Unemployment in a Cournot Oligopoly Model with Ford Effects*

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1 Introduction

Involuntary unemployment is due to imperfect competition. Here[1] we shall examine this proposition in a general Cournot equilibrium model that includes the labour market. However unemployment will not be due to imperfect competition in the labour market, or to the downward rigidity of wages, but really to the oligopolistic competition of producers. We shall see that indeed, under some circumstances, the wage may go all the way down to zero without restoring full employment. These circumstances include an inelastic supply of labour and a low elasticity of the demand for goods at low prices, but they are compatible with the existence of a perfect

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[1]A similar analysis for monopolistic competition is given in d'Aspremont, Dos Santos Ferreira and Gérard-Varet (1986). See also Roberts (1986) and Heller (1986). For a related equilibrium approach to monopolistic competition, see Benassy (1988). A different approach to general equilibrium with imperfect competition is the one initiated by Gabszewicz and Vial (1982). See also Gary-Bobo (1987) where Ford effects are analysed.

competitive equilibrium implying full employment at a positive wage. The observed involuntary unemployment can thus be unambiguously attributed to the oligopolistic behaviour of the producers.

Our analysis is mainly devoted to the case of a single productive sector with a single input, although at the end we present a straightforward extension to the multisectorial case. Also we suppose that, when making their decisions, the producers are wage-takers. This can receive two justifications. One is to consider the labour market as competitive and, for each given value of the money wage, to compute a Cournot equilibrium for the producers under some specification of the demand curve they face. Then the wage can be adjusted parametrically for equilibrium in the labour market. Involuntary unemployment results when the wage has to be adjusted down to zero without reaching full employment. This can happen when the equilibrium outputs do not tend to infinity as the wage goes to zero. Indeed, assuming a low elasticity of demand for low prices, the total revenue function of the producers is decreasing when output is large (or price low), and accordingly, for a small enough wage, the profit function becomes decreasing in output. Allowing a relatively small part of the consumers' wealth to be in money (taken as an unproduced good), demand may be low enough to get unemployment even when the money wage tends to zero. We obtain therefore an extreme situation of equilibrium where workers may be producing for their consumption without being compensated for their work.

There is another justification for having the producers take the money wage as given. This is to introduce a temporary equilibrium framework (see Grandmont, 1977) and to consider that the wage has been fixed at a previous stage, either by the firms[2] or by the unions[3] or by both in a bargaining of some sort. In this temporary equilibrium interpretation the possibility of involuntary unemployment can also be shown using the same reasoning. It is then even more in the spirit of Keynes' involuntary unemployment for which there is "*no* method available to labour as a whole" to be fully employed "by making revised *money* bargains with the entrepreneurs" (Keynes, 1936, p. 13). The difference is that total demand is now influenced by consumers' ex-

[2] As in Roberts (1986).

[3] As in Hart (1982).

pectations about their future income and about future prices and so assumptions on consumers' demand concern expectations as well as preferences. Such assumptions will be discussed in some examples.

In both interpretations though, the Cournot equilibrium concept has to be extended to take into account all interdependencies. First, the producers' decisions should be constrained, on the input side, by the total labour supply. Second, on the output side, the demand curve they face should integrate the effect of their decisions on the consumers' wealth. Usually two types of extreme conjectures are considered. Either the producers take wage incomes and distributed profits as a given parameter, which is then adjusted for equilibrium. Or the producers are fully aware of all the effects of their decisions on wage incomes and on distributed profits. Although very requiring, this goes with the idea that the firms know the "objective" demand function.[4] Here we shall adopt an intermediate position.[5] From a Cournot-Nash viewpoint, each producer, when choosing an output level, anticipates the level of output of the other producers. Since he also knows the nominal wage, he is able to anticipate the total wage income. Moreover in the same way as he can compute the effect of his output decision on the total output, he can compute its effect on the total wage income. On the consumers' side, it is natural to suppose that they know whether they are employed or not when they decide on their consumption. So, considering both sides, it seems adequate to suppose that the producers take into account the effect of their decisions on the total wage income. This is what we call the "Ford effect" (because Henry Ford (1922) was probably the first entrepreneur to recognize in his writings the multiplier effect due to wages). However, for distributed profits, we shall stick to the first type of conjectures. The producers (as well as the consumers) are supposed to take distributed profits as given parameters, which are adjusted to the "true" level at equilibrium.

[4]This is to be opposed to the subjective approach as represented by Negishi (1961, 1979). However the importance of the distinction between objective and subjective demand for a general theory of monopolistic competition is emphasized by Triffin (1940). For a recent overview see Hart (1985).

[5]The pure parametric approach was used by Marschak and Selten (1974) and Hart (1982). In d'Aspremont, Dos Santos Ferreira and Gérard-Varet (1989) we adopt the other extreme conjecture. For a general discussion, see d'Aspremont, Dos Santos Ferreira and Gérard-Varet (1986).

In Section 2, the model and the definition of an extended Cournot-Nash equilibrium are introduced and, under some assumptions, the existence of such an equilibrium is proved. Then, in Section 3, we show the possibility of involuntary unemployment and, in Section 4, we consider the multisectorial case. Technical arguments and the detailed computations of an example are respectively deferred to two appendices.

2 An extended Cournot-Nash equilibrium

We shall concentrate at first on the analysis of an economy constituted by an oligopolistic market for a homogeneous product and a labour market. Also we shall suppose the existence of a single nonproduced good. In the temporary equilibrium interpretation of the model, this good is viewed as money.

A finite set $N = \{1, 2, \cdots, i, \cdots, n\}$ of firms is involved in the production of the homogeneous produced good. Each firm in this set is characterized by a production function f_i and has to select a production level y_i and an employment level z_i belonging to the set $\{(y_i, z_i) \in \mathbb{R}^2_+ : y_i \leq f_i(z_i)\}$. We assume that competition between these firms is oligopolistic "à la Cournot" and that they have a complete knowledge of the demand behaviour of the consumers.

The set of consumers is supposed to coincide with the set of potential workers and is represented by an interval $[0, L]$. For simplicity, each individual is assumed to supply one unit of labour whatever the nonnegative money wage $w \geq \underline{w} \geq 0$, where $\underline{w}$ is the reservation wage. Hence L is the total inelastic supply of labour for $w \geq \underline{w}$. Also, each individual will be assumed to have the same positive monetary wealth $m > 0$, to expect the same dividend π to be distributed at the end of the current period and to receive a current wage income equal to the money wage w, if he is employed, or to zero, if he is unemployed. Letting $p > 0$ denote the money price of the produced good, the demand of an employed consumer is denoted $\phi(p, \pi, w, m)$. Accordingly, $\phi(p, \pi, 0, m)$ will denote the demand of an individual receiving zero wage income, either because he is unemployed or because $w = 0$.

Given a choice $z = (z_1, \cdots, z_n) \in \mathbb{R}^n_+$ of employment levels by the firms, denote by $Z \stackrel{\text{def}}{=}$

$\sum_{i \in N} z_i$ the aggregate employment level. The aggregate demand for the produced good is given by the function:

$$\Phi(p, Z, \pi, w, m, L) \stackrel{\text{def}}{=} Z\phi(p, \pi, w, m) + (L - Z)\phi(p, \pi, 0, m).$$

Given a choice $y = (y_1, \cdots, y_n) \in \mathbb{R}_+^n$ of production levels by the firms, denote by $Y \stackrel{\text{def}}{=} \sum_{i \in N} y_i$ the aggregate production level. In the following we shall introduce assumptions allowing the definition of an inverse demand function $\psi(\cdot, Z, \pi, w, m, L)$, which gives, for every amount of aggregate production Y, the price p at which this amount can be sold for given values of Z, π, w, m and L, i.e.

$$\begin{aligned} p &= \psi(Y, Z, \pi, w, m, L) \quad \text{if and only if} \\ Y &= \Phi(p, Z, \pi, w, m, L). \end{aligned}$$

In our approach we shall consider that the producers are wage-takers. This assumption is justified by the fact that, in the following, we analyze the present oligopolistic situation for any fixed level of the money wage w. More precisely the equilibrium concept defined below, and the resulting involuntary unemployment notion, will be, as discussed in the introduction, independent of any particular endogenous determination of the money wage.

