# Commodity exchanges as gradient processes*

**Claude d'Aspremont**
*CORE, Université catholique de Louvain*
**Henry Tulkens**
*Facultés Universitaires Saint-Louis*

## Abstract

The purpose here is to make explicit the sense in which two dynamic processes, due to Malinvaud and others (whose solutions determine an efficient allocation for a given economy), are related to the gradient projection method known in the nonlinear optimization literature. The connections we establish derive from simple observations on first order characterizations of efficient allocations; they also lead to the formulation of another process, that applies to a classical welfare maximization problem; finally, they provide a common basis for an *a priori* justification of each of the three processes involved, which supplements the intrinsic properties that they can be shown to have.

## 1. Introduction

In some recent economic literature, much interest has been devoted to the formulation of differential equations systems whose solutions take their image in the feasible allocation set and converge to some Pareto efficient allocation (see, e.g., Malinvaud 1970–1971, Section 4, or Malinvaud 1972, Chapter 8, Section 4, or the more recent work by Smale 1976 and Champsaur et al. 1977). Typically, these systems of differential equations are formulated in a way closely related to the so-called "marginal" conditions whereby Pareto efficient allocations are usually characterized (at least in the case of economies with a sufficient degree of differentiability), and their economic interest appears from the properties they are proved to have.

Instead of constructing such systems directly, we would like to show in this note how some of them can be derived from a common economic theoretic framework, and by means of a fairly standard technique of nonlinear optimization known as the gradient projection method. Specifically, we shall do this for two processes called here for brevity "M70" and "MDP" respectively (see Sections 4 and 5). Along the way, we shall be led to formulate in Section 3 a somewhat different process – call it "WMP" – which appears to be a most direct application of the gradient projection method to a classical form of welfare maximization (hence the initials).[1]

In Section 2, we begin by defining the economy (which for simplicity is chosen to be of pure exchange), and we note there some properties of the first order conditions characterizing efficiency;

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1. It should be noted that Milleron (1974) already used explicitly this methodology for the particular case of homogeneous utility functions.

these bring to light most naturally the connections existing between all three processes. Other comparative and interpretative remarks are offered in our concluding Section 2.

## 2. Alternative optimality conditions for a pure exchange economy

Consider a pure exchange economy formed by a set $N = \{1, \ldots, i, \ldots, n\}$ of agents, a set $M = \{1, \ldots, h, \ldots, m\}$ of private commodities and satisfying the following assumptions:

(i) The preferences of each agent $i \in N$ are represented by a continuous utility function $u_i(x_i)$ defined on a consumption set equal to $\mathring{\mathbb{R}}^m_+ \overset{\text{def}}{=} \{x_i \in \mathbb{R}^m \mid x_i = (x_{i1}, \ldots, x_{ih}, \ldots, x_{im})$ and $x_{ih} > 0$ for every $h\}$ (*continuity* assumption).

(ii) Each function $u_i$ is $C^2$ and strictly concave (*differentiability* and *convexity* assumptions).

(iii) For each $u_i$, the vector of partial derivatives $(u_{i1}, \ldots, u_{ih}, \ldots, u_{im})$ maps into $\mathbb{R}^m_+$ and is such that $u_{i1} > 0$ (*monotonicity* assumption).

In this model, an allocation is simply a vector $x = (x_1, \ldots, x_i, \ldots, x_n) \in (\mathring{\mathbb{R}}^m_+)^n$. On the other hand, each commodity $h$ is assumed to be available in a fixed aggregate amount $\omega_h > 0$. Let $\omega = (\omega_1, \ldots, \omega_h, \ldots, \omega_m) \in \mathbb{R}^m_+$. We shall denote by

$$\mathring{X} = \left\{ x \in (\mathring{\mathbb{R}}^m_+)^n \mid \forall\, h \in M, \sum_{i \in N} x_{ih} = \omega_h \right\},$$

the set of feasible allocations, and by

$$\begin{aligned} X^* = \{ x \in \mathring{X} \mid \nexists y \in \mathring{X} \text{ such that} :\ \forall\, i \in N, u_i(y_i) \geq u_i(x_i) \\ \text{and } \exists j \in N \text{ with } u_j(y_j) > u_j(x_j) \}, \end{aligned}$$

the set of Pareto efficient allocations. Furthermore:

(iv) For any sequence $(x^k)$, $k = 1, 2, \ldots$, in $\mathring{X}$, if $x^k_{ih} \to 0$ for some $i$ and $h$, then $u_i(x^k_i) \to -\infty$ (*boundary* condition).

