A Measure of Aggregate Power in Organizations*

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Abstract

We present axiomatically an index of aggregate power in organizations, by measuring the overall ability of the agents to induce a change through coalition formation. The axioms are closely related to those used for the Shapley Value. The resulting formula appears as the sum, taken over all possible coalitions, of the products of two probabilities of each coalition: its probability to form and its probability to become winning.

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Introduction

The game-theoretic approach has represented an important step forward in the quantitative analysis of the individual power to influence or to control aggregate outcomes. In the context of cooperative games, the so-called characteristic function has received two types of interpretation.[1]

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[1] As recently stated by Aumann [2], "cooperative solution concepts can be thought of as a sort of average or expected outcome, or an a priori measure of power."

The first one in terms of results (or payoffs) to the different groups (or coalitions) of participants. In this approach, various solution concepts have been proposed, the main one being the core. The second type of interpretation is given in terms of the ability for coalitions to obtain a certain result. This interpretation is already suggested by von Neumann and Morgenstern for the class of "simple games" when the characteristic function only indicates which coalitions are capable of "winning". Better than the concept of core, the Shapley-Shubik index has been considered as a particularly adequate vehicle, in this type of interpretation, for measuring the power of an individual participant in a given organization. The power is the individual capacity to bring about the passage of an issue or more generally, to effect a change in the outcome, by making a coalition successful. However, when comparing two (hypothetical or existing) legislative bodies, boards of corporate managers or markets, measuring individual power is not sufficient to know which of the two ranks higher in terms of the *overall* ability of their participants to induce a change through coalition formation. What would be required is a measure allowing to estimate the probability of such a change. This note suggests an approach which directly axiomatizes some functions expressing this type of aggregate power, the adopted axioms being closely related to those used for the Shapley value [6]. After some definitions and the statement of the axioms, a formula is characterized leading to a basic interpretation. The aggregate power existing in a given organization is measured by the sum, taken over all possible coalitions, of the products of two probabilities for each coalition: its probability to form and its probability to become winning. Finally, starting from a certain class of voting games the measure is further specified and leads to a particular interpretation.

1 Formal definitions and axioms

In the context just described it is natural to consider the following class of characteristic functions. A *characteristic function* is a pair (N, v), where $N = 1, 2, \cdots, n$ represents the set of agents and v is a function from the set of subsets of N, called coalitions, to the interval $[0, 1]$ such that: $v(\emptyset) = 0$ and, for any S and T, if $S \subset T \subset N$ than $v(S) \leq v(T)$.

In the present framework, given a characteristic function (N, v), the number $v(S)$ may be interpreted as the probability for S to change, let us say, the structure of control within the organization – if the coalition S happens to form.

Whenever the range of the function v reduces to the set $\{0, 1\}$, then (N, v) is called a *simple* characteristic function and the coalitions such that $v(S) = 1$ are called *winning* coalitions. A winning coalition is *minimal* if no proper subcoalition is winning. Thus, by extension, for any characteristic function (N, v), the probability $v(S)$ that S change the structure of control may be taken as the probability that S be a winning coalition. A particular class of simple characteristic functions consists of those, denoted (N, v_M), where M is the single minimal winning coalition.

Finally, for any characteristic function (N, v), we shall (as usual) call null any agent i who does not affect the probability that any coalition be winning; i.e., for all $S \subset N$, $v(S \cup \{i\}) = v(S)$.

With these definitions, our aim is now to construct some aggregate power index F, associating to each characteristic function (N, v) the probability $F(N, v)$ that, within the corresponding organization, the structure of control will be changed by some successful coalition. This is done using axioms very similar to the ones used for the Shapley value. The first two axioms assert, respectively, that the agents' names do not matter and that null agents do not affect the aggregate measure.

I. Symmetric Axiom. *For any permutation π of N (i.e., a one-to-one function from N onto N) and for any two characteristic functions (N, v) and (N, w), if, for all $S \subset N$, $w(\pi(S)) = v(S)$, then $F(N, v) = F(N, w)$.*

II. Null Agent Axiom. *If (N, v) and (N', w) are characteristic functions such that $N' = N \cup \{n'\}$, with n' a null agent in (N', w), and if, for all $S \subset N$, $w(S) = v(S)$, then $F(N, v) = F(N', w)$.*

The third axiom can be interpreted in terms of lotteries. Indeed, given two characteristic functions (N, v) and (N, w), one may consider the composite function $(N, \alpha\, v + (1 - \alpha) w)$, where a lottery chooses at random the function v with probability $\alpha \in [0, 1]$ and the function w with probability $(1-\alpha)$. Clearly the overall probability of a change of regime is then $\alpha F(N,v)+(1-\alpha)F(N,w)$. Thus, the

III. Linearity Axiom. *For any two characteristic functions* (N,v) *and* (N,w), *for any* $\alpha \in [0,1]$,

$$F(N, \alpha\, v + (1-\alpha)w) = \alpha\, F(N,v) + (1-\alpha)F(N,w).$$

The next axiom says that if, for every coalition S, its probability $v(S)$ of success increases, the overall probability of success should rise. This is a monotonicity condition which we state here in its weakest form.