We can introduce the profit function Π_i of each producer $i \in N$. It is a function of the strategic choices, by all producers, of their feasible levels of production and employment and it is defined for every possible values of the parameters π, w, m and L:

$$\Pi_i(y_i, y_{-i}, z_i, z_{-i}, \pi, w, m, L) \stackrel{\text{def}}{=} y_i \psi(y_i + \sum_{j \neq i} y_j, z_i + \sum_{j \neq i} z_j, \pi, w, m, L) - wz_i$$

for every $i \in N$, with the conventions:

$$\begin{aligned} y_{-i} &= (y_1, \cdots, y_{i-1}, y_{i+1}, \cdots, y_n) \in \mathbb{R}_+^{n-1}, \\ z_{-i} &= (z_1, \cdots, z_{i-1}, z_{i+1}, \cdots, z_n) \in \mathbb{R}_+^{n-1}. \end{aligned}$$

For any fixed values of the parameters, this would lead, with a constant level of employment, to a classical Cournot-oligopoly model. But, taking into account the variability of the aggregate employment level amounts to assume that each firms realizes the impact of its decisions on the aggregate income and hence on the demand schedule it faces. This feedback income effect is the one we referred to in the introduction as the "Ford effect". Anyhow we may still define an *extended Cournot-Nash equilibrium* for given possible values of the parameters w, m and L as a pair $(y^*, z^*) \in \mathbb{R}^{2n}_+$ such that:

(i) $\forall\, i \in N, (y_i^*, z_i^*)$ is a solution to

$$\max_{(y_i,z_i)} \Pi_i(y_i, y^*_{-i}, z_i, z^*_{-i}, \pi^*, w, m, L)$$

subject to: $0 \le y_i \le f_i(z_i)$

$$0 \le z_i \le L - \sum_{j \ne i} z_j^*,$$

and

(ii) $L\pi^* = \sum_{i \in N} \Pi_i(y^*, z^*, \pi^*, w, m, L).$

Notice the introduction of the second feasibility constraint. This is natural since, by the Nash assumption, the labour demands of all $j \ne i$ are supposed to be given. Notice also the condition (ii) which requires that distributed profits be equal, at the equilibrium, to realized profits. For any possible values of the parameters w, m and L, we denote by $\mathcal{E}(w, m, L) \subset \mathbb{R}^{2n}_+$ the set of extended Cournot-Nash equilibria.

We introduce now a set of assumptions which ensure that this set is nonempty. The main assumptions concern the demand of the individual consumers. We suppose, to remain closer to the standard partial equilibrium Cournot model, that the individual consumer's demand is separable into a price and a wealth component. More specifically, the demand of every consumer having monetary wealth $m > 0$ (which includes past savings, past dividends distributed at the end of the previous period and the present value of expected future income, if we adopt the temporary equilibrium interpretation of the model), getting current wage income w (put $w = 0$ if he is unemployed) and expecting to receive dividends π at the end of the current period, is[6]:

$$\phi(p, \pi, w, m) = h(p)[m + \pi + w],\ \ p > 0, m > 0, \pi \ge 0, w \ge 0.$$

[6]The reader will find in Appendix II examples of a demand function explicitly derived from a consumer's intertemporal utility maximizing program.

The function h above satisfies the following assumption:

Assumption 1. h, defined and continuous for all positive prices, is nonnegative and, whenever positive, twice continuously differentiable and strictly decreasing. Also

$$h(0) \stackrel{\text{def}}{=} \lim_{p \to 0} h(p) \in (0, \infty] \text{ and } \lim_{p \to \infty} h(p) = 0.$$

The next assumption restricts the *marginal propensity to consume* $ph(p)$ as follows:

Assumption 2. For all $p > 0$ and, whenever $h(p) > 0$,

$$ph(p) < \min\{1, \eta(p)\},$$

where $\eta(p) \stackrel{\text{def}}{=} \frac{h'(p)}{h(p)} p$ denotes the (Marshallian) price elasticity of the demand function.

The first condition, restricting the marginal propensity to be less than one, is familiar. The second condition is implied for instance by the assumption that there is, for a consumer, an intertemporal substitution effect which adds to the income effect of a price change, at least so long as current saving is nonnegative.[7] And in the static interpretation of the model, Assumptions 1 and 2 are jointly implied by smooth, convex, homothetic preferences (defined on the nonnegative orthant), such that the indifference curves do not cut the axis for the produced good and such that the marginal rate of substitution is strictly decreasing (whenever both goods are desired)[8]. These are assumptions made by Hart (1982), except that we do not exclude saturation of the produced good.

The last assumptions on consumer's behaviour also apply to the marginal propensity to consume, as follows:

[7]Indeed, $\left.\frac{d\{h(p)[m + \pi + w]\}}{d[(\pi + w)/p]}\right|_{p\,=\,\text{constant}} = ph(p)$ and $\left.\frac{d\{h(p)[m + \pi + w]\}}{d[(\pi + w)/p]}\right|_{\pi + w\,=\,\text{constant}} = \frac{ph(p)[m + \pi + w]}{\pi + w}\eta(p)$, so that $\left.\frac{d\{h(p)[m + \pi + w]\}}{d[(\pi + w)/p]}\right|_{p\,=\,\text{constant}} < \left.\frac{d\{h(p)[m + \pi + w]\}}{d[(\pi + w)/p]}\right|_{\pi + w\,=\,\text{constant}}$ implies $ph(p) < \eta(p)$, if $ph(p)[m + \pi + w] \leq \pi + w$.

[8]Indeed, notice that the ratio of the quantities of the produced and non-produced goods is, for an optimal consumption: $h(p)/[1 - ph(p)]$. Elasticity of substitution is then easily determined, as a function of $p : \sigma(p) = [\eta(p) - ph(p)]/[1 - ph(p)]$. For differentiable strictly convex preferences (whenever the produced good is desired), one must have, for all $p, 0 < \sigma(p) \leq \infty$. As $ph(p) < 1$ is imposed by the budget constraint together with the assumption that the indifference curves do not cut the axis for the produced good, Assumption 2 follows.

Assumption 3.[9]

$$\lim_{p\to\infty} ph(p) = 0;$$

Assumption 4.

$$-\frac{h''(p)p}{h'(p)} < 2\max\{1, \eta(p)\}$$

an inequality which will be conveniently used to prove quasi-concavity of the profit functions.

If we introduce the variable $x \stackrel{\text{def}}{=} h(p)$, we can state alternatively Assumption 4 in the following way: the marginal propensity to consume, px, is concave whenever increasing, both as a function of p (with x depending upon p by h) and as a function of x (with p depending upon x by the inverse of h).

In theorem 3 of Novshek (1985), Assumption 2 – saturation at a nil price – and Assumption 3 – ($\psi_Y + Y\psi_{YY} \leq 0$) – are stronger than what we have here on the demand side[10]. However, by this theorem 4, we know that our assumptions on demand are too weak to get existence while keeping his minimal assumptions on the production side.

Therefore, we assume that the i-th firm has a production function f_i such that:

Assumption 5. For any $i \in N$, f_i is defined on $\mathbb{R}_+$, nonnegative, twice continuously differentiable, concave and strictly increasing. Also

$$f_i(0) = 0.$$

The assumption of strict monotonicity eliminates the possibility of the so-called "Marxian unemployment" which would appear should the marginal productivity become nil at a finite (and less than full) employment level.

[9] Assumption 3 is introduced essentially to obtain continuity of the profit functions at the origin. By using a more general existence theorem it could be weakened to: $\eta(p) > 1$ for arbitrarily large values of p.

[10] Using the expression for ψ_Y and ψ_{YY} as calculated below ($\psi_Y = h/h'Y$ and $\psi_{YY} = -h^2h''/h'^3Y^2$), we have for our model:

$$\psi_Y + Y\psi_{YY} = -\frac{h}{h'Y}\left(\frac{hh''}{h'^2} - 1\right) \leq 0 \text{ iff } \frac{hh''}{h'^2} \leq 1 \text{ if } -\frac{h''p}{h'} \leq \eta,$$

which is stronger than Assumption 4.

Under the above assumption of a multiplicatively separable individual demand function, the aggregate demand for the product becomes:

$$Y = \Phi(p, Z, \pi, w, m, L) = h(p)[Zw + L(m + \pi)].$$

Given Assumption 1, the inverse demand is defined, for $Y > 0$, by:

$$\begin{aligned} p &= \psi(Y, Z, \pi, w, m, L) \\ &= h^{-1}\left[\frac{Y}{Zw + L(m+\pi)}\right] \quad \text{if } Y \leq h(0)[Zw + L(m+\pi)] \\ &= 0 \quad\quad\quad\quad\quad\quad\quad\quad\quad \text{otherwise;} \end{aligned}$$

also, $\psi(0, Z, \pi, w, m, L) = \sup\{p \in \mathbb{R}_+ - \{0\} : h(p) > 0\} = \overline{p} \in (0, \infty]$.

