Linear Inequality Methods to Enforce Partnerships under Uncertainty: An Overview*

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Abstract

We review different (generic) conditions on stochastic outcome functions to enforce either efficient or nearly efficient partnerships. Their logical relationship is explored. Two kinds of conditions are considered. However, the property for an action profile to be "compatible" plays a crucial role in both kinds. Also, two kinds of enforcement mechanisms are considered: enforcement through utility transfers and enforcement through repetition.

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JEL Classification: D2, D7

1. Introduction

Looking at the theory of mechanism design, and the derivation of individual incentives for the participants to a collective organization, the first methodological distinction concerns the different types of informational limitations. This is the distinction between moral hazard, caused by the imperfect observability of actions, and adverse selection, resulting from private holding of relevant information by the individual actors. This first distinction is standard in game theory, where uncertainty about the players' decisions is treated differently from the uncertainty about states of nature.

But there is another distinction, also methodologically important, between two different technical approaches to deriving incentive compatible mechanisms. One is the "differential approach" (using the terminology of Laffont and Maskin, 1980) which, assuming sufficient regularity, consists of solving the differential system associated with the individual optimization first order conditions. The other one, which may be called the "linear approach" and that we will take here, assumes sufficient separability and treats the individual incentive problems as systems of linear inequalities. This second approach was taken first to handle adverse selection in the design of collectively optimal public-decision mechanisms (d'Aspremont and Gérard-Varet, 1979) and in the design of optimal selling rules for a discriminating monopolist (Crémer and McLean, 1985). More recently, it has been used to solve moral hazard problems both in static partnerships (Legros and Matsushima, 1991),[1] with the team joint output assumed to be stochastic, and in repeated partnerships (Fudenberg et al.,

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1. See also Legros (1989) and Matsushima (1989b)

1994),[2] to obtain various "folk theorems" and the possibility for efficiency to be approximated with sufficiently low discounting. In this recent work, adopting the linear approach, two kinds of enforcement mechanisms are considered: enforcement through utility transfers and enforcement through repetition. In a repetitive situation, transferability is obtained through accumulation of payoffs over time: The recursive structure of the dynamic game, and the extended application of the optimality principle of dynamic programming, allow for the decomposition of equilibrium payoffs into "present-period" payoffs and "continuation" payoffs. The continuation payoffs play in the dynamic framework the role played by the transfers in the static framework.

Now, whether in the static or in the dynamic framework, the conditions that are imposed on the probabilistic outcome function to enforce partnership under moral hazard (whether through transfers or repetition) are analogous to other conditions, already used in previous models of adverse selection to restrict the individual probabilistic beliefs. Our purpose here is to review these various conditions and exploit this analogy systematically. This review will be done in a model of team moral hazard, but capitalizing on results obtained in the adverse selection case. Logical relations between these various conditions will be clarified.

The moral hazard problem is well illustrated in the simplest model of team production, where a set of agents (the partners) have each to choose an action at some cost or disutility. Output, as measured in monetary units, is a function of individual actions. It should be shared among the partners, and, to ensure participation, no one should be losing. There is moral hazard (or imperfect monitoring) when actions are not perfectly observable and the sharing rule cannot be directly based on them. Then, as often observed, there are situations where noncooperative behavior prevents the team from producing the collectively optimal level of output, whatever the proposed rule to fully share the total output. This inefficiency of partnership due to moral hazard has been well argued by Alchian and Demsetz (1972) for deterministic outcome functions and was formalized by Holmstrom (1982) under simple regularity conditions. Moreover, even when partnership is repeated and outcome is stochastic, a negative conclusion may be maintained. In the two-agent example developed by Radner et al. (1986), there are two possible observable outcomes[3] – high or low output – each having positive probabilities whatever the agents do (work or shirk). Since any agent choosing to work increases the probability of high output, it is collectively efficient in expected (and discounted) payoff terms that both agents work. But, every equilibrium payoff, and the associated sharing of the surplus, is uniformly bounded away from the efficiency frontier, whatever the discount rate.

As already suggested in Holmstrom (1982), there are two possible ways to avoid these negative conclusions. One is the possibility of exploiting the informational properties of the outcome function (particularly when it is stochastic), the other is the possibility of weakening efficiency into an approximate-efficiency requirement. The possibility of obtaining the first best by exploiting the stochastic character of the outcome function is first demonstrated by Williams and Radner (1994)[4] and then pursued in the work of Legros and Matsushima quoted above. The possibility given by the weakening of efficiency is clearly presented by Legros and Matthews (1993) for deterministic outcome functions and is further exploited by Fudenberg et al. (1994), who combine the two possibilities in looking at stochastic outcome functions. These two papers show that efficiency can be approximated by using mixed strategy equilibria (as an alternative way to make the outcome

2. See also Matsushima (1989a) and Fudenberg and Tirole (1992).
3. Because there are only two possible outcomes, this example will appear to be degenerate. See also Williams and Radner (1994).
4. This article was first circulated as a discussion paper in 1987.

stochastic). Accordingly, in the following, we shall distinguish between two kinds of conditions, some ensuring enforceability of the desired actions exactly, others ensuring only their approximation by mixed strategy equilibria. These two kinds of conditions will appear complementary. The first kind of conditions will be shown to hold generically, leaving however many meaningful cases uncovered, and to be recaptured by the second kind of conditions. Examples are the classical subscription game for the production of a public good or the stochastic Cournot oligopoly model of Green and Porter (1984). Of course, other cases, such as the two-agent example developed by Radner et al. (1986), will remain uncovered (due essentially to the lack of observable outcomes).

Our paper is organized as follows. In Section 2, the partnership model is presented. In Section 3, the linear approach is introduced and, restricting to transferable utilities, transfer schemes are used as enforcement mechanisms. Conditions of the two kinds are presented and analyzed. In Section 4, the same approach and the same two kinds of conditions are used, but without transferability, and with repetition playing the role of enforcement mechanism.

2. A model of partnership

Formally, the "team moral hazard problem" may be defined as an *elementary game* in normal form. A "team" consists of a set $N = \{1, \dots, i, \dots, n\}$ of "partners" (or players), each having to choose simultaneously an *action* a_i in a set A_i. For simplicity we take this set to be finite. A vector $a = (a_1, \dots, a_i, \dots, a_n) \in A = \bigtimes_{i=1}^{n} A_i$ is called an *action profile*. There is a set X of public *outcomes*, also assumed finite, and a *stochastic outcome function*. This is a function associating with each action profile $a \in A$ a probability distribution $p(\cdot \mid a)$ on the set X. With each partner i is associated a utility function $u_i(x, a_i)$ defined[5] on $X \times A_i$. The payoff of partner i can be computed as the expectation $U_i(a) = \sum_{x \in X} u_i(x, a_i) p(x \mid a)$.

Allowing the players to randomize over their actions, we denote by α_i a *mixed action* of partner i that is a probability distribution on A_i, and by $\hat{A}_i$ the set of all mixed actions of player i. Letting $\alpha_i(a_i)$ be the probability assigned to action a_i by a mixed action α_i and considering a mixed action profile $\alpha \in \hat{A} = \bigtimes_{i=1}^{n} \hat{A}_i$, the corresponding probability of the action profile a is given by the product $\alpha(a) = \Pi_{i=1}^{n} \alpha_i(a_i)$. We let

$$p(x \mid \alpha) = \sum_{a} p(x \mid a) \alpha(a).$$

The support of α_i is the set $S(\alpha_i) = \{a_i \in A_i : \alpha_i(a_i) > 0\}$. Similarly, for $\alpha_{-i} \in \bigtimes_{j \neq i} \hat{A}_j$, we let $\alpha_{-i}(a_{-i}) = \prod_{j \neq i} \alpha_j(a_j)$, with $a_{-i} \in \bigtimes_{j \neq i} A_j = A_{-i}$, and

$$p(x \mid a_i, \alpha_{-i}) = \sum_{a_{-i}} p(x \mid a_i, a_{-i}) \alpha_{-i}(a_{-i}).$$

For a mixed action profile $\alpha \in \hat{A}$, the payoff of player i is

$$U_i(\alpha) = \sum_{a \in A} U_i(a) \prod_{j=1}^{n} \alpha_j(a_j)$$

and

$$U_i(a_i, \alpha_{-i}) = \sum_{a_{-i} \in A_{-i}} U_i(a_i, a_{-i}) \prod_{j \neq i} \alpha_j(a_j) = \sum_{x \in X} u_i(x, a_i) p(x \mid a_i, \alpha_{-i}).$$

5. As will be mentioned, some of our results will apply to the more general case where u_i is defined on $X \times A$.

In this elementary game an action profile a^* is collectively *efficient* if there is no other action profile a such that, for every i, $U_i(a) \geq U_i(a^*)$, with at least one inequality holding strictly (a^* and a may be respectively replaced by α^* and α if we allow for mixed actions).