IV. Monotonicity Axiom. *If* (N,v) *and* (N,w) *are characteristic functions such that, for every nonempty coalition* $S \subset N$, $w(S) \geq v(S)$, *then* $F(N,w) \geq F(N,v)$.

Now, defining the characteristic functions (N, v_0^N) and (N, v_1^N) by

$$v_0^N(S) = 0 \quad \text{ and } \quad v_1^N(S) = 1,$$

for every nonempty coalition $S \subset N$, it may be checked that the previous axioms imply that, for any characteristic function (N,v),

$$F(N, v_0^N) \leq F(N,v) \leq \lim_{\#N\to\infty} F(N, v_1^N) = \sup_{(M,w)} F(M,w).$$

The interpretation of F as a probability leads naturally to require the following:

V. Normalization Axiom. $\inf_{N,v)} F(N,v) = 0$ *and* $\sup_{(N,v)} F(N,v) = 1$.

2 The formula

We can give the following characterization of an aggregate index F satisfying the above axioms. In the following derivations we shall denote $\#N$, $\#S$, $\#T$, etc., respectively, by n, s, t, etc.

Theorem 1 *The function F defined on the set of all characteristic functions (N, v) by the formula*

$$F(N, v) = \sum_{S \subset N} \left[\int_0^1 \alpha^s (1-\alpha)^{n-s} d\mu(\alpha) \right] v(S), \tag{1}$$

where μ is a probability measure on $(0, 1]$, satisfies Axioms I-V. Moreover any function satisfying Axioms I-V can be written in this form for some probability measure μ on $(0, 1]$.

Proof. Using Axioms III, IV, and V, the index F is a linear function of the $v(S)$'s, say $\sum_{S \subset N} \alpha_{N,S} v(S)$, such that the coefficients $\alpha_{N,X}$'s are non-negative and sum up to no more than 1. Let the difference to 1 be $\alpha_{N,\emptyset}(v(\emptyset) = 0$ anyway); then the coefficients become a probability vector on subsets of $\{1, \cdots, n\}$; i.e., a probability distribution on $\{0, 1\}^n$. Axiom II implies consistency of those probability distributions (for N' as in II and each $S \subset N$, $\alpha_{N,S} = \alpha_{N',S} + \alpha_{N',S \cup \{n'\}}$); hence Kolmogorov's consistency theorem [5, p.123] asserts they all derive from some probability distribution on $\{0, 1\}^\infty$. Let X_n denote the nth coordinate mapping on $\{0, 1\}^\infty$. Axiom I implies exchangeability of the X_n's. By the Finetti's theorem [5, p.228], there exists a probability distribution μ on $[0, 1]$ such that

$$\alpha_{N,S} = \int_0^1 \alpha^s (1-\alpha)^{n-s} d\mu(\alpha).$$

Finally, Axiom V implies that $\lim_{n\to\infty} \alpha_{N,\emptyset} = 0$, hence $\mu(\{0\}) = 0$. Since it is easy to verify that, for any probability μ on $(0, 1]$, formula (1) satisfies Axioms I-V, the result follows. ∎

Formula (1) shows that, if one chooses a coalition S at random by first selecting α according to μ, and, second, by letting every player i independently join the random coalition S with probability α, then $F(N, v) = E(v(S))$, where E is the mathematical expectation operator. This formula appears as a product of two probabilities, the probability that a coalition will form times the probability that it will be winning. Concerning the first one, our axiom of monotonicity implies that all other things being equal the smaller is the required size of a coalition to be winning, and the larger is the number of possible winning coalitions, the higher is this probability. The second probability,[2] $v(S)$, expresses the ability of the coalition S of

[2]As suggested in the introduction an alternative interpretation is to consider $v(S)$ as the value of coalition S: it is then the "wealth" available to the coalition S, so that the measure corresponds to the expected payoff of the

agents to induce an overall change of outcome or its capacity to take a decision affecting the whole organization. Hence the aggregate index, which combines these two probabilities, can be viewed as the expected capacity of an organization to decide. This is very much like what is called "decisiveness" in the social choice literature.

In terms of a legislative body consisting of a finite number of members, the measure could then be used to express the probability of a given change of legislation. Similarly, in the framework of oligopoly theory,[3] the index can be viewed as the probability that a given coalition S will be able to change the competitive situation, from price-taking to price-making.

3 Further specification and interpretation

To realize the need for further specification of the above formula let us concentrate on simple characteristic functions, i.e., voting situations. Consider, for instance, all three-agent superadditive majority games. Among these games one can distinguish those having only one minimal winning coalition (of size 1, 2, or 3), those having two intersecting minimal winning coalitions (of size 2) and the simple majority case (with minimal winning coalitions $\{1,2\},\{2,3\},\{3,1\}$). Whatever the measure μ, the axioms allow to rank them as follows: simple unanimity is less decisive than a single minimal winning coalition of size 2, which is less decisive than two minimal winning coalitions of size 2, which is less decisive than both the simple majority case and the case of a single minimal winning coalition of one agent ("dictatorship"). However, it is not possible to compare these last two situations without specifying the measure μ. Now consider the majority games (N, v_q) with quota q and *with a large set of agents* N; i.e., $v_q(S) = I(s/n \geq q)$. According to the random α being strictly greater of strictly smaller than the quota q, the random coalition S will be winning or losing with probability close to 1. Thus,

$$F(N, v_q) \cong \mathrm{Prob}(\{\alpha \geq \mathrm{q}\}) = \mu([\mathrm{q}, 1]).$$

whole game. In this case, however, the normalization axiom has to be adapted because there is no reason to have $v(S) \in [0,1]$.