Under our present assumptions, it is well known that to each Pareto efficient allocation $x^*$ in $X^*$ one can associate a strictly positive $n$-vector $\lambda$ such that $x^*$ is a solution of the following nonlinear program:

$$\max_x \sum_{i \in N} \lambda_i u_i(x_i) \text{ subject to} \tag{1}$$

$$\sum_{i \in N} x_{ih} = \omega_h, \ \ \forall\, h \in M, \tag{2}$$

$$x_{ih} > 0, \ \ \forall\, i \in N, \ \ \forall\, h \in M. \tag{3}$$

Also, the first order optimality conditions of this classical "welfare maximization" problem imply that any $x \in \mathring{X}$ maximizes $W(x) \overset{\text{def}}{=} \sum_{i \in N} \lambda_i u_i(x_i)$ if and only if $x$ satisfies the equalities:

$$\lambda_i u_{ih} = \lambda_j u_{jh} \ \ \forall\, h \in M, \ \ \forall\, i, j \in N. \tag{4}$$

Defining as usual for every agent $i$ and any $x_i \in \mathbb{R}^m_+$

$$\pi_{ih} = u_{ih}/u_{i1}, \ \ \forall\, h \in M,$$

as the marginal rate of substitution of $i$, at $x_i$, between commodities $h$ and 1, the equalities (4) imply:

$$\pi_{ih} = \pi_{jh}, \ \ \forall\, h \in M, \ \ \forall\, i, j \in N, \tag{5}$$

which are also known to be necessary and sufficient conditions for $x \in \mathring{X}$ to be a Pareto efficient allocation.

However there is still an equivalent formulation for these conditions. Indeed, (5) implies: $\forall\, i, j \in N$, $\|\pi_i\| = \|\pi_j\|$ where for every $i \in N$, $\|\pi_i\|$ is the Euclidean norm of the vector $\pi_i = (\pi_{i1}, \ldots, \pi_{ih}, \ldots, \pi_{im})$. Hence it is easy to see that (5) implies:

$$\frac{u_{ih}}{\|\nabla\, u_i\|} = \frac{u_{jh}}{\|\nabla\, u_j\|}, \ \ \forall\, h \in M, \ \ \forall\, i, j \in N, \tag{6}$$

where $\nabla\, u_i$ denotes the gradient vector of $u_i$ at $x_i$, since

$$\frac{\pi_{ih}}{\|\pi_i\|} = \frac{u_{ih}}{\|\nabla\, u_i\|}, \ \ \forall\, i \in N, \ \ \forall\, h \in M. \tag{7}$$

Equivalence[2] follows from the fact that (6) implies:

$$\frac{u_{ih}}{u_{jh}} = \frac{\nabla\, u_i}{\|\nabla\, u_j\|} = \frac{u_{i1}}{u_{j1}}, \ \ \forall\, h \in M, \ \ \forall\, i, j \in N.$$

Thus, every Pareto optimal allocation belonging to $\mathring{X}$ may be characterized by any one of the three expressions (6), (5), or (4), with the appropriate vector $\lambda$ in the latter. This simple observation is shown below to provide the link between the three dynamic processes to which we now turn.

## 3. The gradient projection process for the welfare maximization problem

If one considers the nonlinear optimization problem (1)–(3), a simple method for defining a differential system leading to an optimal solution is the following.

For any $x \in \mathring{X}$, the gradient vector at $x$ of the objective function $W(x)$, namely $(W_{ih}) = (\lambda_i\, u_{ih})$, determines the direction (i.e., a vector belonging to $\mathbb{R}^{nm}$, which we denote here by $\dot{x}$) of maximum rate of increase of the function $W$ at point $x$. However, changing $x$ in this direction would not in general ensure that the new point, $x'$, is feasible, that is, the conditions (2) might no longer be satisfied. A technique for preserving feasibility is provided[3] by projecting the gradient vector onto the tangent subspace determined by the (active) constraints (2). Formally, this results from premultiplying the gradient of (1) by the projection matrix

$$P = I - A'(AA')^{-1}A,$$

where $I$ is the $nm \times nm$ identity matrix and $A$ the $m \times nm$ matrix of coefficients of the left hand side of the constraints (2). The structure of $A$, in the present case of a pure exchange economy, yields

