $\theta_k^A = \theta_k^B = 1$, implying $\widetilde{P}_k^A = \widetilde{P}_k^B = 0$, for any public good k). Under semi-cooperation, for

[18] See, for example, Udry (1996).

[19] A first step in that direction is made in Cherchye et al. (online 2012).

[20] These so-called conduct parameters measure the relative weight of competitive toughness and play in the analysis of firm behavior a role similar to the degrees of autonomy in the analysis of household behavior. For more explanation and references, see d'Aspremont and Dos Santos Ferreira (2009).

any h and h', the generic element of $\Theta^A \equiv \left(\left[\partial_{\mathcal{P}} x^A\right] + \left[\partial_{\mathcal{Y}} x^A\right]\left[{}^T g^A\right]\right)\mathbf{\Pi}^B$, namely $\sum_k \left(\left[\partial_{\mathcal{P}_k} x_h^A\right] + \left[\partial_{\mathcal{Y}} x_h^A\right]\left[g_k^A\right]\right)\mathbf{\Pi}^B_{kh'}$, is also nil under separate spheres, since $\mathbf{\Pi}^B_{kh'} = 0$ if $k \in M^A$, and $\left(\left[\partial_{\mathcal{P}_k} x_h^A\right] + \left[\partial_{\mathcal{Y}} x_h^A\right]\left[g_k^A\right]\right) = 0$ if $k \in M^B$ (a variation in the price of a public good to which spouse A does not contribute cannot induce changes in A's demand for any good and $g_k^A = 0$ by definition). The same argument holds of course for the second term Θ^B of the matrix Θ, so that $\Theta = 0$.

As concerns budgetary externality effects, we obtain a similar result: $\left[\widetilde{P}^J\right]\mathbf{\Gamma}^{-J} = \sum_k \left(1 - \theta_k^{-J}\right) P_k^J \mathbf{\Gamma}_k^{-J} = 0$, since $P_k^J = 0$ if $k \in M^{-J}$, and $\mathbf{\Gamma}_k^{-J} = 0$ if $k \in M^J$. Hence, $\Xi = \left[\partial_g x^A\right]\Gamma^B + \left[\partial_g x^B\right]\Gamma^A$. Each matrix $\left[\partial_g x^J\right]$ has at most $n + m^J$ nonzero rows, which, however, cannot be linearly independent since $[(p,P)]\left[\partial_g x^J\right] = 0$ (consumption changes induced by the sole utility externality effect should not modify the expenditure $(p,P)\cdot x^J$, which has to be kept equal to J's income). Hence, the rank of $\left[\partial_g x^J\right]$ is at most equal to $n + m^J - 1$. As to the matrix $\mathbf{\Gamma}^J$, it has at most m^J nonzero rows, so that the rank of the matrix Ξ cannot be higher than

$$\begin{aligned} r_\Xi &= \min\left\{n + m^A - 1, m^B\right\} + \min\left\{n + m^B - 1, m^A\right\} \\ &= m + \min\left\{n - 1 - \left(m^A - m^B\right), 0\right\}. \end{aligned}$$

As already stated, the rank of matrix Δ is at most equal to $r_\Delta = 1$. Now, by applying Euler's identity to the functions ξ and x^J, which are homogeneous of degree 0, we see that $[(p,P)]\left[{}^T\Psi\right] = [(p,P)]\left[{}^T\Sigma\right] = 0$, implying $[(p,P)]\left[{}^T(\Delta - \Xi)\right] = 0$, so that the columns of the matrix $\Delta - \Xi$ are not linearly independent. Hence, the rank of this matrix is at most equal to $n + m - 1$. Taking this upper bound into account and simply adding r_Δ and r_Ξ, we obtain precisely the result that Lechene and Preston (2011, theorem 3.3) have established for the fully noncooperative case:

$$\begin{aligned} r_{\Delta-\Xi} &= \min\left\{n + m - 1, 1 + m + \min\left\{n - 1 - \left(m^A - m^B\right), 0\right\}\right\} \\ &= m + \min\left\{n - \max\left\{m^A - m^B, 1\right\}, 1\right\}. \end{aligned}$$

Joint contribution to public consumption

As regards the matrix Θ of the substitution and income effects of changes in the personalized prices, the kth column of $\left[\partial_{\mathcal{P}} x^J\right] + \left[\partial_{\mathcal{Y}} x^J\right]\left[{}^T g^J\right]$ is again zero for $k \in M^{-J}$ under joint contribution to m^{AB} public goods, as it was in the regime of separate spheres. Hence, in the product of this matrix with matrix $\mathbf{\Pi}^{-J}$, the corresponding kth line of the latter might as well be zero. This matrix has m^J further zero lines, namely any jth line such that $j \in M^J$. Hence, the matrix product is upper bounded by m^{AB} and the rank of Θ (the sum of two such products) by $r_\Theta = 2m^{AB}$.

As to the matrix Ξ of externality effects, we can apply the argument developed for the regime of separate spheres, only with an increase of m^{AB} in the number of nonzero rows of matrices $\left[\partial_g x^J\right]$ and $\mathbf{\Gamma}^{-J}$ $(J = A, B)$. We have also to take into account the possible rank increase introduced by the outer products $\left[\partial_{\mathcal{Y}} x^{-J}\right]\left[\widetilde{P}^J\right]$ $(J = A, B)$. We

thus obtain for the maximum rank of matrix Ξ:

$$\begin{aligned} r_{\Xi} &= \min\left\{n + m^A + m^{AB}, m^B + m^{AB}\right\} + \min\left\{n + m^B + m^{AB}, m^A + m^{AB}\right\} \\ &= m + m^{AB} + \min\left\{n - \left(m^A - m^B\right), 0\right\}. \end{aligned}$$

By taking again into account the homogeneity of degree zero of the demand functions ξ and x^J, and adding the three maximum ranks r_{Δ}, r_{Ξ} and r_{Θ}, we obtain the maximum rank of the deviation matrix:

$$\begin{aligned} r_{\Delta-\Xi+\Theta} &= \min\left\{n + m - 1, 1 + m + 3m^{AB} + \min\left\{n - \left(m^A - m^B\right), 0\right\}\right\} \\ &= m + \min\left\{n - 1, \min\left\{n - \left(m^A - m^B\right), 0\right\} + 1 + 3m^{AB}\right\} \\ &= m + 3m^{AB} + \min\left\{n - \max\left\{m^A - m^B - 1, 1 + 3m^{AB}\right\}, 1\right\}. \end{aligned}$$

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