We can then compute, for $Y > 0$ and $\psi > 0$, the first derivatives:

$$\psi_Y = \frac{h}{h'Y} < 0 \text{ and } \psi_Z = -\frac{h^2}{h'Y} w \geq 0.$$

Existence of the inverse demand function ψ for $Y > 0$ implies that the profit function of the producer $i \in N, \Pi_i(\cdot, y_{-i}, z_i, z_{-i}, \pi, w, m, L)$, as given above, is well-defined for any values of the parameters if $y_j > 0$ for some $j \in N$. By Assumption 3, we can also define:

$$\Pi_i(0, 0, z_i, z_{-i}, \pi, w, m, L) = \lim_{y \to 0} \Pi_i(y_i, y_{-i}, z_i, z_{-i}, \pi, w, m, L) = -wz_i.$$

Now, Assumption 2 entails that the "Ford effect" of a change in the individual producer's employment level on his revenue (i.e. $y_i\psi_Z$) is dominated by the corresponding cost effect (i.e. w), so that the profit function Π_i is decreasing in z_i, for $w > 0$, and $y_i < f_i(z_i)$. Indeed, for $w > 0$,

$$\begin{aligned} \frac{\partial \Pi_i}{\partial z_i} = y_i\psi_Z - w &= -y_i \frac{h^2}{h'Y} w - w \\ &= w\left[\frac{y_i}{Y}\frac{\psi h(\psi)}{\eta(\psi)} - 1\right] < w\left[\frac{y_i}{Y} - 1\right] \leq 0 \end{aligned}$$

by Assumption 2, if $f_i(z_i) > y_i > 0$ and $\psi > 0$, and

$$\frac{\partial \Pi_i}{\partial z_i} = -w < 0$$

if $y_i = 0$ and $\psi > 0$ or if $y_i \geq 0$ and $\psi = 0$.

Hence, for $w > 0$, every producer $i \in N$ will only choose (y_i, z_i) on his efficient frontier, i.e. $y_i = f_i(z_i)$. Thus we may consider that he has only one strategic variable, namely $z_i \in [0, \infty)$. If $w = 0$, $\partial \Pi_i / \partial z_i = 0$, so that inefficient employment levels may be chosen, but they do not dominate the efficient one. As a consequence, given any $m > 0$ and $L > 0$, we introduce for every $i \in N$ the new payoff function $\tilde{\Pi}_i$ (where we omit m and L as arguments in order to shorten notation):

$$\tilde{\Pi}_i(z_i, z_{-i}, \pi, w) \stackrel{\text{def}}{=} \Pi_i(f_i(z_i), (f_j(z_j))_{j \neq i}, z_i, z_{-i}, \pi, w, m, L).$$

To prove the existence of an extended Cournot-Nash equilibrium for given values of the parameters w, m and L, i.e. to prove that $\mathcal{E}(w, m, L) \neq \emptyset$, we consider an associated game where:

i) the payoff of every player $i \in N$ is the associate profit function $\tilde{\Pi}_i$;

ii) his strategies are the levels of employment $z_i \geq 0$;

iii) and for each choice of strategies $z_{-i} \in \mathbb{R}_+^{n-1}$ by the other players, the strategies of player $i \in N$ are constrained to be in the interval $[0, \max\{0, L - \sum_{j \neq i} z_j\}]$.

We denote by $\tilde{\mathcal{E}}(w, m, L) \subset \mathbb{R}_+^n$ the set of equilibria in the associated game, which are n-tuples $z^* \in \mathbb{R}_+^n$ such that:

(i) $\forall\, i \in N,\ z_i^* \in \left[0, L - \sum_{j \neq i} z_j^*\right]$ and $\tilde{\Pi}_i(z_i^*, z_{-i}^*, \pi^*, w) = \max_{z_i \in [0, L - \sum_{j \neq i} z_j^*]} \tilde{\Pi}_i(z_i, z_{-i}^*, \pi^*, w)$, and

(ii) $L\pi^* = \sum_{i \in N} \tilde{\Pi}_i(z^*, \pi^*, w)$.

Now, we are in a position to prove the existence of an extended Cournot-Nash equilibrium under Assumptions 1 to 5. Clearly, to prove the existence of an extended Cournot-Nash equilibrium, it is enough to prove the existence of an equilibrium in the associated game. For this purpose we shall use Debreu (1952) "Social Equilibrium Existence Theorem", introducing a fictitious player to take care of condition (ii). The difficulty is to prove that the payoff function

$\tilde{\Pi}_i$, is quasi-concave in the variable z_i (see Lemmata 1 to 3 and the resulting proposition in Appendix I), where the proof of existence is given). We may state:

Theorem 1 *Consider any $m > 0$ and $L > 0$. Under Assumptions 1 to 5, for every $w \geq 0$ and for $n \geq 2$, there is an extended Cournot-Nash equilibrium, i.e. $\mathcal{E}(w, m, L) \neq \emptyset$. Moreover, for $n = 1$, there is a monopoly equilibrium, which is unique if $w > 0$. Lastly, there exists $\overline{w} \in (0, \infty]$ such that for any $w \in [0, \overline{w})$ no equilibrium is trivial: $(y, z) \in \mathcal{E}(w, m, L)$ implies $Y = \sum_{i \in N} y_i > 0$.*

3 Existence of involuntary unemployment

In this section, we address the more demanding question of the possibility of unemployment, arising at any positive wage. For fixed values of the parameters m and L, for a given money wage w, and for any equilibrium $(y^*, z^*) \in \mathcal{E}(w, m, L)$, a situation of unemployment will obtain if (and only if) $Z^* = \sum_{i \in N} z_i^* < L$. A natural question is whether by some modification of the wage level one could reduce, or even suppress unemployment. Such a question leads to a definition of involuntary unemployment.

For any possible value of the parameters let:

$$\mathcal{Z}(w, m, L) \stackrel{\text{def}}{=} \{Z \in \mathbb{R}_+ : Z = \sum_{i \in N} z_i, \text{ for some } (y, z) \in \mathcal{E}(w, m, L)\}.$$

For fixed values of m and L, $\mathcal{Z}(\cdot, m, L)$ gives the total labour demand equilibrium correspondence. Any selection Z for $\mathcal{Z}(\cdot, m, L)$ is a function from $\mathbb{R}_+$ to itself such that: $\forall\, w \in \mathbb{R}_+, Z(w) \in \mathcal{Z}(w, m, L)$. Then we say that there is *involuntary unemployment given* m and L if, for any selection $\hat{Z}$ for $\mathcal{Z}(\cdot, m, L)$:

$$\inf_{w > 0} \{L - \hat{Z}(w)\} > 0.$$

To illustrate this notion we may suppose that the correspondence $\mathcal{Z}(\cdot, m, L)$ has strong regularity properties and possesses a continuous upper boundary $\mathbf{Z}$, for all positive wages. Then in the example given by Figure 1, the amount of involuntary unemployment is given by the distance I. Of course, as discussed in the introduction, we could in such a situation adjust parametrically the nominal wage to zero and then talk about "underemployment" in the sense of an inefficient employment level. We leave this semantic decision to the reader.

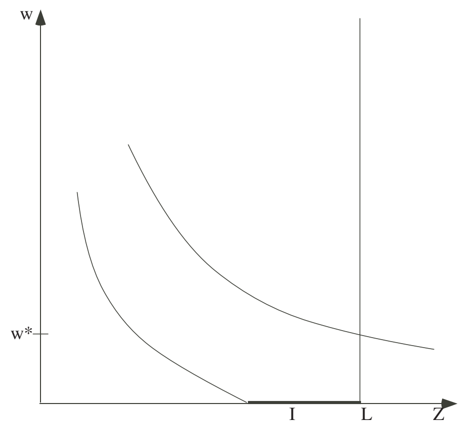

Figure 1:

We have also represented the competitive equilibrium demand for labour D, that is, the demand for labour arising at each value of w, when the price of the produced good is fixed at its market-clearing level. It may very well lead, as shown in the figure, to a positive equilibrium wage w^*.

As it will be shown, Assumptions 1 to 5 are sufficient to ensure that all equilibria involve unemployment, at any money wage above some positive reservation wage, and given appropriate values of the parameters m and L. However, our definition of involuntary unemployment requires a zero reservation wage, and in that case we need two additional assumptions, one on the individual demand function and the other on the production functions:

Assumption 6.

$$\lim_{p \to 0} \eta(p) < 1/n;$$

Assumption 7.

$$\forall\, i \in N, \lim_{x_i \to \infty} f_i(z_i) = \infty.$$

By Assumption 6, the producers' marginal revenue becomes negative at finite production levels, so that the equilibrium output price is bounded away from zero even if the money wage tends to zero. By Assumption 7, employment cannot increase indefinitely while output remains bounded. Given these additional assumptions, we state:

Theorem 2 *Under Assumptions 1 to 5, given any $L > 0$ (alternatively, given any $M = Lm > 0$), and any $\underline{w} > 0$, there is a low enough $\underline{m} > 0$ such that for all $m \in (0, \underline{m})$, $\inf_{w > \underline{w}}\{L - \hat{Z}(w)\} > 0$, for any selection $\hat{Z}$ for $\mathcal{Z}(\cdot, m, L)$. The same holds true for $\underline{w} = 0$, i.e. there is involuntary unemployment, under the additional Assumption 6, given $L > 0$, or under the additional Assumptions 6 and 7, given $M = Lm > 0$.*

Proof: The proof is by contradiction.