2. The case $u_{ih} = u_{jh} = 0$, for some $h \in M$, $h \neq 1$ and $i, j \in N$, is immediate (in both ways).
3. See, for instance, Luenberger (1973, p.247 and ff).

for $P$ a $nm \times nm$ block-diagonal matrix in which the blocks $B_h, h = 1, \ldots, m$, are $n \times n$ matrices of the form:

$$B_h = \begin{bmatrix} 1-\frac{1}{n} & \ldots & -\frac{1}{n} & \ldots & -\frac{1}{n} \\ \vdots & & \vdots & & \vdots \\ -\frac{1}{n} & \ldots & 1-\frac{1}{n} & \ldots & -\frac{1}{n} \\ \vdots & & \vdots & & \vdots \\ -\frac{1}{n} & \ldots & -\frac{1}{n} & \ldots & 1-\frac{1}{n} \end{bmatrix}.$$

A straightforward calculation shows that the feasible direction $\dot{x} = (\dot{x}_{11}, \ldots, \dot{x}_{ih}, \ldots, \dot{x}_{nm})$, obtained at $x$ after premultiplying by $P$ the gradient of $W(x)$, is of the form:

$$\dot{x}_{ih} = \lambda_i \, u_{ih} - \frac{1}{n} \sum_{j \in N} \lambda_j \, u_{jh} \;\; \forall \, i \in N, \;\; \forall \, h \in M; \tag{8}$$

it is naturally called the "gradient projection direction" at $x$. Notice that if (and only if, with our assumptions) $x$ were an optimal solution of (1)–(3), the right hand side of (8) would be the 0-vector, in view of (4); here appears, thus, the connection between the expression of the gradient projection direction and the form of the first order conditions for an optimal solution.

We shall now interpret system (8) as a system of differential equations and call it the *WMP process*, by assuming that each $x_{ih}$ is a function of time, $x_{ih}(t)$, and that each $\dot{x}_{ih}$ denotes the time derivate $dx_{ih}/dt$. For any nonempty interval $T \subseteq \mathbb{R}_+$, containing 0, a solution of this system of differential equations, with respect to some initial condition $x^0 \in \mathring{X}$, is a function $x(\cdot)$ from $T$ to $\mathbb{R}^{mn}$ such that, for any $t \in T$,

$$\dot{x}_{ih}(x(t)) = \lambda_i \, u_{ih}(x_i(t)) - \frac{1}{n} \sum_{j \in N} \lambda_j \, u_{jh}(x_j(t)) \quad \forall \, i \in N, \, \forall \, h \in M, \text{ and } x(0) = x^0.$$

A first property of the dynamic system (8) is provided by the following lemma.

**Lemma 1** *The WMP process (8) is collectively monotone, in the sense that, for any solution $x : T \to \mathbb{R}^{nm}$ of the associated system of differential equations and for any $t \in T$, if $x(t) \in \mathring{X}$ then*

$$\dot{W}(x(t)) = \sum_{i \in N} \lambda_i \, u_{ih}(x_i(t)) \dot{x}_{ih}(x(t)) > 0,$$

*unless $x(t)$ maximizes $W$ on $\mathring{X}$, in which case $\dot{W}(x(t)) = 0$.*

**Proof** This simply derives from the fact that:

$$\sum_{h \in M} \sum_{i \in N} \lambda_i \, u_{ih} \left( \lambda_i \, u_{ih} - \frac{1}{n} \sum_{j \in N} \lambda_j \, u_{jh} \right) = \sum_{h \in M} \sum_{i \in N} \left( \lambda_i \, u_{ih} - \frac{1}{n} \sum_{j \in N} \lambda_j \, u_{jh} \right)^2 > 0$$

for al $x \in \mathring{X}$ except, by condition (4), if $x$ maximizes $W$ on $\mathring{X}$. ■

However, the main interest of system (8) derives from the following proposition.

