Quasi-monopolies*

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I

In this paper we study the remaining fragments of a collusive cartel after it breaks down, that is, after some of its members have undercut the collusively set price in a particular industry. As noted by Scherer (1980), "the very act of fixing the price at a monopolistic level creates incentives for sellers to expand output beyond the quantity that will sustain the agreed-upon price". To make this output expansion effective, it suffices for the chisellers to slightly undercut the monopoly price: this manoeuvre should attract into their business the whole monopoly demand. Our contention is that, after the cartel structure has broken down as a result of such chiselling, a new market structure may emerge, which remains close, but not identical, to the starting collusive arrangement. Accordingly, we call this structure a *quasi-monopoly.*

To illustrate this concept let us consider the following story akin to an illustration proposed by Shitovitz (1973) in a different, but related, context. Imagine a town full of taxis whose drivers are organized into a powerful cartel controlling the whole city. They have agreed to fix a rate per mile equal to, or approaching, the monopoly fare. Some of these taxi-drivers may be tempted to organize a conspiracy against the cartel and undercut the official fare; this will considerably expand the number of their customers, possibly in excess to the number of taxis available to the chisellers! The original cartel will then break down into two fragments. The first

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fragment groups all "loyal" cartel members, the second, the chisellers. Of course, the temptation is great for the first group to retaliate, and in turn to quote a still lower price so as to recover its customers. But as a result of this price war the fare may descend to its competitive level, which could hurt them severely, presumably more severely than if they would stick to the original monopolistic price. In fact, the extent to which the loyal members are hurt depends crucially on the number of chisellers. *If this number is small, the chisellers immediately hit their capacity, and the residual demand would in any case be served by cartel members at the monopoly fare.* This suggests that the loyal cartel members might tolerate the erection of a "splinter group" if the latter is sufficiently small – small enough, that is, to ensure that it is still advantageous to keep the monopoly price rather than enter a price war. The market structure presumably prevailing at the end will consist of two cartels of very unequal size: a small free-rider cartel quoting a price slightly below the monopoly price, and the cartel of "loyal" taxi-drivers still demanding the monopoly price because it knows that its competitor can only supply a very small share of the market. As described above, the final structure is similar to the case of a single cartel quoting the monopoly price, i.e. a quasi-monopoly.

The above illustration suggests that an appropriate theoretical notion could be proposed that would capture the idea of "equilibrium" implicit in the example. Furthermore, this notion should also capture the fact that only asymmetric sizes of the cartels could lead to such an equilibrium. It is indeed the asymmetry in the capacity of the cartels that leads the "big" one to tolerate the existence of the "small" one: the capacity of the latter is limited to such an extent that it can hardly compromise the power of the former.

II

Consider the market for some homogeneous product with a continuous strictly decreasing demand function $D(p)$. Suppose this demand is initially served by a cartel consisting of all firms in the industry. We assume that there is a large number n of such firms, and that they are all identical, each with a limited amount of productive capacity and constant marginal costs; we normalize the output that the capacity for each firm is exactly equal to $1/n$ and we normalize marginal costs to zero. Joint profit maximization has raised successfully the price from its competitive level p_c to the monopoly price p_M.

Imagine now that this cartel agreement is dislocated in the sense that a certain (presumably small) number k of firms, acting in unison, decide to drive down the price p_M. Then we get a new situation where the set of firms is split into *two* cartels, respectively called cartel α and cartel β, the first grouping the $(n-k)$ "loyal" firms with total capacity $\alpha = (n-k)/n$, and the other grouping the k deviant firms, with capacity $\beta = k/n$. If in addition some sharing of the market between the two cartels is specified for every pair of prices p_α and p_β that they can respectively quote, this new situation gives rise to a duopoly with capacity constraints.