In such a game, one cannot expect in general that an efficient action profile (or any other action profile) be *directly* enforceable as a noncooperative equilibrium, in the Nash sense, whereby no player could increase his payoff by a unilateral deviation from the given efficient action profile. The main objective is then to enforce an action profile (efficient or not) as a noncooperative equilibrium *indirectly*, by designing some mechanism based on the publicly observed outcome and not on the individual actions. Introducing such a mechanism amounts to modifying the elementary game. As mentioned in the Introduction, two types of enforcement mechanisms have been considered. The first type of mechanism, limited to a static framework where the utilities are measured in monetary units and are transferable, consists of fixing monetary transfer schemes among the players. They will be examined in the next section. The second type of mechanism, studied in Section 4, requires a dynamic framework and consists of repeating the elementary game: Repetition itself will form the enforcement mechanism.

The elementary game just described encompasses (in discrete terms) several models already extensively analyzed in the literature. One example is the partnership model, as analyzed by Holmstrom (1982), in which each agent supplies an unobservable input to a (deterministic or stochastic) production process, and the output is shared by the team. Another example is given by Green and Porter (1984), oligopoly model, in which firms' output levels are not publicly observable and the market price is a stochastic function of total output. A third example is the classic subscription game for the (stochastic or deterministic) provision of a public good: Every individual subscribes some monetary amount and the total is devoted to covering the cost of a public good (e.g., see Laffont, 1988). A last example is given by the design of international treaties on environmental issues (e.g., see Mäler, 1990; d'Aspremont and Gérard-Varet, 1992), in which the outcomes are states of the natural environment and in which the individual utilities can be decomposed into two parts, the abatement cost (of taking an action) and the damage cost associated with the state of the environment and resulting from all individual actions.

3. Transfer schemes

3.1 A necessary and sufficient condition to enforce full sharing of the surplus

Let us first consider the static interpretation of the framework and assume that the agents have quasi-linear utility functions: There is a good, playing the role of money, which is transferable without any restrictions, and the utilities are measured in units of that good. Under the transferability assumption, a mixed action profile α^* is collectively efficient if and only if the following requirement holds:

$$\alpha^* \in \underset{\alpha \in \hat{A}}{\operatorname{argmax}} \sum_{i \in N} U_i(\alpha).$$

In the design of an incentive mechanism to enforce a desired mixed action profile α^*, we shall not restrict our attention to the case in which it is efficient. However, assuming that every agent has an outside option giving a utility level normalized to 0, we shall always consider that α^* produces a *nonnegative* (resp. *positive*) *total surplus*, that is,

$$\sum_{i \in N} U_i(\alpha^*) \geq 0 \quad \text{(resp. } > 0\text{)}.$$

An important advantage of the transferability assumption is to reduce the mechanism design to the solving of a system of linear inequalities defining an appropriate transfer scheme. A *transfer scheme* is a function $t : X \to \mathrm{Re}^n$ transforming the elementary game into a "new game," with payoff functions given by

$$\sum_{x \in X} [u_i(x, a_i) + t_i(x)] p(x \mid a),$$

and strategy spaces still given by $\{A_i\}$. Such a transfer scheme should be designed to get α^* as a Nash equilibrium of the corresponding game: It should satisfy the following set of linear *incentive compatibility* constraints:

$$U_i(\alpha^*) + \sum_{x \in X} t_i(x) p(x \mid \alpha^*) \geq U_i(a_i, \alpha^*_{-i}) + \sum_{x \in X} t_i(x) p(x \mid a_i, \alpha^*_{-i}) \forall\, i \in N\; \forall\, a_i \in A_i. \tag{IC}$$

The mixed action profile α^* is then said to be *enforceable*.

In addition, for feasibility, the surplus generated should be "fully shared" within the team. In other words, the transfer scheme should satisfy the *budget-balancing* linear equalities

$$\forall\, x \;\; \sum_{i \in N} t_i(x) = 0. \tag{BB}$$

This is a strong property but indispensable to close the model. Weaker restrictions could allow some outside distribution of the surplus (e.g., to the owners, if it is a firm, as suggested by Alchian and Demsetz, 1972; Holmstrom, 1982). Alternatively, one could impose budget-balancing only in expected value terms with respect to the stochastic outcome function $p(\cdot \mid \alpha^*)$, but this would then imply the intervention of some outside party ensuring *ex post* feasibility.

$$\sum_{x \in X} \sum_{i \in N} t_i(x) p(x \mid \alpha^*) = 0. \tag{EB}$$

A last property is that every agent be given at least the incentives to participate: The transfer scheme should satisfy additional linear inequalities, the so-called *individual rationality constraints*

$$\forall\, i \in N \;\; U_i(\alpha^*) + \sum_{x \in X} t_i(x)\, p(x \mid \alpha^*) \geq 0. \tag{IR}$$

The problem of finding, for a given action profile α, a transfer scheme satisfying this set of incentive and feasibility constraints defines the "partnership problem." The "linear approach" to the problem consists of taking this set as a system of linear inequalities in t and applying standard results about the consistency of such systems.[6] To illustrate let us prove the following lemma, which is a simple "theorem of the alternative" applied to the problem, and which will be used repeatedly in the sequel.[7]

6. As mentioned in the Introduction, such a technique was first used to solve the team adverse selection problem, both in the finite and in the infinite case. In both cases, results such as given in Fan (1956) can be used. In the infinite case, though, for topological reasons, only ε-Bayesian incentive compatibility is obtained (d'Aspremont and Gérard-Varet, 1982). Such a technique was first used to solve the team moral hazard problem by Legros (1989) and Matsushima (1989a).

7. Such a lemma is also proved in Legros and Matsushima (1991).

Lemma 1 *Consider any utility profile $\{u_i\}$. Any mixed action profile α^* producing nonnegative surplus is enforceable by a budget-balancing and individually rational transfer scheme $t \in \times_{i=1}^n \mathrm{Re}^X$ if and only if: For every $\lambda \in \times_{i=1}^n \mathrm{Re}_+^{A_i}$, $\lambda \neq 0$, such that, $\forall\, i, j \in N, \forall\, x \in X$,*

$$\begin{aligned}&\sum_{a_i} \lambda_i(a_i)[p(x \mid \alpha^*) - p(x \mid a_i, \alpha^*_{-i})] \\ =&\sum_{a_j} \lambda_j(a_j)[p(x \mid \alpha^*) - p(x \mid a_j, \alpha^*_{-j})],\end{aligned} \tag{1}$$

we must have

$$\sum_{i \in N} \sum_{a_i} \lambda_i(a_i)[U_i(a_i, \alpha^*_{-i}) - U_i(\alpha^*)] \leq 0. \tag{2}$$

Proof We have to solve the following system of linear inequalities in $t \in \mathrm{Re}^{nX}$:

$$\begin{aligned}&\sum_x t_i(x)[p(x \mid \alpha^*) - p(x \mid a_i, \alpha^*_{-i})] \geq [U_i(a_i, \alpha^*_{-i}) - U_i(\alpha^*)] && \forall\, i\, \forall\, a_i \\ &\sum_x t_i(x) p(x \mid \alpha^*) \geq -U_i(\alpha^*) && \forall\, i \\ &\sum_i t_i(x) = 0 && \forall\, x.\end{aligned}$$

By standard results, its consistency is equivalent to: For every $\lambda \in \times_{i=1}^n \mathrm{Re}_+^{A_i}$, $\lambda \neq 0$, for every $\mu \in \mathrm{Re}^X$ and $\gamma \in \mathrm{Re}_+^N$ such that

$$\sum_{a_i} \lambda_i(a_i)[p(x \mid \alpha^*) - p(x \mid a_i, \alpha^*_{-i})] - \mu(x) + \gamma_i p(x \mid \alpha^*) = 0,\ \forall\, i\, \forall\, x, \tag{A1}$$

we must have

$$\sum_i \sum_{a_i} \lambda_i(a_i)[U_i(a_i, \alpha^*_{-i}) - U_i(\alpha^*)] - \sum_i \gamma_i U_i(\alpha^*) \leq 0. \tag{A2}$$

But (A1) implies (by summation over x) that $\sum_x \mu(x) = \gamma_i \geq 0, \forall\, i$, so that $\gamma_i = \gamma_j = \tilde{\gamma}$, for some $\tilde{\gamma} \geq 0$, and (1) holds. Moreover (2) implies (A2), since $\sum_i \gamma_i U_i(\alpha^*) = \tilde{\gamma} \sum_i U_i(\alpha^*) \geq 0$. Therefore, if, $\forall\, \lambda \in \times_{i=1}^n \mathrm{Re}_+^{A_i}$, $\lambda \neq 0$ [(1) $\Rightarrow$ (2)], then $\forall\, \lambda \in \times_{i=1}^n \mathrm{Re}_+^{A_i}$, $\lambda \neq 0$, $\forall\, \mu \in \mathrm{Re}^X$, $\forall \gamma \in \mathrm{Re}_+^N$ [(A1) $\Rightarrow$ (A2)].