**Proposition 1** *For the WMP process (8) and for every $x^0 \in \mathring{X}$, there exists a unique solution $x(\cdot) : [0, \infty) \to \mathring{X}$, which is such that $\lim_{t \to \infty} x(t)$ exists and is a Pareto efficient allocation.*

**Proof**

1. *Existence*

   (a) In view of Lemma 1, if a solution $x(\cdot)$ exists, the fact $x^0 > 0$ implies that $x(t) > 0\,\forall\, t \in T$. This results from the fact that for any $x^0 \in \mathring{X}$, and for any sequence $(x^k)$, $k = 1, 2, \ldots$ in the set
   $$U(x^0) \overset{\text{def}}{=} \{x \in \mathring{X} \mid W(x) \geq W(x^0)\},$$
   $\lim_{k\to\infty} x^k_{ih} > 0$ for any $i$ and $h$. Indeed, suppose that there exists a sequence $(x^k)$, $k = 1, 2, \ldots$ satisfying $x^k \in U(x^0)\,\forall\, k$ and such that $\exists\, j$, $\exists\, h$ with $\lim_{k\to\infty} x^k_{jh} = 0$. By the boundary condition (iv), $\lim_{k\to\infty} u_j(x^k_j) = -\infty$; on the other hand, by the monotonicity assumption,
   $$\sup_{x\in\mathring{X}} u_i(x_i) \leq u_i(\omega) < +\infty\ \ \forall\, i \in N, i \neq j,$$
   since $\forall\, x \in \mathring{X}$, $x_i < \omega$. Hence $\lim_{k\to\infty} \sum_i \lambda_i\, u_i(x^k_i) = -\infty$. But then, $\exists\, K$ such that $\forall\, k \geq K$, $\sum_i \lambda_i\, u_i(x^k_i) < W(x^0)$, which contradicts the definition of the sequence $(x^k)$.

   (b) Since it is immediate that for any $h \in M\ \ \sum_{i\in N} \dot{x}_{ih} = 0$, we may add that $x(t) \in \mathring{X}$ for any $t \in T$, if a solution exists with $x^0 > 0$.

   (c) By the differentiability assumption (ii) the partial derivatives of the right hand side of (8) are continuous in $\mathring{\mathbb{R}}^{nm}_+$; hence, this right hand side is locally Lipschitz in $\mathring{\mathbb{R}}^{nm}_+$. This implies, by (Rouche, 1973, Theorem 5.6, p. 83) that there exists a unique solution $x(\cdot)$, defined on some interval, which cannot be continued on the right.

   (d) By Lemma 1, this solution $x(\cdot)$, defined on some interval $[0, \beta)$, maps into $U(x^0)$, which is a compact subset of $\mathring{\mathbb{R}}^{nm}_+$, and so has at least one accumulation point, say $\overline{x}$. If $\beta < \infty$, then by the second part of (Rouche, 1973, Theorem 5.6), $(\beta, \overline{x})$ must belong to the frontier of $\mathbb{R} \times \mathring{\mathbb{R}}^{mn}_+$. Since by step (a), $\overline{x}_{ih} > 0$, for all $i$ and $h$, we can only have $\beta = +\infty$.

2. **Convergence**

   (a) To prove the second part of the proposition we note as an immediate corollary to Theorem 6.1 in Champsaur et al. (1977), that if a dynamic system has a unique solution for every $x^0 \in \mathring{X}$, say $x(x^0; t)$, which varies continuously with $x^0$ in $\mathring{X}$ and which remains in $\mathring{X}$ for all $t$, and if there is a Lyapunov function[4], then the system is *quasi-stable*[5]. From the system (8), existence of a unique solution $x(x^0; \cdot) : [0, \infty) \to \mathring{X}$ has just been established; moreover, by (Rouche, 1973, Theorem 3.1, p. 105), $x(\cdot; t)$ is continuous in $x^0$ on $\mathring{X}$, since the right hand side of (8) is locally Lipschitz; finally, the collective monotonicity (Lemma 1) and the boundedness of $W$ imply that we can take $L = W$ as a Lyapunov function for (8). Hence (8) is quasi-stable.

4. *A Lyapunov function* for a dynamic system $\dot{x}(x)$ is any continuous function $L$ from $\mathring{X}$ to $\mathbb{R}$ such that for any $x^0 \in \mathring{X}$, and any solution $x(\cdot)$ starting at $x^0$, $\lim_{t\to\infty} L(x(t))$ exists and, whenever $x(t)$ is constant for all $t \in [0, \tau] \subset [0, \infty)$ with $\tau > 0$, then $x^0$ is an equilibrium. An *equilibrium* of a dynamic system is a state $x \in X$ of the system such that $\dot{x}(x) = 0$.
5. A dynamic system is said to be *quasi-stable* iff any accumulation point of any solution is an equilibrium.