Our notion of quasi-monopoly develops around the answers to the following questions. First, what asymmetry in the capacities k/n and $(n-k)/n$ can be tolerated without retaliation by the "loyal" firms? On the other hand, what maximal capacity k/n can the deviant cartel afford in order to guarantee to its members a higher profit than their original share in the monopoly profit? The temptation for a set of k firms to break the cartel agreement comes from the fact that quoting a price p_β slightly below p_M may lead these firms to a higher profit than their original share of the cartel profit. To be credible, however, this objection to the cartel agreement should not be threatened by an advantageous retaliation for the loyal firms, consisting in undercutting the price p_β itself. In other words, price war should be avoided. In general, the incentive to retaliate decreases when the price p_β becomes small, since retaliation implies in turn the choice of a still smaller price. But at the same time, the advantage of the deviant firms in breaking the initial agreement also diminishes when p_β is driven down. If the deviant group remains small, it is conceivable that their exist values of undercutting strategies p_β that may simultaneously increase the joint profits of its participants and prevent retaliation form the loyal firms. The type of arrangement underlying the above illustration rests exactly on these considerations. More precisely, let $\Pi_\alpha(p_\alpha, p_\beta)$ and $\Pi_\beta(p_\alpha, p_\beta)$ denote the joint profits of, respectively, cartel α and cartel β, under the particular rule of sharing the market demand at prices p_α and p_β. Suppose also that the monopoly price is the price p_M such that $\Pi_\alpha(p_\alpha, p_\beta) + \Pi_\beta(p_\alpha, p_\beta)$ is maximal at (p_M, p_M). Then we call a *quasi-monopoly* a pair of prices $(p_M, \overline{p}_\beta)$, where p_M is the monopoly price, and $\overline{p}_\beta$ meets the following requirements:

$$\Pi_\beta(p_M, \overline{p}_\beta) > \Pi_\beta(p_M, P_M) \tag{1}$$

$$\Pi_\beta(p_M, \overline{p}_\beta) = \max_{p_\beta} \Pi_\beta(p_M, p_\beta) \tag{2}$$

for all p_β such that, $\forall\, p_\alpha$, $\Pi_\alpha(p_M, p_\beta) \geq \Pi_\alpha(p_\alpha, p_\beta)$. Condition (1) guarantees that the deviant firms together get higher profits than their original share in the monopoly profit. Condition (2) ensures that, given p_M, the deviant firms maximize their joint profits but only in the range of p_β values for which it remains optimal for the cartel of loyal firms to keep the collusive monopoly price p_M, given p_β, in order to prevent retaliation by the loyal cartel and hence avoid a price war. Via both conditions (1) and (2) this definition refers to the collusive solution as the starting-point from which the new price equilibrium is generated by a cartel breakdown.[1]

III

The definition of a quasi-monopoly raises several questions. First, as already mentioned, such a definition depends on the particular way in which total demand is shared between the two cartels at a pair of prices (p_α, p_β). It is reasonable to assume that, if $p_\alpha = p_\beta$, then the cartels share the quantity sold in proportion to their capacity and, that, if $p_\alpha \neq p_\beta$, then the cartel quoting the lower price serves the market up to its capacity. But many alternative rationing schemes may be envisaged to distribute this capacity between the customers. In fact, the demand to be served by the cartel quoting the higher price depends on the particular scheme that is effectively used. For instance, the most pessimistic rule to evaluate the residual demand would be to suppose that the cartel quoting the lower price serves first the high-priced part of the

[1] An alternative definition for a two-price equilibrium generated by a cartel breakdown would allow the loyal cartel to pick any price p_α that consists in a best reply to $\overline{p}_\beta$, and not restrict this choice to p_M only. Then we are led to a Stackelberg equilibrium (with cartel α as the follower and the splinter group β as the leader) satisfying in addition condition (1). This concept is then close to the concept developed recently by Gelman and Salop (1982) in an incumbent-entrant model. However, we feel that our condition (2) better prevents the splinter group from taking the risk of a price war, since by this condition the best reply of cartel α is the status quo.

demand curve, or that customers are served in the order induced by the ranking (downward) of their reservation prices. The residual demand to the cartel quoting the higher price is then total demand at that price minus the capacity of the cartel quoting the lower price. This first rule is used by Shapley and Shubik (1969) and Levitan and Shubik (1972). Another rule, suggested in Shubik (1959) and used in Beckmann (1965), consists in drawing at random, from among all customers willing to buy at the lower price, those who will be served. This implies that the residual demand to the cartel quoting the higher price is a *proportion* (the unsatisfied customers) of total demand at that price. In our analysis below we follow this second, less pessimistic, rule. Furthermore, a simple argument shows that no quasi-monopoly solution exists for the first rule, as *any* undercut of the monopoly price entails advantageous retaliation. This is not the case for the second rule for which existence may be obtained whenever the relative number of deviant firms is small enough and the monopoly demand is not exactly equal to the capacity of the industry (see Proposition 2 in the Appendix).