Take any sequence $\{(y^s, z^s, p^s, \pi^s, w^s, m^s, L^s)\}_{s=0}^{\infty}$ such that, for all s, $(y^s, z^s) \in \mathcal{E}(w^s, m^s, L^s)$, $p^s = \psi(Y^s, Z^s, \pi^s, w^s, m^s, L^s)$, $w^s > \underline{w}$, $L^s\pi^s = p^sY^s - w^sZ^s$ and, to obtain the contradiction, $Z^s = L^s$. Also, $\lim_{s\to\infty} m^s = 0$, and either $\lim_{s\to\infty} L^s = L > 0$ or $\lim_{s\to\infty} L^s m^s = M > 0$.

First, consider the case $\underline{w} > 0$. As equilibrium profits are necessarily nonnegative, by Assumption 5, we have, for all s, $L^s\pi^s = p^sh(p^s)L^s(\pi^s + m^s + w^s) - w^sL^s \geq 0$ and hence:

$$\pi^s = \frac{p^sh(p^s)}{1 - p^sh(p^s)}m^s - w^s \geq 0.$$

We must first show that the sequence $\{p^sh(p^s)/[1 - p^sh(p^s)]\}_s$ is bounded from above. This results immediately from Assumptions 2 and 3 unless $\lim_{s\to\infty} p^s = 0$. However, given that, for any $i \in N$, $\partial\tilde{\Pi}_i(z^s, \pi^s, w^s)/\partial z_i \geq 0$ if $z_i^s > 0$, which is a necessary condition for maximization of $\tilde{\Pi}_i(\cdot, z_{-i}^s, \pi^s, w^s)$, we have $p^s > w^s/f_i'(z_i^s)$. (See the proposition in Appendix I, stating that this inequality holds whenever $\tilde{\Pi}_i$ is nondecreasing in z_i and $w > 0$). As $\lim_{s\to\infty} z_j^s > 0$ for at least one producer, say the j-th, $\{w^s/f_j'(z_j^s)\}_s$ and hence $\{p^s\}_s$ is bounded away from zero if $\underline{w} > 0$. So, using the above expression for π^s, we get the contradiction we were looking for, because, as $\lim_{s\to\infty} m^s = 0$, $\lim_{s\to\infty} \sup \pi^s = -\lim_{s\to\infty} \inf w^s \leq -\underline{w} < 0$. Thus, as long as L is not arbitrarily low, full employment cannot be sustained in equilibrium by arbitrarily small values of m, whenever the reservation wage is positive.

Now consider the case $\underline{w} = 0$. It is clear that the same contradiction applies if $\{w^s\}_s$ is bounded away from zero, so that we must suppose $\lim_{s\to\infty} w^s = 0$. Using the above expression for π^s, we get:

$$Y^s = \frac{h(p^s)}{1 - p^s h(p^s)} L^s m^s.$$

If we can show that the sequence $\{h(p^s)/[1 - p^s h(p^s)]\}_s$ is bounded from above, which, outside the trivial case where $h(0) < \infty$, is tantamount to showing that $\{p^s\}_s$ is bounded away from zero, then the contradiction is obtained, for $\lim_{s\to\infty} L^s = L > 0$, because, as $\lim_{s\to\infty} m^s = 0$, $\lim_{s\to\infty} Y^s = 0$ implying, by Assumption 5, $\lim_{s\to\infty} Z^s = 0 < L$ contrary to $Z^s = L^s$ for all s. And, for $\lim_{s\to\infty} M^s = M > 0$, we have $\lim_{s\to\infty} L^s = \infty$ and hence, using Assumption 7, $\lim_{s\to\infty} Y^s = \infty$, whereas $\{L^s m^s h(p^s)/[1 - p^s h(p^s)]\}$ is bounded from above, so that we get again a contradiction.

Thus we must show that $\{p^s\}_s$ is bounded away from zero even if $\lim_{s\to\infty} w^s = 0$. For that purpose, we shall use again the first-order condition for maximization of $\tilde{\Pi}_i(\cdot, z^s_{-i}, \pi^s, w^s)$ if $z^s_i > 0$ (as it must be for at least one producer): $\partial\tilde{\Pi}_i(z^s, \pi^s, w^s)/\partial z_i \geq 0$. By Lemma 2 (see Appendix I), we know that this condition implies $f_i(z^s_i)/Y^s < \eta(p^s)$. By summing over all the producers, recalling that $\partial\tilde{\Pi}_i/\partial z_i < 0$ only if $z^s_i = 0$, and taking limits, we obtain, if $\lim_{s\to\infty} p^s = 0$:

$$1 = \lim_{s\to\infty} \sum_{i\in N} \frac{f_i(z^s_i)}{Y^s} \leq n \lim_{s\to\infty} \eta(p^s) < 1, \text{by Assumption 6.}$$

Hence, Assumption 6 leads to a contradiction if we assume that $\lim_{s\to\infty} p^s = 0$. Thus, $\{p^s\}_s$ is bounded away from zero and the result stated in the theorem, for $\underline{w} = 0$, follows. ■

Our definition of involuntary unemployment and the corresponding statement of Theorem 2 does not directly take into account equilibria at a nil wage. In this case, a similar result obtains if we exclude inefficient equilibria. More precisely, there is $\overline{m} > 0$ such that for all $m \in (0, \overline{m})$, given $L > 0$ or, alternatively, given $M = Lm > 0$, any *efficient* equilibrium $(y, z) \in \mathcal{E}(0, m, L)$ is such that unemployment prevails. By excluding inefficient equilibria, we rule out $(y, z) \in \mathcal{E}(0, m, L)$ such that $y_i < f_i(z_i)$ for some $i \in N$, or $Y > h(0)Lm$.

We conclude this section by emphasizing that Assumption 6, which plays a crucial role as a condition leading to the existence of involuntary unemployment, becomes more restrictive as the number of firms increases, i.e. as the economy becomes more competitive. At the limit, when the economy is perfectly competitive, it cannot be fulfilled in conjunction with Assumption 2. In this limit case, involuntary unemployment is still possible, but it requires demand saturation in the output market. Indeed, if this market is perfectly competitive and unemployment remains as w tends to zero, we get, by first order conditions for profit maximization:

$$\forall\, i \in N, f_i'(z_i)p = w$$

and also

$$Y = \frac{h(p)Lm}{1 - ph(p)} \text{ and } Z < L.$$

Thus, as $f_i'(z_i) > 0$ by Assumption 5, p tends to zero as w tends to zero and $\lim_{p \to 0} h(p) < \infty$, otherwise the level of production and hence of employment would tend to infinity. On the contrary, when competition is imperfect and under Assumption 6, the equilibrium price does *not* tend to zero with the money wage, so that unemployment may remain even if $\lim_{p \to 0} h(p) = \infty$.

4 An extension to the multisectorial case

In this section we propose a straightforward extension of our analysis to a particular case of an economy with several sectors. Consider an economy with $K + 2$ goods: K produced goods, labour and money. On the production side the economy is divided into K productive sectors. With each sector $k = 1, \cdots, K$ is associated a set N^k of firms, producing the same good. We have $N = \cup_{k=1}^{K} N^k$. Each firm $i \in N^k$ in sector k has a production function f_i defined as previously, for the one product case, so that no intermediate goods are considered. Whatever the general money wage $w \geq 0$, labour is supplied on a single market at the level $L > 0$. If the price in sector k is p^k and if firm $i \in N^k$ produces y_i with z_i units of labour, its profit is given by $p^k y_i - wz_i$ and is supposed to remain undistributed until the end of the period.

On the consumption side, each identical consumer is assumed to have an initial positive monetary wealth $m > 0$, to expect a dividend π, a wage income equal to w if he is employed and equal to zero if not. The behaviour of every consumer is described by K demand functions,

$$\phi^k(p^k, \pi, w, m) = h^k(p^k)[m + \pi + w], \ k = 1, \cdots, K,$$

where w is to be put equal to zero, if he is unemployed. Thus, the demand for the k-th product is assumed to be independent of the other products prices, an assumption which, together with the absence of intermediate goods, keeps the extension to the multisectorial case straightforward. At actual employment level $Z \leq L$, the aggregate demand for product k is given by:

$$\begin{aligned} \Phi^k(p^k, Z, \pi, w, m, L) &\stackrel{\text{def}}{=} Z\phi^k(p^k, \pi, w, m) + (L - Z)\phi^k(p^k, \pi, 0, m) \\ &= h^k(p^k)[Zw + L(m + \pi)]. \end{aligned}$$

At $Y^k > 0$, the inverse demand for product k is defined by:

$$\begin{aligned} p^k &= \psi^k(Y^k, Z, \pi, w, m, L) \\ &= (h^k)^{-1}\left[\frac{Y^k}{Zw + L(m + \pi)}\right] \quad \text{if } Y^k \leq h^k(0)[Zw + L(m + \pi)] \\ &= 0 \quad\quad \text{otherwise;} \end{aligned}$$

also, $\psi^k(0, Z, \pi, w, m, L) = \sup\{p^k \in \mathbb{R}_+ - \{0\} : h^k(p^k) > 0\}$.