(b) Since by (4) any equilibrium of the system (8) is a Pareto efficient allocation, it remains only to show that $\lim_{t\to\infty} x(x^0;t)$ exists for any $x^0 \in \mathring{X}$. But this is immediate since, by our convexity assumptions, there is only one Pareto efficient allocation $x^*$ such that $L(x^*) = \lim_{t\to\infty} L(x(t)) = \lim_{t\to\infty} W(x(t))$.

■

**Corollary 1** *For every $x^0 \in \mathring{X}$, the unique solution $x(\cdot) : [0,\infty] \to \mathring{X}$ is such that $W\left(\lim_{t\to\infty} x(t)\right) = \max_{x\in\mathring{X}} W(x)$.*

## 4. The "M70" process as a gradient process

In footnote 4 (p. 215) of his paper on public goods, Malinvaud (1970–1971) sketches out a dynamic process which we call here the "M70 Process," and which in the case of our private goods pure exchange economy may be written as follows (all variables $x_i$ are kept assumed to be functions of time).
*M70 Process*:[6]

$$\dot{x}_{ih} = \frac{u_{ih}}{\|\nabla u_i\|} - \frac{1}{n}\sum_{j\in N}\frac{u_{jh}}{\|\nabla u_j\|} \quad \forall\, i \in N,\ \forall\, h \in M. \tag{9}$$

Comparing (8) with (9), the latter is immediately seen to be a simple modification of the former, in which every coefficient $\lambda_i$ is now a particular function of $x_i$, namely:

$$\lambda_i(x_i) = (\|\nabla u_i\|)^{-1},\ \ \forall\, i \in N. \tag{10}$$

The choice of the vector $\lambda$ simply corresponds to a normalization of the gradient vector $(u_{i1},\ldots,u_{ih},\ldots,u_{im})$ of the utility function of each agent $i$.

Referring to the original welfare maximization problem (1)–(3), and to the reasoning that led to the formulation of the WMP process (8) associated with it, the M70 process appears to be also a "gradient projection process," but not exactly for the problem (1)–(3) though. Instead, it is such a process *for a continuous sequence* of such problems, each of which is redefined at each $x(t)$ in terms of coefficients $\lambda(x(t))$ defined by (10). It is interesting to note that, while the WMP process (8) was formulated by means of the form (4) of the first order conditions for a maximum, it is the use of the alternative form (6) of these conditions that yields the M70 process (9), which is the gradient projection of a modified optimization problem.

Another way of viewing the M70 dynamic system is to consider it as the system which, for each $x \in \mathring{X}$, determines the direction of the "locally" best improvement among all the feasible ones; more precisely[7], this direction is proportional to the optimal solution of the following "local" programming problem:

$$\max_{\dot{x}\in\mathbb{R}^{nm}} \sum_{i\in N}(\|\nabla u_i\|)^{-1}\sum_{h\in M} u_{ih}\dot{x}_{ih} \tag{11}$$

subject to

$$\sum_{i\in N}\dot{x}_{ih} = 0,\ \ \forall\, h \in M,\ \|\dot{x}\| = 1. \tag{12}$$

6. It should also be pointed out that in fact, Malinvaud (1970–1971) uses for the second term of (9) a *weighted* average, instead of the simple average we use here.
7. See (Luenberger, 1973, Exercise 9, p. 274).

Turning now to the properties of the M70 process, we note first the remarkable fact that the introduction of a sequence of welfare maximization problems by using $\lambda$'s which are functions of $x$, allows for a stronger monotonicity property:

**Lemma 2** *The M70 process (9) is individually monotone in the sense that, for any solution $x : T \to \mathbb{R}^{nm}$ of the associated system of differential equations and for any $t \in T$, if $x(t) \in \mathring{X}$ then*

$$\dot{u}_i(x(t)) = \sum_{h \in M} u_{ih}(x_i(t))\dot{x}_{ih}(x(t)) > 0, \; \forall\, i \in N,$$

*unless $x(t)$ is a Pareto efficient allocation, in which case $\forall\, i \in N$, $\dot{u}_i(x(t)) = 0$.*