This is related to a second question concerning the definition of a quasi-monopoly. As noted above, the breakdown of the original cartel agreement gives rise to a duopoly with capacity constraints; and one may wonder why we should not restrict our attention to the non-cooperative price equilibrium of this duopoly game, rather than propose the alternative concept of quasi-monopoly. Unfortunately, it turns out that the only situation in which a price equilibrium does exist is when the monopoly demand is exactly equal to the capacity of the industry. In this very exceptional situation both the collusive agreement solution and the non-cooperative one coincide and the quasi-monopoly solution does not exist, whatever the relative size of the two cartels. *In all other situations a non-cooperative price equilibrium does not exist.*[2] The proof of these properties is rather tedious and we have preferred to provide them in the Appendix. However, the intuition of the main argument for the non-existence of a price equilibrium is rather simple and has been presented a century ago by Edgeworth (1925). Basically, the argument runs as follows. First, if there exists a non-cooperative price equilibrium, both cartels must quote the

[2]Of course, here we use the proportional rule to share demand at unequal prices. For a parallel analysis with the pessimistic rule, see Levitan and Shubik (1972).

same price at this equilibrium. Otherwise, either the cartel quoting the higher price has zero demand, and hence zero profit, which is impossible; or the cartel quoting the lower price must hit its capacity, and can then increase its profits by increasing slightly its own price, a contradiction. Assume then that the pair of prices (p^*, p^*) is an equilibrium. If $D(p^*) < 1$, at least one of the two cartels does not hit its capacity, and undercutting the price p^* must be profitable. If on the other hand $D(p^*) > 1$, then both cartels hit their capacity, so that one of them may increase slightly its price without losing its customers, thereby increasing its profits. Accordingly, in both cases we get a contradiction. So we are left with the case $D(p^*) = 1$, where demand at p^* is exactly equal to capacity. But then a simple reasoning shows that p^* must be equal to p_M, for otherwise the monopoly price p_M would not maximize the joint profits in the industry.

The absence of non-cooperative equilibrium, which has to be observed if the totally collusive agreement breaks down, indirectly reinforces, we feel, the significance of a quasi-monopoly as an observable arrangement. Not only is such a totally collusive agreement unstable because it is not immune against advantageous price-cuts by deviant firms, but also, should it break down under such pressures, the situation cannot afterwards stabilize at a non-cooperative solution. Moreover, we have seen that if these deviant firms are few in number, and represent together a small share only of the industry capacity, then the competitive price may not be "attractive" either: a price war is not a good deal for the loyal firms. The remaining solutions are either permanent price fluctuations, or possibly the emergence of a quasi-monopoly.

IV

We asserted above that when the original cartel breaks down it must be under the pressure of a *small* group of deviant firms undercutting the monopoly price, and leaving afterwards two "fragments" of asymmetric size at a quasi-monopoly solution. We still have to verify this assertion; namely that no "large" group of deviant firms can gain in leaving the cartel *en bloc*, without inducing retaliation and a price war. At the level of generality at which we are working, the only possible definitions of the words "large" and "small" are, respectively, "greater than or equal to $n/2$", and "strictly smaller than $n/2$". Accordingly, we start by proving that, *at a quasi-monopoly, the group β of deviant firms is strictly smaller than $n/2$.*

Suppose, contrary to this assertion, that $k \geq n-k$. Notice that using $\overline{p}_\beta$ leads cartel β to serve demand up to its capacity: otherwise cartel α would get a zero residual demand, and thus a zero profit, violating condition (2). Accordingly,

$$\Pi_\beta(p_M, p_\beta) = \frac{k}{n}\overline{p}_\beta > \Pi_\beta(p_M, p_M) = \frac{k}{n}\{p_M \cdot D(p_M)\}$$

where the strict inequality follows from condition (1). Since cartel β hits its capacity at $\overline{p}_\beta$, *a fortiori* cartel α must hit its capacity at any price strictly smaller than $\overline{p}_\beta$. (Recall that we have assumed $k \geq n-k$.) Consequently,