The profit Π_i of a producer $i \in N^k$, $k \in \{1, \cdots, K\}$, may then be defined for any values of the parameters $\pi \geq 0, w \geq 0$, $m > 0, L > 0$ as before:

$$\Pi_i(y_i, y_{-i}, z_i, z_{-i}, \pi, w, m, L) \stackrel{\text{def}}{=} y_i\psi^k(Y^k, Z, \pi, w, m, L) - wz_i$$

for $(y, z) \in \mathbb{R}_+^{2n}$. Given this notation, the extended Cournot-Nash equilibrium can be defined precisely as before.

The extension of our results is also straightforward. If we impose the same assumptions, substituting p^k, h^k and $\eta^h, k = 1, \cdots, K$ for p, h and η, respectively, we can keep the same arguments in all the proofs, as a simple inspection suffices to show. There is one exception. In the proofs of both Theorems 1 and 2, we used the fact that $ph(p)/[1 - ph(p)]$ is bounded from above as long as p is bounded away from zero. In the multisectorial case, we must have at equilibrium:

$$L\pi = \sum_{k=1}^{K}\{p^k h^k(p^k)[Zw + L(\pi + m)] - wZ^k\}$$

leading to

$$\pi = \frac{\sum_{k=1}^{tK} p^k h^k(p^k) m}{1 - \sum_{k=1}^{K} p^k h^k(p^k)} - \frac{Z}{L} w,$$

so that it is the overall marginal propensity to consume $\sum_{k=1}^{K} p^k h^k(p^k)$ that must be kept smaller than one.

Thus, in order to obtain Theorems 1 and 2, besides imposing to each sector all the Assumptions 1 to 7 when necessary, we must modify Assumption 2 in the following way:

Assumption 2*. For all $p^k > 0$ and whenever $h^k(p^k) > 0 (k = 1, \cdots, K)$,

$$p^k h^k(p^k) < \eta^k(p^k).$$

Also,

$$\sum_{k=1}^{K} p^k h^k(p^k) < 1.$$

5 Conclusion

In this paper we have analysed the possibility of involuntary unemployment in an extended Cournot model. The present extension was done under a particular conjecture concerning the indirect effects of the producers' decisions on total demand through consumers' wealth: the producers are supposed to be fully aware of the "Ford effect", that is the indirect effect through wage income, but to treat distributed profits parametrically. It turns out that this is the most difficult case since both some indirect effect and some parametric adjustment have to be taken care of. In a companion paper[11], we have obtained (more easily) the possibility of involuntary unemployment under the conjecture that the producers take into account all indirect effects, i.e. both through wage income and distributed profits. More recently, integrating a previous version of the present paper and Hart's (1982) model (with syndicates maximizing wage receipts), Silvestre (1987) obtains the same possibility while treating parametrically the overall consumers' wealth. This he obtains for the symmetric case and under an assumption closely related to Assumption 6 above. Silvestre stresses the fact that this assumption is more realistic than Hart's

[11] d'Aspremont, Dos Santos Ferreira and Gérard-Varet (1989).

corresponding assumption that the marginal revenue of any producer is always positive, even for large quantities (that is even for prices close to zero), when all are producing the same amount. In his case, as here, when the output is large (or price low) enough, marginal revenue becomes negative. Finally it is interesting to compare the examples proposed by Silvestre (1987) to ours (see Appendix II) since, in order to get both existence of equilibrium (in the symmetric case) and involuntary unemployment, with all indirect effects treated parametrically, he uses a utility function in the produced and non-produced goods of the C.E.S. type modified by an additive term in the non-produced good. In our case, with Ford effects, we have a constant elasticity of substitution in present and future consumption (using the temporary equilibrium framework). The small elasticity of demand at prices close to zero that we need results from a combination of either intertemporal complementarity and inelastic price expectations or intertemporal substitutability and elastic price expectations. This together with a small individual wealth, as influenced by pessimistic consumers' expectations about future income, restricts the Pigou effect and induces involuntary unemployment.

Appendix I

In this appendix we prove continuity and quasi-concavity of the payoff functions $\tilde{\Pi}_i$, in the associated game where every producer $i \in N$ has only one strategic variable z_i.

To prove quasi-concavity in z_i of the payoff function $\tilde{\Pi}_i$, we show that $\tilde{\Pi}_i(\cdot, z_{-i}, \pi, w)$ is (strictly) concave so long as the "Ford effect" of an employment increase is dominated by the corresponding supply effect and, besides, the marginal revenue remains positive (Lemma 3) and that it is decreasing as soon as the "Ford effect" becomes the dominating effect (Lemma 1) or the marginal revenue becomes negative (Lemma 2). To do this, we need first some additional notation.

Let $\tilde{z}_i(z_{-i}, \pi, w)$ be the lowest employment level of producer $i \in N$ at which, given other employment levels z_{-i}, dividend π and wage w, the supply effect f_i' of an employment increase ceases to dominate the "Ford effect" $h(\psi)w$. More precisely, for any $(z_{-i}, \pi, w) \in \mathbb{R}_+^{n+1}$, let:

$$\overline{z}_i(z_{-i}, \pi, w) \stackrel{\text{def}}{=} \sup \Bigg\{ z_i \in [0, \infty] : \forall\, \zeta_i \in (0, z_i), \\ Q_i(\zeta_i, z_{-i}, \pi, w) \stackrel{\text{def}}{=} f_i'(\zeta_i) - \frac{f_i(\zeta_i) + \sum_{j \neq i} f_j(z_j)}{(\zeta_i + \sum_{j \neq i} z_j) w + L(\pi + m)} w > 0 \Bigg\}.$$

Observe that, whenever $\overline{z}_i(z_{-i}, \pi, w) < \infty$, the supply and the "Ford effect" balance at this point. Notice also that we have, by Assumption 5, $\overline{z}_i(z_{-i}, \pi, 0) = \infty$ for every $(z_{-i}, \pi) \in \mathbb{R}_+^n$.

Now, let $\tilde{z}_i(z_{-i}, \pi, w)$ be the lowest employment level of producer $i \in N$ at which, given other employment levels z_{-i}, dividend π and wage w, the marginal revenue $R_i \stackrel{\text{def}}{=} \psi + f_i \psi_Y$ ceases to be positive. More precisely, for any $(z_{-i}, \pi, w) \in \mathbb{R}_+^{n+1}$, let:

$$\tilde{z}_i(z_{-i}, \pi, w) \stackrel{\text{def}}{=} \sup \Bigg\{ z_i \in [0, \infty] : \forall\, \zeta_i \in (0, z_i), \\ R_i(\zeta_i, z_{-i}, \pi, w) \stackrel{\text{def}}{=} \psi \left[1 - \frac{f_i(\zeta_i)}{f_i(\zeta_i) + \sum_{j \neq i} f_j(z_j)} \frac{1}{\eta(\psi)} \right] > 0 \Bigg\}$$

where ψ is a short notation for $\psi(f_i(\zeta_i) + \sum_{j \neq i} f_j(z_j), \zeta_i + \sum_{j \neq i} z_j, \pi, w, m, L)$. Again, notice that whenever $\tilde{z}_i(z_{-i}, \pi, w) < \infty$, $R_i = 0$ at this point.

Lemma 1 *Under Assumptions 1, 2 and 5, for any $i \in N$, $(z_{-i}, \pi) \in \mathbb{R}_+^n$, $w > 0$, if the interval $[\overline{z}_i(z_{-i}, \pi, w), \infty)$ is nonempty then:*

a) $Q_i(z_i, z_{-i}, \pi, w) \leq 0$ for any $z_i \in [\overline{z}_i(z_{-i}, \pi, w), \infty)$;

b) the payoff function $\tilde{\Pi}_i(\cdot, z_{-i}, \pi, w)$ is decreasing in this interval.

Proof: Take any $w > 0$ and $(z_{-i}, \pi) \in \mathbb{R}_+^n$. Assume $\overline{z}_i(z_{-i}, \pi, w) < \infty$. As already noticed, for $z_i = \overline{z}_i(z_{-i}, \pi, w), Q_i(z_i, z_{-i}, \pi, w) = 0$. But

$$\begin{aligned} \frac{\partial}{\partial z_i} Q_i(z_i, z_{-i}, \pi, w) &= f_i'' - \frac{w Q_i(z_i, z_{-i}, \pi, w)}{w \sum_{j \in N} z_j + L(\pi + m)} \\ &\leq -\frac{w Q_i(z_i, z_{-i}, \pi, w)}{w \sum_{j \in N} z_j + L(\pi + m)} \qquad \text{(by Assumption 5),} \end{aligned}$$

so that, by a continuity argument, $Q_i(z_i, z_{-i}, \pi, w)$ never becomes positive again when z_i increases, because its derivative can only remain positive as long as $Q_i(z_i, z_{-i}, \pi, w) < 0$. Now, suppose $Q_i(z_i, z_{-i}, \pi, w) \leq 0$, i.e. $f_i' \leq h(\psi)w$, and consider the derivative of the payoff function:

$$\frac{\partial \tilde{\Pi}_i}{\partial z_i} = \psi f_i' \left[1 - \frac{f_i}{Y\eta(\psi)}\right] + \psi h(\psi) w \left[\frac{f_i}{Y\eta(\psi)}\right] - w.$$