**Proof** To simplify the notation, let, $\forall\, i \in N$, $\forall\, h \in M$, $v_{ih} = u_{ih}/\|\nabla u_i\|$. Clearly, $\dot{u}_i \geq 0$ iff $\dot{u}_i/\|\nabla u_i\| \geq 0$ (recall that $\|\nabla u_i\| > 0$). Since $\|v_i\| = \|v_j\| = 1$, we get:

$$\begin{aligned}\frac{\dot{u}_i}{\|\nabla u_i\|} &= \sum_{h \in M} v_{ih}\Bigl(v_{ih} - \frac{1}{n}\sum_{j \in N} v_{jh}\Bigr) = \frac{1}{n}\sum_{j \in N}\Bigl(\sum_{h \in M} v_{ih}^2 - \sum_{h \in M} v_{ih}\, v_{jh}\Bigr) \\ &= \frac{1}{n}\sum_{j \in N}(\|v_i\|^2 - |v_i \cdot v_j|) = \frac{1}{n}\sum_{j \in N}(\|v_i\|\,\|v_j\| - |v_i \cdot v_j|) \geq 0\end{aligned}$$

by Cauchy-Schwarz inequality. Moreover, since for any $j \in N$,

$$\|v_i\|^2 = |v_i \cdot v_j| = \|v_j\|^2 = 1 \text{ iff } v_i = v_j,$$

we must have, by conditions (5), $\dot{u}_i/\|\nabla u_i\| > 0$, unless $x$ is Pareto efficient. ■

With this result, we may now prove the following proposition, just as in Proposition 1 for the WMP process.

**Proposition 2** *For the M70 process (9) and for every $x^0 \in \mathring{X}$, there exists a unique solution $x(\cdot) : [0, \infty) \to \mathring{X}$, which is such that $\lim_{t\to\infty} x(t)$ exists and is a Pareto efficient allocation.*

**Proof** The first part of the proof is analogous to the argumentation in Proposition 1, redefining $U(x^0)$ as

$$\{x \in \mathring{X} \mid u_i(x_i) \geq u_i(x_i^0)\; \forall\, i \in N\}$$

and replacing Lemma 1 by Lemma 2 in paragraphs 1.1 and 1.4, and using in paragraph 1.3 the fact that $\lambda_i$ is continuously differentiable in $\mathring{\mathbb{R}}_+^{nm}$.

For the second part, we may again use Theorem 6.1 of Champsaur et al. (1977), taking as Lyapunov function $\sum_{i \in N} u_i(x_i)$. ■

## 5. The "MDP" process as a gradient process

In Chapter 8 of his microeconomics textbook, Malinvaud (1972, pp. 190–192), develops another dynamic process for a pure exchange economy, which has come to be known as the "MDP process" in its version with public goods[8]. In our present notation, this process reads:

8. See Champsaur (1976, p. 293).

*MDP Process:*

$$\dot{x}_{ih} = \pi_{ih} - \frac{1}{n}\sum_{j\in N}\pi_{jh} \;\; \forall\, h \in M,\; h \neq 1,\;\; \forall\, i \in N, \tag{13}$$

$$\dot{x}_{i1} = -\sum_{\substack{h\in M\\ h\neq 1}} \pi_{ih}\dot{x}_{ih} + \delta_i \sum_{\substack{h\in M\\ h\neq 1}}\sum_{j\in N}\left[\pi_{jh} - \frac{1}{n}\sum_{k\in N}\pi_{kh}\right]^2 \;\forall\, i \in N, \tag{14}$$

where the constants $\delta_i$ satisfy $\delta_i \geq 0, \forall\, i \in N$ and $\sum_{i\in N}\delta_i = 1$. (Note that here again we use simple arithmetic averages instead of weighted ones.)

Comparing this system with (9) and with (8) above, and remembering the efficiency conditions (5), one sees that as far as all commodities $h \neq 1$ are concerned, this is again a process formulated on the basis of the first order conditions for a Pareto efficient allocation, the version (5) being used this time. However, the equations (14) of the process do not bear much resemblance with these conditions. Because it is known (as recalled below; see Proposition 3) that for this process too the unique solution converges to a Pareto efficient allocation, it is natural to ask oneself which further modification of the original welfare maximization problem yields the MDP process as a gradient process. In the same spirit as our preceding development regarding the M70 process, a sequence of changing problems is involved; but in addition, the original maximization problem undergoes a substantial transformation that we presently develop.