$$\Pi_\alpha(\overline{p}_\beta - \varepsilon, \overline{p}_\beta) = \frac{n-k}{n}(\overline{p}_\beta - \varepsilon)$$

for all $\varepsilon > 0$. Combining this equation with the above inequality, we obtain

$$\Pi_\alpha(\overline{p}_\beta - \varepsilon, \overline{p}_\beta) = \frac{n-k}{n}(\overline{p}_\beta - \varepsilon) > \frac{n-k}{n}\{p_M \cdot D(p_M)\} = \Pi_\alpha(p_M, p_M)$$

if ε is chosen sufficiently small. But the inequality

$$\Pi_\alpha(p_M, p_M) > \Pi_\alpha(p_M, \overline{p}_\beta) \tag{3}$$

must also hold. For otherwise we would have simultaneously

$$\Pi_\beta(p_M, \overline{p}_\beta) > \Pi_\beta(p_M, p_M)$$

(by (1)) and

$$\Pi_\alpha(p_M, \overline{p}_\beta) \geq \Pi_\alpha(p_M, p_M),$$

contradicting the fact the pair of prices (p_M, p_M) maximizes joint profits. Consequently, if ε is chosen sufficiently small,

$$\Pi_\alpha(\overline{p}_\beta - \varepsilon, \overline{p}_\beta) > \Pi_\alpha(p_M, \overline{p}_\beta).$$

But this contradicts condition (2) of the definition of a quasi-monopoly.

Notice in passing that the property that has just been proved excludes as a possible outcome of the failure of the collusive agreement a situation that could have been viewed *a priori* as symmetric to that of a quasi-monopoly; namely, a situation where a *large* group of deviant firms would exit from the original cartel and quote a slightly *higher* price than the collusive price. Free-riders must be small if they are to be effective.

V

The inequality (3), used in the above argument, according to which the profit of any firm in cartel α is larger at the monopoly solution (p_M, p_M) than at the quasi-monopoly solution $(p_M, \overline{p}_\beta)$, has significant implications concerning the stability of such an arrangement.[3] Indeed, by condition (1) it implies that the profit at $(p_M, \overline{p}_\beta)$ is larger for any deviant firm than for any loyal firm and hence confirms *ex post* the advantage gained by the splinter group. An individual firm in this group would not exchange its place with a loyal firm. Moreover, since the inequality (3) holds for any size of k such that the quasi-monopoly solution exists, no firm in the splinter group would consider leaving this group and joining the loyalists. By condition (1) again, it would imply a loss of profit. By the same token, any firm in the loyal cartel would find it advantageous to join the deviant firms. However, as the above argument shows, it is not possible for the splinter group to grow indefinitely. Indeed, for some size k strictly smaller than $n/2$, any additional membership for the splinter group would imply the choice of a price that either makes the monopoly solution more profitable for all firms (by violation of condition (1))

[3] For the stability analysis of another kind of arrangement, namely the price-leadership collusive agreement, see d'Aspremont et al. (1983). On the other hand, additional properties concerning the stability of the quasi-monopoly solution have been demonstrated by d'Aspremont and Jaskold Gabszewicz (1980) in the framework of a particular example. In particular, it is shown in this example that the payoffs at the quasi-monopoly solution dominate the "Stackelberg profits" of both cartels (i.e. the profit each one would receive if acting as a leader) and cannot be Pareto-dominated by agreeing on any other pair of prices. Using Theorem 2 in Moulin (1981), these properties may be interpreted as saying that the quasi-monopoly solution can be self-enforced by threats that are best-reply decision rules. Hence it confirms our first intuition that both cartels, in choosing their quasi-monopoly price strategies, anticipate retaliatory threats from their competitor.

or gives an incentive to the remaining loyalists to retaliate by undercutting it (by violation of condition (2)).

On the other hand, our Proposition 3 in the Appendix examines the relationship between the quasi-monopoly solution $(p_M, \overline{p}_\beta)$ and the number β, i.e. the relative number of firms belonging to the splinter group. It establishes that the price $\overline{p}_\beta$ is a strictly decreasing function of β whenever the quasi-monopoly solution exists for several values of k. An immediate implication of Proposition 3 is that the profit of each individual firm in the splinter group monotonically decreases as k grows; indeed, whatever β, each of them works at capacity. Therefore, there is a countervailing effect to the tendency of the splinter group to grow. While new loyalists want to join, those already accepted wish to limit new memberships as much as possible so as to stop the erosion of their profits.