To show the converse, observe that (1) implies (A1). Indeed (1) implies that, for some $\kappa \in \mathrm{Re}^X$, $\sum_{a_i} \lambda_i(a_i)[p(x \mid \alpha^*) - p(x \mid a_i, \alpha^*_{-i}] = \kappa(x), \forall\, i, \forall\, x$, which leads to (A1) by posing $\mu \equiv \kappa$ and $\gamma_i = 0$. Then (A2) coincides with (2), and the result follows. ∎

From the proof of Lemma 1, it appears that individual rationality does not represent an effective restriction. This is also established directly by Legros and Matsushima (1991), who obtain individual rationality for any given utility profile $\{u_i\}$ and any action profile α^* producing nonnegative surplus, as soon as the action profile is enforceable by a budget-balancing transfer scheme t. Indeed, for any sharing coefficients $\{\delta_i\}$, with $\delta_i \geq 0$ and $\sum_{i=1}^n \delta_i = 1$, it suffices to construct the new transfer scheme t', such that, $\forall\, i \in N, \forall\, x \in X$,

$$t'_i(x) = t_i(x) - U_i(\alpha^*) + \sum_{x \in X} t_i(x) p(x \mid \alpha^*) + \delta_i \sum_j U_j(\alpha^*). \tag{3}$$

By construction, this new transfer scheme satisfies individual rationality, is still budget-balancing and maintains incentive compatibility (since it amounts to adding a term constant in a_i).

It is important to notice that the condition given in Lemma 1 is a *necessary and sufficient* condition and that it restricts, unseparably, *both* the outcome function $p(\cdot \mid \cdot)$ and the utility profile $\{u_i\}$. Legros and Matsushima (1991) use such a lemma to derive an alternative, somewhat more interpretable, necessary and sufficient condition, but still restricting unseparably both the outcome function and the utility profile. They first construct a measure β of the "likelihood of a deviation" with respect to some (efficient) pure action profile a^* as

$$\beta(\alpha) = \frac{\sum_i [U_i(\alpha_i, a^*_{-i}) - U_i(a^*)]}{n[1 - \sum_x \min_i p(x \mid \alpha_i, a^*_{-i})]} \text{ if } \sum_x \min_i p(x \mid \alpha_i, a^*_{-i}) < 1$$

(if not, $\beta(\alpha) = 0$). Then the alternative condition, necessary and sufficient to have a^* enforceable by a budget-balancing and individually rational transfer scheme, is that[8]

$$\sup_\alpha \beta(\alpha) < \infty.$$

For the specification of both this necessary and sufficient condition and the one given by Lemma 1, a crucial restriction is the budget-balancing condition (BB). If we replace (BB) by the weaker restriction of expected-budget-balancing (EB), then, using an argument similar to the one of Lemma 1, we get[9]

Theorem 1 *For any utility profile $\{u_i\}$, any efficient pure action profile a^* producing nonnegative surplus is enforceable by an individually rational, expected-budget-balancing transfer scheme.*

Proof To solve the system of inequalities and equalities in $t \in \mathrm{Re}^{nX}$, defined by (IC), (IR) and (EB), we consider, as in the proof of Lemma 1, the corresponding dual conditions: For every $\lambda \in \bigtimes_{i=1}^n \mathrm{Re}_+^{A_i}$, every $\mu \in \mathrm{Re}$, and every $\gamma \in \mathrm{Re}_+^N$ such that

$$\sum_{a_i \neq a_i^*} \lambda_i(a_i)[p(x \mid a^*) - p(x \mid a_i, a^*_{-i})] + (\gamma_i - \mu)p(x \mid a^*) = 0 \;\; \forall\, i \,\forall\, x, \tag{4}$$

we must have

$$\sum_i \sum_{a_i \neq a*_i} \lambda_i(a_i) \sum_x [u_i(x, a_i)p(x \mid a_i, a^*_{-i}) - u_i(x, a_i^*)p(x \mid a^*)] - \sum_i \gamma_i \sum_x u_i(x, a_i^*)p(x \mid a^*) \leq 0. \tag{5}$$

But (4) implies (by summation over x) that $\mu = \gamma_i \geq 0, \forall\, i$, so that

$$\sum_{a_i \neq a_i^*} \lambda_i(a_i)[p(x \mid a^*) - p(x \mid a_i, a^*_{-i})] = 0 \;\; \forall\, i \,\forall\, x. \tag{6}$$

8. Legros and Matsushima (1991, Theorem 2).
9. The analogue to this theorem for the adverse selection case is Theorem 10 in d'Aspremont and Gérard-Varet (1982).

By efficiency we may write

$$
\begin{aligned}
&\sum_i \sum_{a_i \neq a_i^*} \lambda_i(a_i) \sum_x [u_i(x, a_i) p(x \mid a_i, a_{-i}^*) - u_i(x, a_i^*) p(x \mid a^*)] \\
\leq &\sum_i \sum_{a_i \neq a_i^*} \lambda_i(a_i) \sum_x \left[\sum_{j \neq i} u_j(x, a_j^*) p(x \mid a^*) - \sum_{j \neq i} u_j(x, a_j^*) p(x \mid a_i, a_{-i}^*) \right] \\
= &\sum_i \sum_x \sum_{j \neq i} u_j(x, a_j^*) \sum_{a_i \neq a_i^*} \lambda_i(a_i) [p(x \mid a^*) - p(x \mid a_i, a_{-i}^*)].
\end{aligned}
\tag{7}
$$

This last expression is null by (6). Finally, since

$$\sum_i \gamma_i \sum_x u_i(x, a_i^*) p(x \mid a^*) = \mu \sum_x \sum_i u_i(x, a_i^*) p(x \mid a^*) \geq 0,$$

we get (5). ■

In this proof, efficiency of the chosen pure action profile is essential. It shows that allowing the budget to balance only in expected value is a way to solve the moral hazard problem, but relying on some (unspecified) external agency to ensure *ex post* feasibility.

Returning to the strong budget-balancing property, we shall now take another route and use Lemma 1 to analyze alternative *sufficient* conditions imposed *only* on the stochastic outcome function, independently from any restriction imposed on the utility profile. For that purpose, as mentioned in the introduction, we distinguish two kinds of conditions among those introduced in the literature. One kind will include conditions sufficient to enforce any efficient pure actions and stronger conditions, sufficient to enforce any actions (efficient or not). The other kind will only be enough to obtain approximate efficiency via mixed actions.

3.2 Sufficient conditions of the first kind: enforcing efficient actions

The first condition that we define will be shown to be sufficient to enforce any efficient pure action profile a^*. This is simply an analogue to the "compatibility condition" initially introduced to solve the team adverse selection problem.[10] However, it has never been considered in the team moral hazard problem, and it will turn out to be the weakest of its kind (among those that have appeared in the literature).

A mixed action profile α^* is *compatible with* $p(\cdot \mid \cdot)$, or $p(\cdot \mid \cdot) \in C_{\alpha^*}$, if: For every $\lambda \in \bigtimes_{i=1}^n \mathrm{Re}_+^{A_i}$ such that (1) of Lemma 1 holds, we have

$$\sum_{a_i} \lambda_i(a_i) [p(x \mid \alpha^*) - p(x \mid a_i, \alpha_{-i}^*)] = 0 \;\; \forall\, i \in N \; \forall\, x \in X. \tag{2*}$$

Then the condition to be imposed on the stochastic outcome function $p(\cdot \mid \cdot)$ is simply stated as:

Compatibility Condition (C*). Any pure action profile $a^* \in A$ is compatible with $p(\cdot \mid \cdot)$.

We let also $C^* = \cap_{a^*} C_{a^*}$ denote the subset of all stochastic outcome functions $p(\cdot \mid \cdot)$ satisfying the compatibility condition. We shall see below various ways to interpret this condition. But, using Lemma 1, we immediately obtain:

10. This is also called condition C. See d'Aspremont and Gérard-Varet (1979, 1982). In the second reference the condition is stated in a slightly more general and more adequate form (see Johnson et al., 1990, footnote 2).

Theorem 2 *Suppose* C* *holds. For any utility profile* $\{u_i\}$*, any efficient pure action profile* a^* *producing nonnegative surplus is enforceable by an individually rational and budget-balancing transfer scheme, i.e., satisfying* (IC)*,* (IR) *and* (BB).