Consider some arbitrary allocation in $\mathring{X}$, say $\overline{x}$, and the associated utility vector $\overline{v} = (\overline{v}_1, \ldots, \overline{v}_n)$ given by $\overline{v}_i = u_i(\overline{x}_i), \forall\, i \in N$. To the welfare maximization problem (1)–(3), let a set of $n$ additional constraints be added, namely:

$$u_i(x_{i1}, \ldots, x_{im}) \geq \overline{v}_i \qquad \forall\, i \in N. \tag{15}$$

By construction, $\overline{x}$ is a feasible solution for the augmented problem (1)-(3) and (15), with strict equality holding for each one of the new constraints (15). Note at this point that for every solution $x \in \mathring{X}$ for which such equalities hold, each one of the latter can be made explicit in $x_{i1}$ rather than in $v_i$, using the monotonicity assumption, and be written as

$$x_{i1} = \overline{f}_i(x_{i2}, \ldots x_{im}; \overline{v}_i) \;\; \forall\, i \in N. \tag{16}$$

On the other hand, for every other solution of the augmented problem for which $u_i(x_i) > \overline{v}_i$ holds for some $i$, there is for each such $i$ an amount of commodity 1, say $y_{i1} \geq 0$, such that

$$x_{i1} = \overline{f}_i(x_{i2}, \ldots, x_{im}; \overline{v}_i) + y_{i1}. \tag{17}$$

Now, since $\sum_{i\in N} x_{i1} = \omega_1$ must hold for any solution which belongs to $\mathring{X}$, we may write that every feasible solution $x$ of the augmented problem satisfies

$$\sum_{i\in N} x_{i1} = \sum_{i\in N}\overline{f}_i(x_{i2}, \ldots, x_{im}; \overline{v}_i) + \sum_{i\in N} y_{i1} = \omega_1,$$

or

$$\sum_{i\in N} y_{i1} = \left(\omega_1 - \sum_{i\in N}\overline{f}_i(x_{i2}, \ldots, x_{im}; \overline{v}_i)\right). \tag{18}$$

This in turn implies that there exists a vector $\delta = (\delta_1, \ldots, \delta_i, \ldots, \delta_n)$ with $\delta_i \geq 0, \forall\, i \in N$, and $\sum_{i\in N} \delta_i = 1$ such that

$$y_{i1} = \delta_i\left(\omega_1 - \sum_{j\in N} \overline{f}_j(x_{j2}, \ldots, x_{jm}; \overline{v}_j)\right) \;\; \forall\, i \in N. \tag{19}$$

Finally, to any $\overline{x} \in \mathring{X}$, and for every $\delta \in \mathbb{R}^n_+$ such that $\sum_{i\in N} \delta_i = 1$, one may associate the following nonlinear optimization problem:

$$\max_{x\in\mathbb{R}^{nm}} \sum_{i\in N} \lambda_i u_i(x_i) \tag{20}$$

subject to

$$x_{i1} = \overline{f}_i(x_{i2}, \ldots, x_{im}; \overline{v}_i) + \delta_i\left(\omega_1 - \sum_{j\in N} \overline{f}_j(x_{j2}, \ldots, x_{jm}; \overline{v}_j)\right), \;\; \forall\, i \in N, \tag{21}$$

$$\sum_{i\in N} x_{ih} = \omega_h, \;\; \forall\, h \in M_-, \tag{22}$$

$$x_{ih} > 0, \;\; \forall\, i \in N, \;\; \forall\, h \in M, \tag{23}$$

where $M_-$ denotes the set $M \setminus \{1\}$.