Finally, it is worthwhile to point out at last stability property of a quasi-monopoly solution, namely that there is no incentive for any member(s) of the splinter group to under the price $\overline{p}_\beta$. This property follows immediately from the fact that, in the splinter group, each firm hits its capacity at the price $\overline{p}_\beta$.[4]

VI

If, as correctly asserted, a totally collusive agreement is intrinsically unstable, a natural question arises: what happens if, in spite of coercive measures and the like, the cartel finally fails in maintaining the discipline of its members? The usual answer is that either a price war must be observed, driving prices to the competitive level, or ceaseless price fluctuations must arise between the competitive and monopoly levels. Here we have suggested a third possible

[4] Of course, this does not guarantee that there is no incentive for members of the splinter (respectively loyalists) group to break off by quoting a price *between* p_M and $\overline{p}_\beta$. But when we get a new situation where the set of firms would be split in three, or more, cartels, and where a certain rule to share residual demand among them should be specified. The resulting price constellation would consist only of prices in the interval $(p_M, \overline{p}_\beta)$, a price profile still closer to the monopoly solution than the quasi-monopoly one. In this context, quasi-monopoly can be viewed as a first approximation to such a price profile.

scenario. If individual firms suffer capacity constraints, individual or "small-group" exits from the cartel cannot hurt the latter very much since, in any case, the chisellers are blocked by their capacity restrictions. Accordingly, such "small" exits must indeed be expected, since it pays those who decided to exit; but on the other hand, no retaliation must follow, since a price war would hurt the cartel more than tolerance would. The result of these combined forces is the quasi-monopoly.

The extent to which this market structure is empirically observed remains an open question which it could be useful to examine. A fruitful departure point could be, for instance, the market for credit in a country like Belgium. It is observed in this country that the credit market is controlled by a large cartel grouping the major banks of the country, which jointly determine their interest rates on deposits. This cartel is surrounded by two or three much smaller banks, which offer slightly higher interest rates on deposits than those offered by the banks affiliated with the cartel.

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Appendix

We here consider the problem of the existence of a non-cooperative price equilibrium of the two-cartel non-cooperative game described in Sections II and III, and relate it to the existence of a quasi-monopoly. To define the pay-offs of this game, we must specify how total market demand is shared between cartels α and β at a pair of prices (p_α, p_β). To that effect we use the proportional rule discussed in Section III.

$$\begin{aligned}\text{If } p_\alpha = p_\beta, \quad D_\alpha(p_\alpha, p_\beta) &= \frac{n-k}{n} D(p); \qquad p \overset{\text{def}}{=} p_\alpha = p_\beta \\ D_\beta(p_\alpha, p_\beta) &= \frac{k}{n} D(p).\end{aligned} \tag{A.1}$$

In other words, both cartels share demand in proportion to their capacities if they quote the same price and if demand is less than total capacity.

$$\begin{aligned}&\text{If } p_\alpha \neq p_\beta \text{ (say, } p_\alpha > p_\beta), \\ &D_\alpha(p_\alpha, p_\beta) = \max\left\{0, \frac{D(p_\beta) - k/n}{D(p_\beta)} D(p_\alpha)\right\}.\end{aligned} \tag{A.2}$$

That is to say, the cartel quoting the higher prices serves a residual demand equal to zero whenever the capacity of the other cartel is sufficient to serve all the quantity demanded at the higher price. Otherwise residual demand is equal to a proportion of the total quantity demanded at a higher price, and this proportion is equal to the proportion of consumers who have not been served at the lower price; in any case, cartel α cannot sell more than its capacity.

Furthermore, let us assume that the receipt function $pD(p)$ is strictly concave and reaches its maximum at p_M, and that $D(p_M) \leq 1$.

Proposition 1

(a) If $D(p_M) < 1$, then there exists no non-cooperative equilibrium.

(b) If $D(p_M) = 1$, the only equilibrium pair is (p_M, p_M); i.e. there exists a non-cooperative price equilibrium if, and only if, total demand at the monopoly price is equal to the capacity of the industry.