Proof Take any utility profile $\{u_i\}$ and efficient pure action profile a^* producing a nonnegative surplus. If, for $\lambda \in \bigtimes_{i=1}^{n} \mathrm{Re}_{+}^{A_i}$, (1) in Lemma 1 holds, then C* implies (2*). Using efficiency as in the proof of Theorem 1 (inequality (7)), we get (2). ■

Observe that in this theorem, based on C*, the utility profile is not restricted (except that it should produce nonnegative surplus). However, only efficient pure action profile can be enforced as required. As shown in d'Aspremont and Gérard-Varet (1992); d'Aspremont et al. (1997), for its adverse selection analogue, condition C* can be expressed equivalently in "primal" terms, leading to an alternative proof of Theorem 2. Indeed, condition C* is equivalent to[11]

Condition C. $\forall\, a^* \in A, \forall\, p \in \mathrm{Re}^X, \exists\, t \in \mathrm{Re}^{nX}$ such that $\sum_i t_i(x) = \rho(x)$, and

$$\sum_x t_i(x) p(x \mid a^*) \geq \sum_x t_i(x) p(x \mid a_i, a^*_{-i}) \ \ \forall\, a_i \in A_i \, \forall\, i \in N.$$

This condition is easy to interpret. It means that, for any given budget, and the identically zero utility profile, any joint action profile should be enforceable by a transfer scheme balancing this budget. Then, an alternative way to prove Theorem 1 is to start by constructing transfers of the Clarke–Groves–Vickrey type by letting, relative to the given utility profile $\{u_i\}$ and the efficient action profile a^* (producing nonnegative surplus),

$$t_i^0(\cdot) \equiv \sum_{j \neq i} u_j(\cdot, a_j^*).$$

Efficiency implies that the (IC)-constraints are satisfied but that the budget-balancing constraint might be broken. However, taking $\rho(\cdot) \equiv -\sum_i t_i^0(\cdot)$, and applying Condition C, there is a transfer scheme t satisfying all equalities and inequalities listed in the condition. Then, defining $t^* \equiv t^0 + t$, we get budget-balancing while preserving incentive compatibility. Finally, to ensure individual rationality, we may apply the construction given by (3) above.

Fudenberg et al. (1994) – hereafter denoted FLM – introduce a stronger condition. It is more restrictive than condition C* (as we will see in Theorem 3), but leads to an alternative interpretation of it. To state this other condition we need additional notation. Consider a mixed action profile $\alpha \in \hat{A}$ and, for every $i, j \in N$, the following matrices, of dimension $|A_i| \times |X|$, $|A_j| \times |X|$, and $(|A_i| + |A_j|) \times |X|$, respectively:

$$\Pi_i(\alpha_{-i}) = [p(x \mid a_i, \alpha_{-i})]_{a_i \in A_i, x \in X}$$
$$\Pi_j(\alpha_{-j}) = [p(x \mid a_j, \alpha_{-j})]_{a_j \in A_j, x \in X}$$
$$\Pi_{ij}(\alpha) = \begin{bmatrix} [p(x \mid a_i, \alpha_{-i})]_{a_i \in A_i, x \in X} \\ [p(x \mid a_j, \alpha_{-j})]_{a_j \in A_j, x \in X} \end{bmatrix}.$$

A mixed action profile α is said to be *pairwise identifiable for* i *and* j whenever the rank of matrix $\Pi_{ij}(\alpha)$ equals [rank($\Pi_i(\alpha_{-i})$)+ rank($\Pi_j(\alpha_{-j})$)] $- 1$. We can now state FLM's condition:

11. Following the convention introduced by d'Aspremont and Gérard-Varet (1992); d'Aspremont et al. (1997), the letter denoting the dual version of a condition will be starred.

Pairwise Identifiability (PI). For every pair i, j in N, every pure action profile a in A is pairwise identifiable for i and j.

The idea of pairwise identifiability of an action profile is that the probability distributions, induced by one player's deviating mixed action, be distinguishable from the probability distributions induced by some other player's deviation, in order to avoid that "the two players' deviations cannot be distinguished statistically" (see FLM). A similar interpretation can be given to the compatibility condition, since, whenever violated for some action profile a^*, it leads to the existence of a profile of (deviating) mixed actions[12] λ such that, for any pair i, j and any a_i, a_j,

$$\sum_{a_i} \lambda_i(a_i)p(\cdot \mid a_i, a^*_{-i}) - p(\cdot \mid a^*) = \sum_{a_j} \lambda_j(a_j)p(\cdot \mid a_j, a^*_{-j}) - p(\cdot \mid a^*) \neq 0.$$

For every player i, the probability distribution induced by the deviation λ_i is different from the one induced by a^*_i, but undistinguishable from the probability distributions induced by any other player's deviation. The logical relation between pairwise identifiability and compatibility is precisely established by the following.

Theorem 3 *For a stochastic outcome function $p(\cdot \mid \cdot)$, the mixed action profile α is pairwise identifiable for some pair $i, j \in N$ if and only if α is compatible with $p(\cdot \mid \cdot)$.*

Proof To prove the "only if" part, assume that (1) holds (with $\alpha^* = \alpha$) and that, for some i, j, rank $\Pi_{ij}(\alpha) =$ rank $\Pi_i(\alpha_{-i})+$ rank $\Pi_j(\alpha_{-j}) - 1$. Then there is $\lambda = [\lambda_i, \lambda_j, \overline{\lambda}]^t \in \mathrm{Re}^{|A_i|+|A_j|+1}$, with $\lambda_i = (\lambda_i(a_i))_{a_i}$, $\lambda_j = (\lambda_j(a_j))_{a_j}$, solving the following homogeneous system:

$$\sum_{a_i} p(x \mid a_i, \alpha_{-i})\lambda_i(a_i) - \sum_{a_j} p(x \mid a_j, \alpha_{-j})\lambda_j(a_j) - p(x \mid \alpha)\overline{\lambda} = 0 \;\; \forall\, x \in X.$$

It implies, by summation over x, that $\overline{\lambda} = \sum_{a_i} \lambda_i(a_i) - \sum_{a_j} \lambda_j(a_j)$. In matrix notation it becomes $\tilde{\Pi}_{ij}(\alpha)\lambda = 0$, with

$$\tilde{\Pi}_{ij}(\alpha) = [(p(\cdot \mid a_i, \alpha_{-i}))_{a_i}, (-p(\cdot \mid a_j, \alpha_{-j}))_{a_j}, (-p(\cdot \mid \alpha))],$$

the corresponding $|X| \times (|A_i| + |A_j| + 1)$ matrix. Since the last column $(-p(\cdot \mid \alpha))$ can be written as $(\sum_{a_i} \alpha(a_i)p(\cdot \mid a_i, \alpha_{-i}))$, we get rank $\tilde{\Pi}_{ij}(\alpha) =$ rank $\Pi_{ij}(\alpha)$.

To show that α is compatible with $p(\cdot \mid \cdot)$, we need to show that (2*) holds. In fact, using (1), it is enough to show that the following two subsystems can be solved, respectively, in $\lambda_i = (\lambda_i(a_i))_{a_i}$ and $\overline{\lambda}_i$ (implying that $\overline{\lambda}_i = \sum_{a_i} \lambda_i(a_i)$), and in $\lambda_j = (\lambda_j(a_j))_{a_j}$ and $\overline{\lambda}_j$ (implying that $\overline{\lambda}_j = -\sum_{a_j} \lambda_j(a_j)$). The first is given by

$$\sum_{a_i} p(x \mid a_i, \alpha_{-i})\lambda_i(a_i) - p(x \mid \alpha)\overline{\lambda}_i = 0 \;\; \forall\, x \in X,$$

or, in matrix notation, with $\tilde{\Pi}_i(\alpha_{-i}) = [(p(\cdot \mid a_i, \alpha_{-i}))_{a_i}, (-p(\cdot \mid \alpha))]$ the corresponding $(|X| \times |A_i| + 1)$-matrix: $\tilde{\Pi}_i(\alpha_{-i})[\lambda_i, \overline{\lambda}_i]^t = 0$. The second subsystem is

$$-\sum_{a_j} p(x \mid a_j, \alpha_{-j})\lambda_j(a_j) - p(x \mid \alpha)\overline{\lambda}_j = 0 \;\; \forall\, x \in X,$$

12. The normalization to 1 of $\sum_{a_i} \lambda_i(a_i)$ is obtained without loss of generality since Condition C* imposes no value for every $\lambda_i(a^*_i)$.

or

$$\tilde{\Pi}_j(\alpha_{-j})[\lambda_j, \overline{\lambda}_j]^t = 0,$$

with $\tilde{\Pi}_j(\alpha_{-j}) = [(-p(\cdot \mid a_j, \alpha_{-j}))_{a_j}, (-p(\cdot \mid \alpha))]$ the corresponding $(|X| \times |A_j| + 1)$-matrix.