[3] Such a framework is considered in d'Aspremont and Jacquemin [1].

This gives a direct interpretation in terms of decisiveness for the measure μ, and the axioms determine F uniquely for all characteristic functions from its values for the majority games with a large number of agents.[4]

To show that let us consider two different types of voting situations. In the first one the quota is supposed to be less than $\frac{1}{2}$. It is then clear that a vote on any kind of issue should be decisive. Therefore, for all $q \leq \frac{1}{2}$, $F(N, v_q) \cong \mu([q, 1]) = 1$, hence $\mu([0, \frac{1}{2})) = 0$. In the second type the quota is supposed to be strictly greater than $\frac{1}{2}$. Then one might have nondecisive votes and this depends upon the issues on the agenda. This leads to two interpretations. A general one is to see F as a characteristic of the whole set of issues: the value of F for a certain quota q is equal to *the proportion of issues* for which the number of favorable votes is at least q%. The higher is this proportion the higher is F. A more requiring interpretation is to consider F as the probability that *no issue at all* on the agenda will be blocked. Then if we assume

[4]Let us remark that formula (1) seems to have a close relationship to a formula which has been discovered by Dubey *et al.* [3] for the semivalue. And indeed, any aggregate index F can be decomposed as a sum of semivalues. To see this, it is enough to replace $v(S)$ in formula (1) by the sum of the Shapley values of the same characteristic function restricted to S. Easy computations then lead to the other formula

$$F(N, v) = \sum_{i \in N} F_i(N, v), \tag{2}$$

where, for every $i \in N$, F_i is defined by

$$F_i(N, v) = \sum_{T \subset N \setminus \{i\}} \left[\frac{t!(n-t-1)!}{n'} \int_0^1 \sum_{s=t+1}^{N} \left(\begin{array}{c} n \\ s \end{array} \right) \alpha^s (1-\alpha)^{n-s} d\mu(\alpha) \right] [v(T \cup \{i\}) - v(T)].$$

It may be verified that F_i qualifies as a semivalue (i.e., an index satisfying the conditions (1)-(4) of [3]) if the measure μ is renormalized adequately. On the other hand, taking any semivalue, its sum over individuals will not in general satisfy all the axioms I-V. For instance, for the sum of the Banzhaf indices (see [4]), the weight coefficient of the small coalitions is negative, and hence does not satisfy Axiom IV. However, if we consider a semivalue characterized (according to formula (1.1) in [3]) by a probability measure on [0,1] having a nonincreasing density function, then its sum over individuals satisfy our axioms. SInce any nonincreasing density on [0,1] is a convex combination of densities of the form $(1/\alpha_0)I(0 \leq \alpha \leq \alpha_0)$, $0 \leq \alpha_0 \leq 1$, this result is easy to verify. In addition one may show (but this is less straightforward) that the condition of a semivalue characterized by a probability measure having a nonincreasing density function is not only sufficient but also necessary for the result.

that the scope of the agenda is broad enough to include at least one disputed issue (i.e., for which no qualified majority exists), the probability of blocking must be 1. Therefore, with $q > \frac{1}{2}, F(N, v_q) \cong \mu([q,1]) = 0$, hence, $\mu(\{\frac{1}{2}\}) = 1$. It results that a random coalition will be losing or winning with probability close to 1 (with a large number of agents) according to the quota q being strictly above or below $\frac{1}{2}$. To the extent that such a property might appear to be reasonable, this would lead to a particular specification of μ and a complete characterization of F,

$$F(N, v) = \sum_{S \subset N} \frac{1}{2^n} v(S);$$

i.e., F is simply the average of the $v(S)$'s. This means that for voting situations F is increasing with the number of winning (not necessarily minimal) coalitions and that for two situations having the same number of winning coalitions the value of F will be identical. For example in the above three-agent situations, the majority case is as decisive as dictatorship.

4 Conclusion

Measuring the power characterizing an organization is a difficult task and has been the object of much research in social science. The problem is multidimensional because various aspects such as the nature and the objectives or organizations, the characteristics of the participants, their strategic capabilities, and their degree of consensus can differ, at a given moment of time as well as over time.

In this note we have proposed a measure of aggregate power which differs from the standard approaches based on individual indices. Our measure seems well adapted to static cooperative situations where the formation of winning coalitions plays a crucial role for the control of the outcomes, be they passages of bills, changes in stockholders' policies or monopolization of markets. It is an axiomatized function based on two multiplicative criteria, the probability that a coalition will form and the probability that it will be winning.

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