Proof. According to the argument given on p. 6, if (p_α^*, p_β^*) is an equilibrium pair, we must have $p_\alpha^* = p_\beta^* = p^*$ and $D(p^*) = 1$. Let us show that $p^* = p_M$, so that $D(p_M) = 1$. First, it is clear that, since $D(p_M) \leq 1$, we must have $p^* \leq p_M$ (recall that $D(p^*) = 1$). Suppose now that $p^* < p_M$; then $D(p_M) < 1$, and

$$\begin{aligned}\Pi_\alpha(p_M, p^*) &= \left\{\frac{D(p^*) - k/n}{D(p^*)}\right\} p_M D(p_M) = \frac{n-k}{n}\{p_M D(p_M)\} \\ \Pi_\alpha(p^*, p^*) &= \frac{n-k}{n} p^* = \frac{n-k}{n}\{p^* D(p^*)\}.\end{aligned}$$

Therefore, by the very definition of p_M,

$$\Pi_\alpha(p_M, p^*) > \frac{n-k}{n}\{p^* D(p^*)\},$$

a contradiction to the fact that (p^*, p^*) is an equilibrium. Consequently, if (p^*, p^*) is an equilibrium, it must be that $D(p_M) = 1$, which proves part (a) of the proposition.

To prove part (b), it only remains to show that (p_M, p_M) is an equilibrium pair whenever $D(p_M) = 1$. To argue by contradiction, suppose that for some cartel, say cartel α, there exists a price $\tilde{p}_\alpha$, such that:

$$\Pi_\alpha(\tilde{p}_\alpha, p_M) > \Pi_\alpha(p_M, p_M) = \frac{n-k}{n} p_M.$$

Now, if $\tilde{p}_\alpha > p_M$, then $D(\tilde{p}_\alpha) < D(p_M) = 1$, and we may write

$$\Pi_\alpha(\tilde{p}_\alpha, p_M) = \frac{n-k}{n} \tilde{p}_\alpha D(\tilde{p}_\alpha) > \Pi_\alpha(p_M, p_M) = \frac{n-k}{n} p_M D(p_M).$$

This is a contradiction to the assumption that (p_M, p_M) maximizes joint profits. Suppose, alternatively, that $\tilde{p}_\alpha < p_M$. Thus, we must have

$$\Pi_\alpha(\tilde{p}_\alpha, p_M) = \frac{n-k}{n} \tilde{p}_\alpha > \Pi_\alpha(p_M, p_M) = \frac{n-k}{n} p_M,$$

which is impossible. We therefore prove part (b). ■

Proposition 2 *Whenever the relative number k/n of deviant firms is small enough, and if the monopoly demand is not exactly equal to the capacity of the industry (i.e. $D(p_M) \neq 1$), there exists a quasi-monopoly.*

Proof. First of all we have only to consider the case $D(p_M) < 1$, since $D(p_M) = 1$ is excluded and $D(p_M) > 1$ is not allowed by assumption.

Let us choose $\beta = k/n$ so as to satisfy $\beta < D(p_M) < \alpha < 1$ and define $\tilde{p}_\beta$ to be the smallest price p satisfying the equation

$$pD(p) = \left\{ \frac{D(p) - \beta}{D(p)} \right\} \{p_M D(p_M)\}. \tag{A.3}$$

The existence of $\tilde{p}_\beta$ is ensured by the assumption on the receipt function $pD(p)$ and the fact that the right-hand side is continuous, and always inferior to $p_M D(p_M)$. Clearly, $\tilde{p}_\beta < p_M$, and hence, by the definition of $\tilde{p}_\beta$,

$$\left\{\frac{p_M D(p_M) - \tilde{p}_\beta D(\tilde{p}_\beta)}{p_M}\right\} < \frac{D(\tilde{p}_\beta)}{D(p_M)}\left\{\frac{p_M D(p_M) - \tilde{p}_\beta D(\tilde{p}_\beta)}{p_M}\right\} = \beta.$$