Clearly, again, rank $\tilde{\Pi}_i(\alpha_{-i})$ = rank $\Pi_i(\alpha_{-i})$ and rank $\tilde{\Pi}_j(\alpha_{-j})$ = rank $\Pi_j(\alpha_{-j})$, so that rank $\tilde{\Pi}_{ij}(\alpha)$ = rank $\tilde{\Pi}_i(\alpha_{-i})$+ rank $\tilde{\Pi}_j(\alpha_{-j}) - 1$, and, from known results on linear transformations,

$$\begin{aligned} &\dim(\text{Ker}\, \tilde{\Pi}_i(\alpha_{-i})) + \dim(\text{Ker}\, \tilde{\Pi}_j(\alpha_{-j})) \\ =&[|A_i| + 1 - \text{rank}\, \tilde{\Pi}_i(\alpha_{-i})] + [|A_j| + 1 - \text{rank}\, \tilde{\Pi}_j(\alpha_{-j})] \\ =&|A_i| + |A_j| + 1 - rank\tilde{\Pi}_{ij}(\alpha) = \dim(\text{Ker}\, \tilde{\Pi}_{ij}(\alpha)). \end{aligned}$$

Thus, defining two subspaces of Ker $\tilde{\Pi}_{ij}(\alpha)$,

$$\begin{aligned} K_i &= \{\ell \in \text{Re}^{|A_i|+|A_j|+1} : [\ell_i, \overline{\ell}_i]^t \in \text{Ker}\, \tilde{\Pi}_i(\alpha_{-i}), \ell_j = 0\} \\ K_j &= \{\ell \in \text{Re}^{|A_i|+|A_j|+1} : [\ell_j, \overline{\ell}_j]^t \in \text{Ker}\, \tilde{\Pi}_j(\alpha_{-j}), \ell_i = 0\}, \end{aligned}$$

we get that Ker $\tilde{\Pi}_{ij}(\alpha) = K_i \oplus K_j$. In other terms, if $\lambda \in$ Ker $\tilde{\Pi}_{ij}(\alpha)$, then λ can be decomposed uniquely into the sum of $[\lambda_i, 0, \overline{\lambda}_i]^t \in K_i$ with $[0, \lambda_j, \overline{\lambda}_j]^t \in K_j$, where $\overline{\lambda}_i = \sum_{a_i} \lambda_i(a_i)$ and $\overline{\lambda}_j = -\sum_{a_j} \lambda_j(a_j)$. By (1), this is enough to get (2*).

To prove the converse (the "if" part), it is enough now to observe that if

$$\dim(\text{Ker}\, \tilde{\Pi}_i(\alpha_{-i})) + \dim(\text{Ker}\, \tilde{\Pi}_j(\alpha_{-j})) < \dim(\text{Ker}\, \tilde{\Pi}_{ij}(\alpha)),$$

for some pair i, j, then there is some $\lambda \in \text{Ker}\, \tilde{\Pi}_{ij}(\alpha)$ which can be decomposed in such a way that, for $\overline{\lambda}_i = \sum_{a_i} \lambda_i(a_i)$ and $\overline{\lambda}_j = -\sum_{a_j} \lambda_j(a_j)$, $\overline{\lambda} = \overline{\lambda}_i - \overline{\lambda}_j$, but either $[\lambda_i, \overline{\lambda}_i]^t \notin \text{Ker}\, \tilde{\Pi}_i(\alpha_{-i})$ or $[\lambda_j, \overline{\lambda}_j]^t \notin \text{Ker}\, \tilde{\Pi}_j(\alpha_{-j})$. ■

This result leads to still another way to state condition C*: *For every pure action profile a^*, there exists a pair of players for which a^* is pairwise identifiable.* Also, that PI implies C* is an immediate consequence of the "only if" part of this theorem:

Corollary 1 *Condition* PI *implies condition* C*.

However it should be stressed that C* is effectively weaker than PI (which is a condition imposed to *all pairs* of players), since there are meaningful cases where the first holds and not the second. Consider the following class of stochastic outcome functions having a "product structure."[13] To illustrate, suppose there are four partners and that any outcome x in X is decomposable into two statistically independent components, the first component being influenced by the actions of the first two partners, the second by the actions of the other two: $x = (y, z) \in Y \times Z = X$ and

$$p(x \mid a_1, a_2, a_3, a_4) = q(y \mid a_1, a_2) r(x \mid a_3, a_4).$$

For $i, j = 1, 2$, $i \neq j$, let $Q_i(a_j) = [q(y \mid a_i, a_j)]_{a_i \in A_i, y \in Y}$, and

$$Q_{12}(a_1, a_2) = \begin{bmatrix} [q(y \mid a_1, a_2)]_{a_1 \in A_1, y \in Y} \\ [q(y \mid a_1, a_2)]_{a_2 \in A_2, y \in Y} \end{bmatrix}.$$

13. For results using such a structure see, e.g., Holmstrom (1982), Radner (1985), and Fudenberg et al. (1994).

For any given a^* and $i, j = 1, 2$, $i \neq j$, rank($\Pi_i(a^*_{-i})$) = rank($Q_i(a^*_j)$) and rank($Q_{12}(a^*_1, a^*_2)$) = rank($\Pi_{12}(a^*)$). If, on the one hand, we assume that q does not satisfy PI, because

$$\text{rank}(Q_{12}(a^*_1, a^*_2)) < \text{rank}(Q_1(a^*_2)) + \text{rank}(Q_2(a^*_1)),$$

then

$$\text{rank}(\Pi_{12}(a^*)) < \text{rank}(\Pi_1(a^*_{-1})) + \text{rank}(\Pi_2(a^*_2)),$$

so that p does not satisfy PI either. On the other hand, if we assume that r satisfies Condition C, then, for any $\rho : Y \times Z \to \mathrm{Re}$ and any $y \in Y$, there is $t^y : Z \to \mathrm{Re}^4$ such that $t^y_1 \equiv 0$, $t^y_2 \equiv 0$, $t^y_3(z) + t^y_4(z) = \rho(y, z)$ and

$$\sum_z t^y_i(z) r(z \mid a^*_3, a^*_4) \geq \sum_z t^y_i(z) r(z \mid a_i, a^*_j),$$

for any $i, j = 3, 4$, $i \neq j$, and any $a_i \in A_i$. Multiplying each side by $q(y \mid a^*_1, a^*_2)$ and summing over y, we get that p satisfies Condition C. Therefore the product stochastic outcome function as constructed satisfies Condition C but not PI.

3.3 Enforcement without using efficiency

Up to now we have concentrated our efforts on the enforcement of efficient action profiles, and efficiency was used in the argument to get incentive compatibility. However, there might be reasons for which some actions must be chosen that are not efficient for the team. For example, the team might be embedded in a larger community and the recommended actions might be the ones corresponding to efficient actions for the whole community, taking into account all external effects. This raises the issue whether it is possible to ensure the enforceability of any action profile (efficient or not from the team point of view) by a budget-balancing and individually rational transfer scheme. We then need to impose a stronger restriction on the stochastic outcome function. By analogy with the adverse selection case,[14] we propose:

Condition B. $\forall\, a^* \in A, \exists\, t \in \mathrm{Re}^{nX}$ such that

$$\sum_i t_i(x) = 0$$

$$\sum_x t_i(x) p(x \mid \alpha^*) > \sum_x t_i(x) p(x \mid a_i, \alpha^*_{-i}) \;\; \forall\, a_i \in A_i, a_i \neq a^*_i,\; \forall\, i \in N.$$

Suppose Condition B holds. Then for any utility profile $\{u_i\}$ we can multiply the given transfer scheme t by an arbitrarily large positive number M and obtain

$$\sum_i M t_i(x) = 0 \;\; \forall\, x \in X,$$

$$M \sum_x t_i(x)[p(x \mid a^*) - p(x \mid a_i, a^*_{-i})] > \sum_x [u_i(x, a_i) p(x \mid a_i, a^*_{-i}) - u_i(x, a^*_i) p(x \mid a^*)]$$

$\forall\, a_i \in A_i, a_i \neq a^*_i, \forall\, i \in N.$

14. This condition was introduced in d'Aspremont and Gérard-Varet (1982). It was adapted to the moral hazard problem by Matsushima (1989b). See also Legros and Matsushima (1991).

So the transfer scheme Mt satisfies (IC) and (BB). Also, by the construction given in (3), we get a transfer scheme t' satisfying (IC), (BB), and (IR). Moreover, the converse holds. It is enough to consider the utility profile where $u_i \equiv 0$ for all $i \in N$. Then (BB) and (IC) (holding strictly) imply Condition B. We have thus proved the following:

Theorem 4 *Condition* B *holds if and only if, for any utility profile* $\{u_i\}$ *and any* a^* *producing nonnegative surplus,* a^* *is strictly enforceable* (IC)*, with all inequalities strict) by a budget-balancing* (BB) *and individually rational* (IR) *transfer scheme.*

Obviously the same theorem holds for the case[15] where each u_i depends directly on the actions of all players ($u_i = u_i(x, a)$), although logically, neither version of the theorem is stronger than the other. Indeed for one direction the condition allows us to wipe out any counterincentive effect due to the utilities. In the other direction, the argument is based on the identically null utility profile.

Let B_{a^*} denote the set of outcome functions $p(\cdot \mid \cdot)$ such that the system defined by (1), with $\alpha^* = a^*$, admits only solutions where, $\forall\, i$, $\lambda_i(a_i) = 0$, if $a_i \neq a_i^*$. The dual version of Condition B is then:

Condition B*. For any pure action profile $a^* \in A, p(\cdot \mid \cdot) \in B_{a^*}$.