If $\frac{f_i}{Y\eta(\psi)} \leq 1$, then

$$\frac{\partial \tilde{\Pi}_i}{\partial z_i} \leq \psi h(\psi) w - w = w[\psi h(\psi) - 1] < 0 \text{ by Assumption 2.}$$

If $\frac{f_i}{Y\eta(\psi)} > 1$, then:

$$\frac{\partial \tilde{\Pi}_i}{\partial z_i} \leq \psi h(\psi) w \left[\frac{f_i}{Y\eta(\psi)}\right] - w = w \left[\frac{\psi h(\psi)}{\eta(\psi)} \frac{f_i}{Y} - 1\right] < 0, \text{ by Assumption 2.}$$

■

Lemma 2 *Under Assumptions 1, 2, 4 and 5, for every $i \in N$, $(z_{-i}, \pi, w) \in \mathbb{R}_+^{n+1}$, the profit function $\tilde{\Pi}_i(\cdot, z_{-i}, \pi, w)$ is nonincreasing whenever $z_i \geq \tilde{z}_i(z_{-i}, \pi, w)$, and decreasing if $w > 0$.*

Proof: Clearly for any $(z_{-i}, \pi, w) \in \mathbb{R}_+^{n+1}$, if $\tilde{z}_i(z_{-i}, \pi, w) \geq \overline{z}_i(z_{-i}, \pi, w)$ then, by Lemma 1, $\tilde{\Pi}_i$ is decreasing for $z_i \geq \tilde{z}_i(z_{-i}, \pi, w)$. So, consider the case $\tilde{z}_i(z_{-i}, \pi, w) < \overline{z}_i(z_{-i}, \pi, w)$. Remark first that if

$$\sum_{j \in N} f_j(z_j) \geq \left[w \sum_{j \in N} z_j + L(\pi + m)\right] h(0),$$

then

$$\tilde{\Pi}_i(z_i, z_{-i}, \pi, w) = -w z_i,$$

satisfying the result of Lemma 2.

If not and $R_i = \psi \left[1 - \frac{f_i}{Y} \frac{1}{\eta(\psi)}\right] \leq 0$, with $\psi > 0$, then:

$$\begin{aligned} \frac{\partial \tilde{\Pi}_i}{\partial z_i} &= \psi f_i' \left[1 - \frac{f_i}{Y\eta(\psi)}\right] + w \left[\frac{\psi h(\psi)}{\eta(\psi)} \frac{f_i}{Y} - 1\right] \\ &\leq w \left[\frac{\psi h(\psi)}{\eta(\psi)} \frac{f_i}{Y} - 1\right] \leq 0 \\ &\qquad (< 0 \text{ for } w > 0) \text{ by Assumption 2.} \end{aligned}$$

It remains to consider the case where, for $z_i \in [\tilde{z}_i(z_{-i}, \pi, w), \overline{z}_i(z_{-i}, \pi, w)]$, $R_i = \psi \left[1 - \frac{f_i}{Y} \frac{1}{\eta(\psi)}\right]$ > 0. However, this case never occurs, as we will presently show. By Assumption 4, when $R_i \geq 0$, i.e. $\frac{f_i}{Y\eta(\psi)} \leq 1$,

$$\begin{aligned} \frac{f_i}{Y}\frac{h''(\psi)h(\psi)}{[h'(\psi)]^2} &= \left[\frac{f_i}{Y\eta}\right]\left[-\frac{h''\psi}{h'}\right] \\ &\leq \max\left\{0, -\frac{h''\psi}{h'}\right\} < 2 \text{ if } \eta(\psi) \leq 1 \\ &\leq \max\left\{0, \frac{1}{\eta}\left[-\frac{h''\psi}{h'}\right]\right\} < 2 \text{ if } \eta(\psi) \geq 1. \end{aligned}$$

Therefore, for $R_i \geq 0$ and $z_i \leq \overline{z}_i(z_{-i}, \pi, w)$, implying $f_i' - h(\psi)w \geq 0$,

$$\begin{aligned} \frac{\partial R_i}{\partial z_i} &= f_i'[2\psi_Y + f_i\psi_{YY}] + \psi_Z + f_i\psi_{YZ} \\ &= \frac{h(\psi)}{h'(\psi)Y}\left\{\left(2 - \frac{f_i h''(\psi)h(\psi)}{Y[h'(\psi)]^2}\right)(f_i' - h(\psi)w) + h(\psi)w\left(1 - \frac{f_i}{Y}\right)\right\} \leq 0 \end{aligned}$$

since

$$\psi_{YY} = \frac{-[h(\psi)]^2 h''(\psi)}{Y^2[h'(\psi)]^3} \text{ and } \psi_{YZ} = \frac{-[h(\psi)]^2 w}{Y^2 h'(\psi)}\left(1 - \frac{h''(\psi)h(\psi)}{[h'(\psi)]^2}\right).$$

This implies that, for any $(z_{-i}, \pi, w) \in \mathbb{R}_+^{n+1}$, once R_i has become nonpositive, at $z_i = \tilde{z}_i(z_{-i}, \pi, w)$, it remains so in the whole interval $[\tilde{z}_i(z_{-i}, \pi, w), \overline{z}_i(z_{-i}, \pi, w)]$. To conclude, $\tilde{\Pi}_i$ is nonincreasing in the interval, and decreasing if $w > 0$. By Lemma 1, so it is in $[\overline{z}_i(z_{-i}, w), \infty]$, when this interval is nonempty. ■

Lemma 3 *Under Assumptions 1 to 5, for every $i \in N$, $(z_{-i}, \pi, w) \in \mathbb{R}_+^{n+1}$, the payoff function $\tilde{\Pi}_i(\cdot, z_{-i}, \pi, w)$ is strictly concave in the interval* $(0, \min\{\overline{z}_i(z_{-i}, \pi, w), \tilde{z}_i(z_{-i}, \pi, w)\})$.

Proof: Take $(z_{-i}, \pi, w) \in \mathbb{R}_+^{n+1}$ and $z_i \in (0, \min\{\overline{z}_i(z_{-i}, \pi, w), \tilde{z}_i(z_{-i}, \pi, w)\})$. Then, using:

$\psi_{ZZ} = \frac{[h(\psi)]^3\psi^2}{Y^2 h'(\psi)}\left(2 - \frac{h''(\psi)h(\psi)}{[h'(\psi)]^2}\right)$, compute:

$$\begin{aligned}
\frac{\partial^2 \tilde{\Pi}_i}{\partial z_i^2} &= f_i''[\psi + f_i\psi_Y] + (f_i')^2[2\psi_Y + f_i\psi_{YY}] + 2f_i'[\psi_Z + f_i\psi_{YZ}] + f_i\psi_{ZZ} \\
&= f_i''[\psi + f_i\psi_Y] + \frac{h(\psi)}{h'(\psi)Y}\left\{(f_i')^2\left(2 - \frac{f_i h''(\psi)h(\psi)}{Y[h'(\psi)]^2}\right)\right. \\
&\quad -2f_i' h(\psi)w\left(1 + \frac{f_i}{Y} - \frac{f_i h''(\psi)h(\psi)}{Y[h'(\psi)]^2}\right) \\
&\quad \left. +(h(\psi)w)^2\left(\frac{2f_i}{Y} - \frac{f_i h''(\psi)h(\psi)}{Y[h'(\psi)]^2}\right)\right\} \\
&= f_i''[\psi + f_i\psi_Y] + \frac{h(\psi)}{h'(\psi)Y}(f_i' - h(\psi)w)^2\left(2 - \frac{f_i h''(\psi)h(\psi)}{Y[h'(\psi)]^2}\right) \\
&\quad +2\frac{h(\psi)}{h'(\psi)Y}\left(1 - \frac{f_i}{Y_i}\right)(f_i' - h(\psi)w)h(\psi)w.
\end{aligned}$$

Notice first that, for z_i in the specified interval,

$$f_i' - h(\psi)w > 0.$$

Also, $\psi + f_i\psi_Y > 0$ implying $\frac{f_i}{Y\eta(\psi)} < 1$, so that:

$$\frac{f_i h''(\psi)h(\psi)}{Y[h'(\psi)]^2} < 2, \text{using Assumption 4, as shown in the proof of Lemma 2.}$$

From these inequalities it is easy to check that:

$$\frac{\partial^2 \tilde{\Pi}_i}{\partial z_i^2} < 0, \text{using Assumption 5.}$$

■

Putting together the results of the above lemmata, we get:

Proposition 1 *Under Assumptions 1 to 5, for every $i \in N$, the payoff function $\tilde{\Pi}_i$ is continuous and, for any $(z_{-i}, \pi,) \in \mathbb{R}_+^{n+1}$, the function $\tilde{\Pi}_i(\cdot, z_{-i}, \pi, w)$ is quasi-concave. If $w > 0$, the function $\tilde{\Pi}_i(\cdot, z_{-i}, \pi, w)$ is non-decreasing only if $\psi f_i'(z_i) > w$. When $n = 1$ or if $z_{-i} = 0$, it has a unique maximum at $\tilde{z}_i(\tilde{z}_i > 0$ if $\overline{p}[1 - 1/\lim_{p\to\overline{p}} -\eta(p)]\lim_{z_i\to 0} f_i'(z_i) > w \geq 0$, with $\overline{p} = \sup\{p \in \mathbb{R}_+ - \{0\} : h(p) > 0\} \in (0, \infty])$.*

Proof: Continuity of $\tilde{\Pi}_i$ follows directly from its definition and the continuity of the inverse demand function, except possibly when z tends to 0 and $\psi(0, 0, \pi, w, m, L) = \infty$. But then, by

Assumption 3, $\lim_{z\to 0} \tilde{\Pi}_i(z,\pi,w) = L(\pi+m)\lim_{p\to\infty} ph(p) = 0$, so that $\tilde{\Pi}_i(\cdot,\pi,w)$ is continuous at the origin.