Therefore, as β approaches zero, $\tilde{p}_\beta D(\tilde{p}_\beta)$ approaches $p_M D(p_M)$. By the continuity (which results from our assumptions) of the inverse of the receipt function $pD(p)$ restricted to the interval $[0, p_M]$, we immediately get that $\tilde{p}_\beta$ tends to p_M as β tends to 0. Consequently, if we choose β small enough, $\{D(\tilde{p}_\beta) - D(\tilde{p}_\beta)^2\}$ is close to $\{D(p_M) - D(p_M)^2\}$, a strictly positive number, and we must have: $\beta < \{D(\tilde{p}_\beta) - D(\tilde{p}_\beta)^2\}$. Hence, for β small enough,

$$\Pi_\beta(p_M, \tilde{p}_\beta) = \beta\tilde{p}_\beta = \beta\frac{\{D(\tilde{p}_\beta) - \beta\}}{D(\tilde{p}_\beta)^2} p_M D(p_M) > \beta p_M D(p_M) = \Pi_\beta(p_M, p_M)$$

so that condition (1) is satisfied. Moreover,

$$\Pi_\alpha(p_\alpha, \tilde{p}_\beta) \leq \left\{\frac{D(\tilde{p}_\beta) - \beta}{D(\tilde{p}_\beta)}\right\} p_\alpha D(p_\alpha) < \left\{\frac{\{D(\tilde{p}_\beta) - \beta}{D(\tilde{p}_\beta)}\right\} p_M D(p_M) = \Pi_\alpha(p_M, \tilde{p}_\beta)$$

for any $p_\alpha > \tilde{p}_\beta$; in addition, for β small enough,

$$\Pi_\alpha(\tilde{p}_\beta, \tilde{p}_\beta) \leq \alpha\tilde{p}_\beta D(\tilde{p}_\beta) < \left\{\frac{D(\tilde{p}_\beta) - \beta}{D(\tilde{p}_\beta)}\right\} p_M D(p_M) = \Pi_\alpha(p_M, \tilde{p}_\beta)$$

and for any $p_\alpha < \tilde{p}_\beta$,

$$\Pi_\alpha(p_\alpha, \tilde{p}_\beta) \leq p_\alpha D(p_\alpha) < \tilde{p}_\beta D(\tilde{p}_\beta) = \left\{\frac{D(\tilde{p}_\beta) - \beta}{D(\tilde{p}_\beta)}\right\} p_M D(p_M) = \Pi_\alpha(p_M, \tilde{p}_\beta).$$

In conclusion the set

$$P_\beta = \{p_\beta : \Pi_\alpha(p_M, p_\beta) \geq \Pi_\alpha(p_\alpha, p_\beta),$$

for any p_α, and $\Pi_\beta(p_M, p_\beta) > \Pi_\beta(p_M, p_M)\}$ is nonempty (since $\tilde{p}_\beta \in P_\beta$).

It remains to show that $\tilde{p}_\beta$ maximizes $\Pi_\beta(p_M, p_\beta)$ on P_β, i.e. $\tilde{p}_\beta = \overline{p}_\beta$. This we do by showing that $P_\beta \subset \{p_\beta : p_\beta \leq \tilde{p}_\beta\}$. Clearly, any p_β in P_β cannot be strictly greater than p_M, since then $\Pi_\beta(p_M, p_\beta)$ is zero (recall that $D(p_M) < \alpha$). But suppose that $p_M \in P_\beta$. Then we get a contradiction, since, for $p_M > p_\alpha > p_M D(p_M)$, either $D(p_\alpha) \geq \alpha$ and

$$\Pi_\alpha(p_M, p_M) = \alpha p_M D(p_M) < \alpha p_\alpha = \Pi_\alpha(p_\alpha, p_M),$$

or $D(p_M) < D(p_\alpha) < \alpha$ and

$$\Pi_\alpha(p_M, P_M) = \alpha p_M D(p_M) < p_\alpha D(p_\alpha) = \Pi_\alpha(p_\alpha, p_M).$$

Finally, suppose $p_\beta \in P_\beta \cap \{p_\beta : \tilde{p}_\beta < p_\beta < p_M\}$. Then we also get a contradiction, for $\tilde{p}_\beta < p_\beta - \varepsilon < p_\beta < p_M$, as

$$\begin{aligned}
\Pi_\alpha(p_\beta - \varepsilon, p_\beta) &= (p_\beta - \varepsilon) \cdot D(p_\beta - \varepsilon) > \tilde{p}_\beta D(\tilde{p}_\beta) \\
&= \left\{ \frac{D(\tilde{p}_\beta) - \beta}{D(\tilde{p}_\beta)} \right\} p_M D(p_M) \\
&= \Pi_\alpha(p_M, \tilde{p}_\beta).
\end{aligned}$$