Thus Condition B* (or B) is stronger than Condition C* (or C), but entails (and is even equivalent to) enforceability of all action profiles by budget-balancing, individually rational transfer schemes. Clearly such conditions require that the set X of observable outcomes be of sufficient size with respect to the sets of actions. More formally, if the cardinality of X is large enough, relative to the number of actions available to some partners, the set B* of all stochastic outcome functions satisfying B, and hence the larger set C*, are very large indeed: They contain an open and dense subset of all stochastic outcome functions. That is, Conditions B and C are generic.[16]

Theorem 5 *Assuming that there exists a pair* $i, j \in N$ *such that* $|X| \geq |A_i| + |A_j| - 1 \geq 2$*, Conditions* B *and* C *hold generically.*

Proof For $i, j \in N$ and $a^* \in A$, assuming $|X| \geq |A_i| + |A_j| - 1$, the matrix defining the homogeneous system (1) contains a square sub matrix of dimension $(|A_i| + |A_j| - 1)^2$ which is regular if and only if $p(\cdot \mid \cdot) \in B_{a^*}$. Hence,[17] B_{a^*} is an open and dense subset of all stochastic outcome functions. Since the intersection of open and dense subsets is open and dense, the set $[\cap_{a*} B_{a^*}]$ is also an open and dense subset. The result follows because $[\cap_{a^*} B_{a^*}] \subset B^* \subset C^*$. ■

3.4 Sufficient conditions of the second kind

Although the compatibility condition is generic, and weaker than both pairwise identifiability and Condition B, there are still many meaningful cases where it is not satisfied. For instance, an important

15. This is the case treated by Legros and Matsushima (1991, Proposition 3). For the model with adverse selection, see Johnson et al. (1990), who call this the mutually-payoff- relevant case, and d'Aspremont and Gérard-Varet (1992).
16. In the adverse selection case, the genericity of C* was shown by d'Aspremont et al. (1990).
17. See, e.g., Hirsch and Smale (1974, 7.3)

case is the one where

$$p(x \mid a) = g\biggl(x \mid \sum_{i=1}^{n} a_i\biggr), \ \forall\, x\, \forall\, a, A_i = \{0, 1, \ldots, K\}, \ \forall\, i,$$

with $g(\cdot \mid r) \neq g(\cdot \mid r')$ whenever $r \neq r'$. In this case, for any $a^* \in A$ such that $0 \leq a_j^* - a_1^* < K$, $\forall\, j \neq 1$, it is possible to find $a' \in A$ satisfying $(a_j' - a_j^*) = (a_1' - a_1^*) \neq 0$, so that, $\forall\, j \neq 1$,

$$\sum_{i \neq j} a_i^* + a_j' = \sum_{i \neq 1} a_i^* + a_1' \neq \sum_i a_i^*.$$

Notice this set of action profiles includes more than the symmetric profiles, where each player uses the same action. Also, letting $\lambda_j(a_j') = \lambda_1(a_1') \neq 0$, $\forall\, j \neq 1$, and $\lambda_j(a_j) = 0, \forall\, j, \forall\, a_j \neq a_j'$, we see that Condition C* is violated: (1) holds but not (2*).

An example of such a case is given by the classical subscription game for the production of a public good (assuming transferable utilities), where the only actions are subscriptions in some private good made by the consumers and where the total amount subscribed is used as an input to produce the public good. Another example is the discretized version of the stochastic Cournot oligopoly model of Green and Porter (1984), where, to obtain the collusive outcome, the actions are the quantities to be produced and the market price is a function of the total quantity.

Because the compatibility condition does not hold in such models, one may look, as is done by FLM, for conditions of another kind, enforcing only approximate efficiency for the partnership, i.e., such conditions are sufficient to enforce a mixed action profile arbitrarily close to some efficient profile.

We need some definitions. First, a mixed action profile α is said to be *nearly-efficient* for $\varepsilon > 0$ if

$$\left| \max_{a \in A} \sum_{i \in N} U_i(a) - \sum_{i \in N} U_i(\alpha) \right| < \varepsilon.$$

Second, FLM define a mixed action profile α *to have pairwise full rank for a pair* i, j of players, whenever the rank of the matrix $\Pi_{ij}(\alpha)$ is equal to $|A_i| + |A_j| - 1$. They have the following condition (FLM, Condition 6.2):

Local Pairwise Full Rank (LPF). For every pair $i, j \in N$, there exists a mixed action profile α which has pairwise full rank.

This is a weak condition since it only restricts some mixed action profiles. As observed by FLM, this condition is the conjunction of two weaker conditions. One is to impose, for *every pair* $i, j \in N$, the existence of some mixed action profile α having pairwise identifiability. In fact we shall need only an even weaker condition, i.e., to impose the same property only to *some pair* of players. But, by Theorem 3, this is equivalent to having α compatible with $p(\cdot \mid \cdot)$.

Local Compatibility (LC). There exists a mixed action profile α which is compatible with $p(\cdot \mid \cdot)$.

The second weaker condition included in LPF is based on another property applying to a mixed action profile α, namely, to *have individual full rank for* i in the sense that the matrix $\Pi_i(\alpha_{-i})$ should have full rank. The condition is[18]:

Local Individual Full Rank (LIF). For every $i \in N$, there exists a mixed action profile α which has individual full rank for i.

Finally, using FLM Lemma 6.2, our Theorem 3 and Lemma 1, it is now possible to show that the conjunction of LC and LIF is sufficient to enforce nearly efficient mixed joint actions.

Theorem 6 *Assume that* LC *and* LIF *hold. Then, for any utility profile* $\{u_i\}$ *and for any joint mixed action producing positive surplus (in particular, for any efficient pure joint action* a^**), there is an arbitrarily close joint mixed action* α^* *which is enforceable by an individually rational and budget-balancing transfer scheme, i.e., satisfying* (IC)*,* (IR)*, and* (BB)*. Moreover for an efficient pure joint action* a^* *an any* $\varepsilon > 0$*,* α^* *may be chosen to be nearly efficient and produce positive surplus.*

Proof First, by Theorem 3, LC is equivalent to the existence of some mixed action profile α having pairwise identifiability for some pair of players i, j. So, it may be shown that the conjunction of LC and LFI imply that there exists an open and dense set of mixed action profiles each of which is pairwise identifiable for this pair of players and has individual full rank for all players. The argument is the same as in Lemma 6.2 of FLM. Therefore, for any joint mixed action producing positive surplus, it implies that there exists an arbitrarily close joint mixed action α^* (hence all producing positive surplus) which is pairwise identifiable for i, j and has individual full rank for all $k \in N$. Now use Lemma 1 for this α^* and consider any $\lambda \in \bigtimes_{i=1}^{n} \mathrm{Re}_+^{A_i}$, $\lambda \neq 0$, satisfying (1). Because α^* is pairwise identifiable for i, j we get by Theorem 3 (and (1)) that α^* is compatible with $p(\cdot \mid \cdot)$:

$$\sum_{a_k} \lambda_k(a_k)[p(x \mid \alpha^*) - p(x \mid a_k, \alpha^*_{-k})] = 0 \ \ \forall\, k \in N\, \forall\, x \in X.$$

Since α^* has individual full rank for every k, every solution to these equations satisfies

$$\lambda_k(a_k) = \theta_k\ \alpha^*(a_k)\ \ \forall\, a_k,\ \text{for some}\ \theta_k \geq 0,$$

implying, for every k,

$$\sum_{k \in K} \sum_{a_k} \lambda_k(a_k)[U_k(a_k, \alpha^*_{-k}) - U_k(\alpha^*)] = 0,$$

so that (2) of Lemma 1 holds. Therefore α^* is enforceable by an individually rational and budget-balancing transfer scheme. If α^* is chosen arbitrarily close to an efficient joint action a^* producing positive surplus, then, for any $\varepsilon > 0$, we can take α^* to be nearly efficient and to produce positive surplus. The result follows. ■

18. In Fudenberg et al. (1994), a stronger condition is defined, Condition 6.3, which requires that any pure action profile has individual full rank.

4. Repetition

The two kinds of conditions imposed on the stochastic outcome functions can also be used in a dynamic framework, in which the elementary (or stage) game is infinitely repeated and the total payoffs over time are discounted. In this framework, the purpose is to study the set of payoffs in the elementary game that can be attained, or nearly attained, as average discounted equilibrium payoffs in the infinitely repeated game, when the players are sufficiently patient (i.e., the discount factor is sufficiently close to 1). The assumption of quasi-linear utilities is thus not needed anymore, since immediate transfers in the basic game can be replaced by intertemporal transfers, realized along the equilibrium path in the repeated game. FLM base their analysis on two conditions: pairwise identifiability and pairwise full rank. We show in this section that their results can be generalized using, respectively, the compatibility condition and the conjunction of local compatibility with local individual full rank.