Quasi-concavity of $\tilde{\Pi}_i(\cdot,z_{-i},\pi,w)$ results directly from concavity in the interval $(0,\min\{\overline{z}_i(z_{-i},\pi,w)\tilde{z}_i(z_{-i},\pi,w)\})$ (Lemma 3) and the fact that this function is nondecreasing in the interval $[\min\{\overline{z}_i(z_{-i},\pi,w),\tilde{z}_i(z_{-i},\pi,w)\},\infty)$ (Lemmata 1 and 2).

Now, recalling that:

$$\frac{\partial\tilde{\Pi}_i}{\partial z_i} = \psi f_i'\left[1-\frac{f_i}{Y\eta(\psi)}\right] + \psi h(\psi)w\left[\frac{f_i}{Y\eta(\psi)}\right] - w,$$

and that, for $w>0$, $\partial\tilde{\Pi}_i/\partial z_i \geq 0$ implies $f_i' > h(\psi)w$ (by Lemma 1), and $f_i/Y\eta(\psi) < 1$ (by Lemma 2), we get:

$$\frac{\partial\tilde{\Pi}_i}{\partial z_i} \geq 0 \Rightarrow \psi f_i' > w \text{ whenever } w > 0.$$

Finally, if $\overline{p} = \psi(0,0,\pi,w,m,L) = \infty$ or if $\lim_{z_i\to 0} f_i'(z_i) = \infty$, we have $\lim_{z_i\to 0}[\partial\tilde{\Pi}_i/\partial z_i] = \infty$, so that $\tilde{\Pi}_i(\cdot,0,\pi,w)$ has a maximum at $\hat{z}_i \in (0,\min\{\overline{z}_i(0,\pi,w),\tilde{z}_i(0,\pi,w)\})$ (using Lemmata 1 and 2), which is unique by strict concavity of $\tilde{\Pi}_i(\cdot,0,\pi,w)$ in this interval (Lemma 3). If $\overline{p} < \infty$, we have $\lim_{z_i\to 0}[\partial\tilde{\Pi}_i/\partial z_i] = \overline{p}[1 - 1/\lim_{p\to\overline{p}} -\eta(p)]\lim_{z_i\to 0} f_i'(z_i) - w$. If this is positive, the same result follows. Notice that $\lim_{p\to\overline{p}} -\eta(p)$ is bounded only if $\lim_{p\to\overline{p}} -h'(p) = 0$, in the case where $\overline{p} < \infty$. ■

Proof of Theorem 1. In fact we apply Debreu's existence theorem (1952) to a game of $n+1$ players. The first n players are the firms in N. Each $i \in N$ has a quasi-concave payoff function $\tilde{\Pi}_i(\cdot,z_{-i},\pi,w)$ (see the proposition above) and, hence its best reply correspondence is convex-valued. Moreover, for each $i \in N$, the correspondence

$$z_{-i} \to \left[0, \max\left\{0, L - \sum_{\substack{j\neq i\\ j\in N}} z_j\right\}\right]$$

from $\mathbb{R}_+^{n-1}$ to $\mathbb{R}_+$ restricting its admissible strategies is continuous and has a compact graph so that (using again the proposition above) the function

$$\max_{z_i \in \left[0,\max\left\{0,L-\sum_{\substack{j\neq i\\ j\in N}} z_j\right\}\right]} \tilde{P}_i(z_i,z_{-i},\pi,w)$$

is continuous in (z_{-i}, π). The last player, say player 0, is fictitious and is introduced to obtain condition (ii) in the definition of the equilibrium. The strategic variable of this player is the (nonnegative, by Assumption 5) profit variable $\pi \in \mathbb{R}_+$. Now, in equilibrium, one should have:

$$L\pi = ph(p)[Zw + L(m+\pi)] - wZ = ph(p)L(m+\pi) - (1 - ph(p))Zw.$$

Hence

$$\pi \leq \frac{ph(p)}{1 - ph(p)} m,$$

by Assumption 2. Therefore, using the fact that the price at equilibrium should not be smaller than

$$\underline{p} \stackrel{\text{def}}{=} \psi \left(\sum_{i \in N} f_i(L), 0, 0, w, m, L \right)$$

and using Assumptions 2 and 3, we may restrict π to the set

$$[0, \overline{\pi}], \overline{\pi} = \max_{p \in [\underline{p}, \infty]} \left\{ \frac{ph(p)}{1 - ph(p)} \right\} m < \infty$$

(notice that $\underline{p} = 0$ implies $h(0) < \infty$ and hence $\underline{p}h(\underline{p}) = 0$). Then, as a payoff function for the fictitious player, we take

$$\tilde{\Pi}_0(z, \pi, w) - \left[\sum_{i \in N} \tilde{\Pi}_i(z, \pi, w) - L\pi \right]^2.$$

By Assumption 2 and the properties of each $\tilde{\Pi}_i$ it is immediate that $\tilde{\Pi}_0$ is continuous, and strictly quasi-concave in π by construction. Indeed, to prove strict quasi-concavity in π it suffices to show that

$$\frac{\partial \tilde{\Pi}_0(z, \pi, w)}{\partial \pi} = 0 \text{ implies } \frac{\partial^2 \tilde{\Pi}_0(z, \pi, w)}{\partial \pi^2} < 0.$$

Now, if we use the notation

$$\tilde{\Pi}_0(z, \pi, w) \stackrel{\text{def}}{=} -[\xi(z, \pi)]^2$$

we can write

$$\frac{\partial \tilde{\Pi}_0}{\partial \pi} = -2\xi(z, \pi)\xi_\pi(z, \pi) \text{ and } \frac{\partial^2 \tilde{\Pi}_0}{\partial \pi^2} = -2[\xi_\pi(z, \pi)]^2 - 2\xi(z, \pi)\xi_{\pi\pi}(z, \pi).$$

Using the definition of $\tilde{\Pi}_i$, $i \in N$, we can calculate:

$$\xi_\pi(z,\pi) = \psi_\pi Y - L = \left(-\frac{h^2}{h'} - 1\right) L = \left(\frac{\psi h}{\eta} - 1\right) L < 0 \text{ (by Assumption 2).}$$

So $(\partial\tilde{\Pi}_0/\partial\pi) = 0$ implies $\xi(z,\pi) = 0$ and hence

$$\frac{\partial^2\tilde{\Pi}_0}{\partial\pi^2} = -2[\xi_\pi(z,\pi)]^2 < 0.$$

Moreover, we see that the sign of $(\partial\tilde{\Pi}_0/\partial\pi)$ is equal to the sign of $\xi(z,\pi)$, so that $(\partial\tilde{\Pi}_0/\partial\pi)$ is nonnegative for $\pi = 0$ and nonpositive for $\pi = \overline{\pi}$, entailing that π maximizes $\tilde{\Pi}_0(z,\cdot,w)$ only if $\xi(z,\pi) = 0$. Thus by Debreu (1952) there exists a social equilibrium in the corresponding $(n+1)$-player game and, hence, an extended Cournot equilibrium.

In the monopoly case ($n = 1$) we have a two-player game. For $w > 0$ and any $\pi \in [0,\overline{\pi}]$ the firm's payoff function has a unique maximum (see the proposition above), say $z(\pi)$, and it can be shown that there is a unique value of $\pi \in [0,\overline{\pi}]$ satisfying $\tilde{\Pi}_0(z(\pi),\pi,w) = 0$. Indeed, either $\partial\tilde{\Pi}_1(z(\pi),\pi,w)/\partial z = 0$, by the first-order condition of maximization if $0 < z(\pi) < L$, so that $\xi_z(z,\pi) = 0$ and $d\xi(z(\pi),\pi)/d\pi = \xi_\pi < 0$, or $\partial\tilde{\Pi}_1(z(\pi),\pi,w)/\partial z > 0$ (resp. < 0), with $z(\pi) = L$ (resp. $= 0$) and local constancy of z. Hence, again, $d\xi(z(\pi),\pi)/d\pi = \xi_\pi < 0$. By continuity of $z(\cdot)$ and hence of $\xi(z(\cdot),\cdot)$, and as $d\xi(z(\pi),\pi)/d\pi < 0$, we get the uniqueness result.