Hence, $\overline{p}_\beta$ should simply maximize the function βp_β on $P_\beta \subset \{p_\beta : p_\beta \leq \tilde{p}_\beta\}$. Since $\tilde{p}_\beta \in P_\beta$ we get $\overline{p}_\beta = \tilde{p}_\beta$. ∎

This second proposition does not include the case in which $D(p_M) = 1$. In fact, it may be verified that, in such a case, there cannot exist a quasi-monopoly solution. However, by Proposition 1, there is then only one reasonable solution, namely (p_M, p_M), which is both the non-cooperative and the cooperative solution.

Proposition 3 *The quasi-monopoly price $\overline{p}_\beta$ is strictly decreasing function of β.*

Proof. Suppose $(p_M, \overline{p}_\beta)$ is a quasi-monopoly solution. We start by showing that $\overline{p}_\beta = \tilde{p}_\beta$ where $\tilde{\beta}$ is a solution of the equastion (A.3) in Proposition 2.

First, notice that $\overline{p}_\beta$ satisfies the equation

$$\min\{\alpha, D(\overline{p}_\beta)\} \cdot \overline{p}_\beta = \min\left\{\alpha, \frac{D(\overline{p}_\beta) - \beta}{D(\overline{p}_\beta)} \cdot D(p_M)\right\} \cdot p_M. \tag{A.4}$$

Indeed by (2) the left-hand side is smaller than or equal to the right-hand side. Moreover, if the inequality was strict, then the splinter group β could increase its profit by increasing $\overline{p}_\beta$, thus violating (2).

Second, $\overline{p}_\beta$ is in fact such that

$$\min\{\alpha, D(\overline{p}_\beta)\} \cdot \overline{p}_\beta = \left\{\frac{D(\overline{p}_\beta) - \beta}{D(\overline{p}_\beta)}\right\} D(p_M) p_M. \tag{A.5}$$

Suppose the contrary. Using (A.4), this means that cartel α works at capacity at the quasi-monopoly solution. Then,

$$\alpha < \left\{ \frac{D(\overline{p}_\beta) - \beta}{D(\overline{p}_\beta)} \right\} D(p_M) < D(p_M) < D(\overline{p}_\beta)$$

where the last inequality follows from $\overline{p}_\beta < p_M$. Hence (A.4) becomes $\alpha \overline{p}_\beta = \alpha p_M$, a contradiction.

Finally, we show that

$$\overline{p}_\beta D(\overline{p}_\beta) = \left\{ \frac{D(\overline{p}_\beta) - \beta}{D(\overline{p}_\beta)} \right\} p_M D(p_M).$$

Suppose the contrary; then by (A.5),

$$(1-\beta) < D(\overline{p}_\beta) \text{ and } (1-\beta)\overline{p}_\beta = \left\{ 1 - \frac{\beta}{D(\overline{p}_\beta)} \right\} \cdot D(p_M) p_M.$$

By (1) and the fact that β is at capacity, we have

$$\beta \overline{p}_\beta > \beta p_M D(p_M).$$

Consequently, $\overline{p}_\beta > p_M D(p_M)$, so that, by the preceding equality,

$$(1-\beta) < \left\{ 1 - \frac{\beta}{D(\overline{p}_\beta)} \right\}$$

or, equivalently, $D(\overline{P}_\beta) > 1$. Combining $D(\overline{p}_\beta) > 1$ and $\overline{p}_\beta > p_M D(p_M)$, we get

$$\overline{p}_\beta D(\overline{p}_\beta) > p_M D(p_M)$$

which contradicts the optimality property of p_M.

Hence $\overline{p}_\beta$ is a solution of (A.3). By the strict concavity of $pD(p)$ and the fact $D(p)$ is decreasing, $\overline{p}_\beta$ is the unique solution of (A.3) in the interval $[0, p_M]$. Moreover, the left-hand side of (A.3) is invariant with β, and the graph of the right-hand side with $\beta = (k+1)/n$ is strictly below the graph of the same right-hand side with $\beta = k/n$. Accordingly, $\overline{p}_\beta$ is a strictly decreasing function of β. ∎

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