The repeated game is constructed as follows. At each stage $t = 0, 1, \ldots, \infty$ and for each player i, the action space is given by $\{A_i\}$ and the payoff function given by $U_i(a_i) = \sum_x u_i(x, a_i)p(x \mid a)$, the payoff of the elementary game. At each stage, the game results in an outcome $x_t \in X$ which is publicly observable. Also, only the outcome (and not the others' actions) is observed by each agent at each stage. A *public history at stage* t is a sequence $(x^0, x^1, \ldots, x^{t-1})$ in $X_t = X \times \ldots \times X$ and a *public strategy* σ_i is an infinite sequence $\sigma_i^1, \sigma_i^2, \ldots, \sigma_i^t, \ldots, \sigma_i^\infty$, where every σ_i^t is a map from the set of all public histories at stage t to $\hat{A}_i$: for any sequence $(x^0, x^1, \ldots, x^{t-1})$ in X_t, $\sigma_i^t(\cdot \mid x^0, x^1, \ldots, x^{t-1})$ is a mixed action of player i. A vector of public strategies σ induces a probability distribution over the set of public histories at stage t, allowing to define the expected payoff of agent i at stage t, which we denote by $U_i^t(\sigma)$. Then, given a common discount factor $\delta \in (0, 1)$, the payoff of agent i in the repeated game is defined as the payoff $(1-\delta)\sum_{t=0}^\infty \delta^t U_i^t(\sigma)$. Because of the factor $(1-\delta)$, this is an "average payoff per period," comparable to the stage game payoff. The equilibrium concept used by FLM is the *perfect public equilibrium* (PPE). It is a vector of public strategies σ such that, for each $t = 0, 1, \ldots$, and for each public history $(x^0, x^1, \ldots, x^{t-1})$ in X_t, the continuation of these strategies, from stage t on, forms a Nash equilibrium of the remaining game. Since, when the other agents use public strategies, no agent can gain by playing a nonpublic strategy and the beliefs about the others' past actions are irrelevant, a PPE is a true perfect Bayesian equilibrium. The set of discounted average payoff vectors that correspond to PPEs, for a discount factor δ, is denoted $E(\delta)$. Also, in the stage game, because there is a finite number of pure joint actions, the set of payoff vectors generated by all joint mixed actions is a polytope in Re^N. Hence a joint mixed action α^* is Pareto efficient if and only if $\alpha^* \in \operatorname{argmax}_\alpha \sum_{i\in N} \beta_i U_i(\alpha)$, for some $\beta = (\beta_1, \beta_2, \ldots, \beta_N) \gg 0$.

Even restricting to public strategies, FLM get "folk theorems" for the infinitely repeated game. This is to show that "large sets" of payoff vectors of the stage game, including nearly efficient ones, can be attained through repetition as perfect Bayesian equilibrium payoffs of the repeated game, at least when the players are sufficiently patient. Consider any Nash equilibrium $\alpha^0 \in \hat{A}$ of the stage game, with corresponding payoff vector $v^0 \equiv (U_1(\alpha^0), \ldots, U_n(\alpha^0))$. We shall take as a "large set" of payoff vectors to be attained[19] the set V^0 which is the convex hull of v^0 and all the Pareto-efficient payoff vectors in the stage game Pareto-dominating v^0. V^0 is a polytope generated

19. A stronger version of the folk theorem corresponds to taking as the large set of payoff vectors to be attained (or nearly attained) the set of all payoff vectors that are above the min-max payoff vector. This would require stronger assumptions, such as Condition 6.3 in FLM (see Footnote 18 here above).

by v^0 and all payoff vectors, say $v^k, k = 1, 2, \ldots, K$, corresponding to Pareto-efficient pure action profiles, $a^1, a^2, \ldots, a^K$. V^0 is assumed to have a nonempty interior.

The method for constructing, in the repeated game, PPE's giving discounted average payoff vectors that are in the set V^0 for some discount factor δ, relies on the dynamic-programming decomposition of equilibrium payoffs into "current" and "continuation" payoffs, as in Abreu et al. (1986). More precisely, a payoff vector $v \in \mathrm{Re}^n$ is said to be *decomposable* for a subset $W \subset \mathrm{Re}^n$, a discount factor δ and a mixed action profile α^*, if there exists a set of (continuation) payoff vectors $w(\cdot) \in \mathrm{Re}^{nX}$ such that

$$w(x) \in W \qquad \forall\, x \in X$$

and, $\forall\, i \in N, \forall\, a_i \in A_i$,

$$\begin{aligned} v_i &= (1-\delta)U_i(\alpha^*) + \delta \sum_x w_i(x) p(x \mid \alpha^*) \\ &\geq (1-\delta)U_i(a_i, \alpha^*_{-i}) + \delta \sum_x w_i(x) \sum_{a_{-i}} p(x \mid a_i, a_{-i}) \prod_{j \neq i} \alpha^*_j(a_j). \end{aligned}$$

In the following theorem, it will be enough to restrict to W's that are regular hyperplanes $\{w \in \mathrm{Re}^n \mid \sum_i b_i w_i = c\}$, with $b_i > 0$ for all i. This will allow us to rely on the analogy between the problem of finding continuation payoffs satisfying these inequalities (including the constraint $w(x) \in W$, $\forall\, x \in X$) for a mixed action profile $\alpha^* \in A$ and the problem of finding transfer schemes to enforce α^* (including the budget balancing constraint) as considered in the previous sections. Indeed, it amounts to replacing, in the (BB)-constraints, each U_i by $[((1-\delta)/\delta) b_i U_i]$ (a simple change of units) and replace, in the (BB)-constraints, each $t_i(x)$ by $[b_i w_i(x) - (c/n)]$.

FLM define a subset $V \subset \mathrm{Re}^n$ to be *locally self-decomposable* if, for each $v \in V$, there is a $\delta \in [0,1)$ and an open set W containing v such that every $u \in V \cap W$ is decomposable for V itself, for δ and some mixed action profile α^*. We shall need the following result (FLM, Lemma 4.2), where $E(\delta)$ denotes the set of discounted average payoff vectors resulting from PPEs in the infinitely repeated game with discount factor δ.

Lemma 2 *If $V \subset \mathrm{Re}^n$ is compact, convex, and locally self-decomposable, then there exists $\delta' < 1$ such that $V \subset E(\delta)$ for all $\delta \in (\delta', 1)$.*

Now, in order to approximate the polytope V^0, we shall restrict our attention to a certain class of polytopes which are "reduced copies" of V^0. Let $\overline{\omega}$ be the barycenter of V^0 and, for $\theta \in (0,1)$, let $v^k_\theta = \theta\overline{\omega} + (1-\theta)v^k$, for each $k = 0, 1, 2, \ldots, K$. Then a *reduced copy* of V^0 is the polytope V^0_θ generated by the family $\{v^k_\theta : k = 0, 1, \ldots, K\}$.

Our final theorem generalizes Theorem 6.1 of FLM, replacing the conditions of pairwise identifiability and local pairwise full rank used in FLM by, respectively, the compatibility Condition C and the conjunction of local compatibility and local individual full rank. As far as it is based on the compatibility condition, it has to rely on Pareto-efficient payoffs resulting from pure action profile in the stage game and hence requires a new argument.

Theorem 7 *If either Condition* C *or the conjunction of* LC *and* LIF *is satisfied, then for any subset V^0_θ, a reduced copy of V^0, with nonempty interior and θ arbitrarily small, there exists $\delta' < 1$ such that, for all $\delta \geq \delta', V^0_\theta \subset E(\delta)$.*

Proof By Lemma 2, and because V_θ^0 is compact and convex by construction, it is enough to show that V_θ^0 is locally self-decomposable. Consider two cases. First consider W_θ^0, the set of points in V_θ^0 that are Pareto dominated by some other point in V_θ^0. Let $P_\theta^0 \equiv V_\theta^0 \setminus W_\theta^0$ denote the Pareto frontier of V_θ^0. By construction of V^0, and because V_θ^0 is a reduced copy of it, for any point $v \in W_\theta^0$, the line going through v^0 (the Nash payoff) and v crosses the boundary of V_θ^0 at some point $\hat{u}$. Since the interval $[v, \hat{u}]$ on this line is, by convexity, contained in V_θ^0, there exist $\delta \in (0,1)$ and $u' \in (v, \hat{u}) \subset$ Interior V_θ^0 satisfying $v = (1-\delta)v^0 + \delta u'$. Now, given this δ, there exists also a neighborhood W of v such that, for every $u \in V_\theta^0 \cap W$, $u = (1-\delta)v^0 + u''$, for some $u'' \in V_\theta^0$. Hence for any $u \in V_\theta^0 \cap W$, since v^0 is the payoff corresponding to the Nash equilibrium α^0 and u'' is taken as a lump-sum payment, u is decomposable for V_θ^0, δ, and α^0.

Second, consider P_θ^0, the set of Pareto-optimal points in V_θ^0. Observe that, for every $v^k \in V^0$, corresponding to the Pareto-efficient pure action profile $a^k, k = 1, 2, \ldots, K$, there are positive coefficients $\{\beta_i^k : i \in N\}$ such that $v^k \in argmax_{u \in V^0} \sum_i \beta_i^k u_i$. Therefore, by Condition C, for every k there exist continuation payoffs $\{w^k(x)\}$ such that v^k is decomposable for the hyperplane $\{w \in \mathrm{Re}^n \mid \sum_i \beta_i^k w_i = 0\}$, for any $\delta \in (0,1)$, and for the pure action profile a^k. Now consider a point $v \in P_\theta^0$. We want to show that there exist $k \geq 1$, $\delta \in (0,1)$, and u' in the interior of V_θ^0 such that $v = (1-\delta)v^k + \delta u'$. If v is an extreme point ($v = v_\theta^m$, for some $m \geq 1$), then we can take $k = m, \delta = (1-\theta)$, and $u' = \varpi$. If v is not an extreme point then it can be written as a convex combination of some extreme point v_θ^k and a point v' which is a nonextreme point of P_θ^0 and belongs to the same face of the polytope V_θ^0 as v and v_θ^k. Consider the triangle generated by v_0^k, v' and ϖ. Except for the edge containing v, all points of this triangle are in the interior of V_θ^0 and, by construction, the point v_θ^k is on the line joining ϖ and v^k. Therefore the line joining v^k to v intersects the edge between v' and ϖ at a point u' interior to V_θ^0. (See Figure 1).