Finally, consider the reservation price

$$\overline{p} = \sup\{p \in \mathbb{R}_+ - \{0\} : h(p) > 0\}.$$

Using again the proposition above, it is clear that no equilibrium is trivial, for $w \in [0,\overline{w})$, where

$$\overline{w} = \overline{p}[1 - 1/\lim_{p\to\overline{p}^-}\eta(p)]\max_{i\in N}[\lim_{z_i\to 0} f_i'(z_i)].$$

Indeed for at least one producer marginal revenue exceeds marginal cost when production vanishes.

Appendix II

We propose in this appendix examples of a demand function which is explicitly derived from a consumer's intertemporal utility maximizing program. Let us consider, to begin with, the one product case and assume, for the sake of simplicity, that the consumer has a two-period horizon and that his preferences are represented by a C.E.S. utility function. We can then write his program in the following form:

$$\max_{(c_1,c_2)\in\mathbb{R}^2_+} \left(c_1^{\sigma-1/\sigma} + \delta c_2^{\sigma-1/\sigma}\right)^{\sigma/\sigma-1}$$

s.t. $p_1c_1 + p_2c_2 \leq W$, with $\delta > 0$, $\sigma > 0$, and $\sigma \neq 1$, where c_t is the t-th period planned consumption, p_t is the t-th period output price, and W is the present value of the consumer's wealth. The consumer faces a single aggregate intertemporal budget constraint, so that he is supposed to be able to make intertemporal financial transfers in either direction. This simplifies the problem and entails differentiability of the demand function by avoiding corner solutions. But it is not essential and we will show that our examples may not violate the current budget constraint, in the case the consumer is not allowed to borrow.

The solution in c_1 to the consumer's problem is:

$$c_1 = \frac{W}{p_1 + \delta^\sigma p_1^\sigma (p_2)^{1-\sigma}}.$$

Let us assume that the expectation of the future price is a function of the observed price: $p_2 = P(p_1)$, where P is a twice continuously differentiable, nonnegative real-valued function, strictly increasing and such that $P(0) = 0$. Let us also define $\varepsilon(p) \stackrel{\text{def}}{=} P'(p)p/P(p)$. Then we have (omitting the subscript for the first period):

$$h(p) = \frac{1}{p}\frac{1}{1+\delta^\sigma[P(p)/p]^{1-\sigma}}$$

and

$$\eta(p) = 1 - \delta^\sigma \frac{(1-\sigma)[1-\varepsilon(p)][P(p)/p]^{1-\sigma}}{1+\delta^\sigma[P(p)/p]^{1-\sigma}}.$$

Also:

$$\frac{\eta'(p)p}{\eta(p)} = \frac{(1-\sigma)\delta^\sigma[P(p)/p]^{1-\sigma}}{\eta(p)(1+\delta^\sigma[P(p)/p]^{1-\sigma})}\left[\frac{(1-\sigma)[1-\varepsilon(p)]^2}{1+\delta^\sigma[P(p)/p]^{1-\sigma}} + \varepsilon'(p)p\right].$$

It is easy to check that Assumptions 1 to 4 are satisfied in the following cases:

(i) $\sigma < 1, \lim_{p\to\infty} \frac{P(p)}{p} = \infty$ and $\varepsilon'(p) \geq 0 (\varepsilon'(p) > 0$ if $\varepsilon(p) = 1)$, or

(ii) $\sigma > 1, \lim_{p\to 0} \varepsilon(p) \leq \left(\frac{\sigma}{\sigma - 1}\right), \lim_{p\to\infty} \dfrac{P(p)}{p} = 0$ and $\varepsilon'(p) \leq 0 (\varepsilon'(p) < 0$ if $\varepsilon(p) = 1)$.

Finally, Assumption 6 is entailed by the condition:

(i*) $\sigma < 1, \lim_{p\to 0} \frac{P(p)}{p} = \infty$ and $\lim_{p\to 0} \varepsilon(p) < \left(\frac{1}{n} - \sigma\right) \frac{1}{1-\sigma}$, or

(ii*) $\sigma > 1, \lim_{p\to 0} \dfrac{P(p)}{p} = 0$ and $\lim_{p\to 0} \varepsilon(p) > \left(\sigma - \frac{1}{n}\right) \frac{1}{\sigma - 1}$.

As expected, this condition is the more stringent the larger the number of firms (or, alternatively, the larger the elasticity of intertemporal substitution) as (i*) implies: $\sigma < 1/n$, and (ii) and (ii*) together imply:

$$\frac{\sigma - (1/n)}{\sigma - 1} < \lim_{p\to 0} \varepsilon(p) \leq \frac{\sigma}{\sigma - 1}.$$

The reader should also notice that Assumption 6 requires, for prices close to zero, a combination either of intertemporal complementarity and inelastic price expectations or of intertemporal substitutability and elastic price expectations.

Examples of expectation functions satisfying all the above conditions are:

$$P(p) = ap^{\alpha} + bp^{\beta}, \text{ with } 0 < \beta < \frac{(1/n) - \sigma}{1 - \sigma} \leq 1 < \alpha, \text{ in the case } \sigma < 1,$$

and in the case $\sigma > 1$

$$P(p) = ap^{-\alpha} + bp^{-\beta}, \text{ with } 0 < \alpha < 1 \leq \frac{\sigma - (1/n)}{\sigma - 1} < \beta \leq \frac{\sigma}{\sigma - 1}.$$

Two last points should be emphasized. First, if μ is the proportion of currently disposable wealth in the lifetime wealth W, we may impose:

$$\delta^{\sigma} \inf_{p\in(0,\infty)} \left[\frac{P(p)}{p}\right]^{1-\sigma} \geq \frac{1-\mu}{\mu}$$

and get $ph(p) \leq \mu$ for any p, so that the current budget constraint is satisfied when the consumer is not allowed to borrow. The second point is that in both instances of our example, we get $\lim_{p\to 0} ph(p) = 0$ and yet $\lim_{p\to 0} h(p) = \infty$ (unless $\beta = \frac{\sigma}{\sigma-1}$, the extreme case where demand is saturated at $p = 0$).

We must at last consider the extension to the multisectorial case. The demand function we have introduced in Section 4 can be derived from a similar intertemporal program with the particular utility function:

$$U(c_1, c_2) = \sum_{k=1}^{K} \lambda^k \ln u^k(c_1^k, c_2^k),$$

with $\lambda^k > 0$ for $k = 1, \cdots, K$ and $\displaystyle\sum_{k=1}^{K} \lambda^k = 1$, where the sectoral functions u^k may be C.E.S. utility functions as above.

The consumer's program can now be solved in two stages. In the first stage, we take the sectoral expenditures as given and consider K programs and the associated indirect utility functions:

$$V^k(p^k, x^k) \stackrel{\text{def}}{=} \max_{(c_1^k, c_2^k) \in \mathbb{R}_+^2} u^k(c_1^k, c_2^k)$$
$$\text{s.t.} \;\; p^k\, c_1^k + P^k(p^k) c_2^k \leq x^k.$$

Clearly, strict quasi-concavity and homogeneity of degree one of the function u^k lead to a unique solution which depends linearly on x^k. Also, V^k is linear in x^k, so that we may substitute $v^k(p^k)x^k$ for $V^k(p^k, x^k)$.

In the second stage, we must consider the program

$$\max_{x \in \mathbb{R}_+^K} \sum_{k=1}^{K} \lambda^k \ln[v^k(p^k)x^k],$$
$$\text{s.t.} \;\; \sum_{k=1}^{K} x^k \leq W.$$

As $\ln[v^k(p^k)x^k] = \ln v^k(p^k) + \ln x^k$, the solution to this program can be obtained by maximizing the function $\sum_{k=1}^{K} \lambda^k \ln x^k$, which differs from the original objective function by a constant (in terms of the decision variable x). As is well known, log-linearity leads to the solution: $x^k = \lambda^k W$ for any k.

By linearity in x^k of the solutions to the programs of the first stage we finally get the functional form:

$$\tilde{h}^k(p^k)x^k = h^k(p^k)W$$

where $h^k(p^k) = \lambda^k \tilde{h}^k(p^k)$ and $\tilde{h}^k$ is the function h considered above, but depending upon parameters δ^k and σ^k which generally differ from sector to sector. Notice that if $\tilde{h}^k$ satisfies Assumptions 1 to 4, these assumptions are still satisfied by $h^k = \lambda^k \tilde{h}^k$ for any positive λ^k less than one. Furthermore, the same is true of Assumption 2*. Indeed, $p^k \tilde{h}^k(p^k) < 1$ implies $p^k h^k(p^k) < \lambda^k$ and, by summing over k, we get: $\sum_{k=1}^K p^k h^k(p^k) < 1$.

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