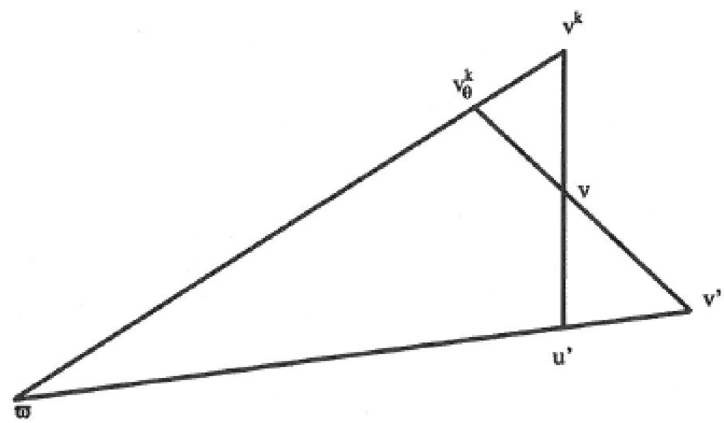

FIGURE 1

So, for some $\delta \in (0,1)$, $v = (1-\delta)v^k + \delta u'$. As above, given this δ, there exists a neighborhood W of v such that, for every $u \in V_\theta^0 \cap W$, $u = (1-\delta)v^k + \delta u''$ for some $u'' \in V_\theta^0$. Then, for every $u \in V_\theta^0 \cap W$, we may construct new continuation payoffs $\hat{w}_i^k(x) \equiv u_i'' + w_i^k(x) - \sum_x p(x \mid a^k) w_i^k(x)$. Since $\sum_x p(x \mid a^k)\hat{w}_i^k(x) = u_i''$, for all i, and since the difference $[u_i'' - \sum_x p(x \mid a^k) w_i^k(x)]$ is constant in x, u is decomposable for V_θ^0, δ, and a^k.

Therefore V_θ^0, which is compact and convex by construction, is locally self-decomposable, and the first part of the theorem (based on Condition C) follows from Lemma 2.

Under conditions LC and LIF, the above argument must be modified only for points in P^0_θ. Take any such point v and consider the line starting from $\overline{\varpi}$ and going through v and cutting V^0 at a point v'. The line segment $(\overline{w}, v)$ is in Interior V^0_θ. So, there exists a point $\overline{u}$ arbitrarily close to the line segment (v, v') and decomposable for the hyperplane $\{w \in \mathrm{Re}^n \mid \sum_i \beta_i^k w_i = 0\}$, for any $\delta \in (0,1)$, and for a joint mixed action α. And, there exists $\delta \in (0,1)$ and u' in the interior of V^0_θ such that $v = (1-\delta)\overline{u} + \delta u'$. Again, given this δ, there exists a neighborhood W of v such that, for every $u \in V^0_\theta \cap W$, $u = (1-\delta)\overline{u} + \delta u''$, for some $u'' \in V^0_\theta$, so that, by the same argument as above, u is decomposable for V^0_θ, δ, and α. The result follows. ■

5. Conclusion

In this paper, we have reviewed several conditions to be imposed on stochastic outcome functions to enforce either efficient or nearly efficient partnerships. Their logical relationship has been explored. We have distinguished two kinds of conditions. However, the property for an action profile to be "compatible" plays a crucial role in both kinds of conditions. In so doing, we have illustrated the fruitfulness of transporting belief restrictions used under adverse selection to the stochastic outcome functions in team moral hazard. Another lesson carried out by this paper is the importance of weakening efficiency into near-efficiency and of the use of mixed strategies. Then, only conditions of the second kind are required. These are "local conditions," i.e., applying to a single mixed action profile. They have been used for two types of enforcement mechanisms: for static enforcement via transfer schemes and for dynamic enforcement through repetition. For the first type, these conditions allow us to solve approximately problems that have "too much symmetry," such as collusion in stochastic Cournot oligopoly or efficiency in the subscription equilibrium for public goods, which cannot be solved by conditions of the first kind. With the second type, any payoff on the Pareto frontier can be approximated as a noncooperative equilibrium of the supergame.

References

Dilip Abreu, David Pearce, and Ennio Stachetti. Optimal cartel monitoring with imperfect information. *Journal of Economic Theory*, 39:251–269, 1986.

Armen A Alchian and Harold Demsetz. Production, information costs, and economic organization. *American Economic Review*, 62(5):777–795, 1972.

Jacques Crémer and Richard P McLean. Optimal selling strategies under uncertainty for a discriminating monopolist when demands are interdependent. *Econometrica*, 53:345–361, 1985.

Claude d'Aspremont and Louis-André Gérard-Varet. Incentives and incomplete information. *Journal of Public economics*, 11(1):25–45, 1979.

Claude d'Aspremont and Louis-André Gérard-Varet. Bayesian incentive compatible beliefs. *Journal of Mathematical Economics*, 10(1):25–45, 1982.

Claude d'Aspremont and Louis-André Gérard-Varet. Négociations internationales sur l'environnement: principes de coopération et stratégies nationales. In *Les Fonctions Collectives dans une Économie de Marché, Actes du 10e congrès des Économistes Belges de Langue*

Française, Centre Interuniversitaire de Formation Permanente, Charleroi, Chapitre. CIFOP, Charleroi, 1992.

Claude d'Aspremont, Jacques Crémer, and Louis-André Gérard-Varet. Incentives and the existence of pareto-optimal revelation mechanisms. *Journal of Economic Theory*, 51(2):233–254, 1990.

Claude d'Aspremont, Jacques Crémer, and Louis-André Gérard-Varet. The primal approach to mechanism design. Technical report, mimeo, 1997.

Ky Fan. On systems of linear inequalities. In H.W. Kuhn and A.W. Tucker, editors, *Linear Inequalities and Related Systems*. Princeton University Press, Princeton, 1956.

Drew Fudenberg and Jean Tirole. *Game Theory*. MIT Press, Cambridge, MA, 1992.

Drew Fudenberg, David K Levine, and Eric Maskin. The folk theorem with imperfect public information. *Econometrica*, 62(5):997–1039, 1994.

Edward J Green and Robert H Porter. Noncooperative collusion under imperfect price information. *Econometrica*, 52:87–100, 1984.

Morris W Hirsch and Stephen Smale. *Differential equations, dynamical systems and linear algebra*. Academic Press, New York, 1974.

Bengt Holmstrom. Moral hazard in teams. *The Bell Journal of Economics*, pages 324–340, 1982.

Scott Johnson, John W Pratt, and Richard J Zeckhauser. Efficiency despite mutually payoff-relevant private information: The finite case. *Econometrica*, 58:873–900, 1990.

Jean-Jacques Laffont. *Fundamentals of Public Economics*. MIT Press, Cambridge (MA), 1988. (translated by J.P. Bonin and H. Bonin).

Jean-Jacques Laffont and Eric Maskin. A differential approach to expected utility maximizing mechanisms. *Econometrica*, 48:1507–1520, 1980.

Patrick Legros. *Efficiency and stability in partnerships*. PhD thesis, California Institute of Technology, 1989.

Patrick Legros and Hitoshi Matsushima. Efficiency in partnerships. *Journal of Economic Theory*, 55 (2):296–322, 1991.

Patrick Legros and Steven A Matthews. Efficient and nearly-efficient partnerships. *Review of Economic Studies*, 60(3):599–611, 1993.

Karl-Göran Mäler. International environmental problems. *Oxford Review of economic policy*, 6(1): 80–108, 1990.

Hitoshi Matsushima. Efficiency in repeated games with imperfect monitoring. *Journal of Economic Theory*, 48(2):428–442, 1989a.

Hitoshi Matsushima. Necessary and sufficient condition for the existence of penalty rules with budget balancing. Mimeo, Institute of Socio-Economic Planning, University of Tsukuba, 1989b.

Roy Radner. Repeated principal-agent games with discounting. *Econometrica*, 53:1173–1198, 1985.

Roy Radner, Roger Myerson, and Eric Maskin. An example of a repeated partnership game with discounting and with uniformly inefficient equilibria. *Review of Economic Studies*, 53(1):59–69, 1986.

Steven R Williams and Roy Radner. Efficiency in partnership when the joint output is uncertain. In J. Ledyard, editor, *The Economics of Informational Decentralization*. Kluwer Academic, Dordrecht/Norwell, MA, 1994.