It is now possible to use on this problem a combination of both the reduced gradient and the gradient projection techniques. Indeed we may reduce first the problem by eliminating the $x_{i1}$'s in the following way:

$$\begin{aligned} \max_{x\in\mathbb{R}^{n(m-1)}} \overline{W}(x; \lambda, \overline{v}) \stackrel{\text{def}}{=} \sum_{i\in N} \lambda_i\, u_i\Bigg[&\overline{f}_i(x_{i2}, \ldots, x_{im}; \overline{v}_i) \\ &+ \delta_i\left(\omega_1 - \sum_{j\in N} \overline{f}_j(x_{j2}, \ldots, x_{jm}; \overline{v}_j)\right), x_{i2}, \ldots, x_{im}\Bigg] \end{aligned} \tag{24}$$

subject to

$$\sum_{i\in N} x_{ih} = \omega_h \quad \forall\, h \in M_-, \tag{25}$$

$$x_{ih} > 0 \quad \forall\, i \in N, \quad \forall\, h \in M_-. \tag{26}$$

Clearly, the allocation $\overline{x}$, whereby the indifference functions $\overline{f}_i(\cdot)$ appearing in (24) have been defined, is a feasible solution for this problem.

Next, let us compute the gradient of the objective function $\overline{W}$ as given by (24), at the allocation $\overline{x}$. Its typical element is of the form:

$$\left.\frac{\partial \overline{W}}{\partial x_{ih}}\right|_{\overline{x}} = \lambda_i\left[u_{i1}\frac{\partial \overline{f}_i}{\partial x_{ih}} - \delta_i \frac{\partial \overline{f}_i}{\partial x_{ih}} + u_{ih}\right] - \sum_{j\neq i} \lambda_j\, u_{j1}\, \delta_j \frac{\partial \overline{f}_i}{\partial x_{ih}} \quad i \in N, h \in M_-. \tag{27}$$

Upon noticing that by construction of (16)

$$\frac{\partial \overline{f}_i}{\partial x_{ih}} = -\left.\frac{u_{ih}}{u_{i1}}\right|_{\overline{x}_i} = -\pi_{ih}(\overline{x}_i),$$

(27) reduces to, after rearranging terms,

$$\begin{aligned} \left.\frac{\partial \overline{W}}{\partial x_{ih}}\right|_{\overline{x}} &= \lambda_i(-u_{ih} + u_{i1}\,\delta_i\,\pi_{ih} + u_{ih}) + \sum_{j\neq i} \lambda_j\,u_{j1}\,\delta_j\,\pi_{ih} \\ &= a(\overline{x})\pi_{ih}(\overline{x}_i), \end{aligned} \tag{28}$$

where

$$a(\overline{x}) = \sum_{j\in N} \lambda_j\,u_{j1}\,\delta_j. \tag{29}$$

If we compute the projection of this gradient onto the tangent subspace determined, at $\overline{x}$, by the constraints (25), by means of the matrix operator $P$ defined in Section 2 above (which is of dimension $n(m-1)\times n(m-1)$ in this case), we obtain the feasible direction for all $h \in M_-$ as:

$$\dot{x}_{ih}(\overline{x}) = a(\overline{x})\Bigg(\pi_{ih}(\overline{x}_i) - \frac{1}{n}\sum_{j\in N}\pi_{jh}(\overline{x}_j)\Bigg) \quad \forall\, i \in N,\ \ \forall\, h \in M_-. \tag{30}$$

Differentiating (21), expression (30) yields in addition the direction for $h = 1$ as

$$\begin{aligned} \dot{x}_{i1}(\overline{x}) &= -\sum_{h\in M_-} \pi_{ih}(\overline{x}_i)\dot{x}_{ih}(\overline{x}) + \delta_i \sum_{j\in N}\sum_{h\in M_-} \pi_{jh}(\overline{x})\dot{x}_{jh}(\overline{x}) \\ &= -\sum_{h\in M_-} \pi_{ih}(\overline{x}_i)\dot{x}_{ih}(\overline{x}) \\ &= +\delta_i\,a(\overline{x})\sum_{h\in M_-}\sum_{j\in N}\Bigg(\pi_{jh}(\overline{x}_j) - \frac{1}{n}\sum_{k\in N}\pi_{kh}(\overline{x}_k)\Bigg)^2 \quad \forall\, i\in N. \end{aligned} \tag{31}$$

Comparing (30) and (31) thus obtained with (13) and (14), the MDP process appears to determine, at each point $\overline{x}$, say, of its trajectory, the direction of the gradient projection of the transformed welfare maximization problem (24)–(26), with the coefficient $\lambda_i$ chosen so that

$$\lambda_i(\overline{x}) = [u_{i1}(\overline{x}_i)]^{-1} \ \ \forall\, i \in N. \tag{32}$$

Indeed, (32) implies, through (29), that $a(\overline{x}) = 1$ in (30) and (31). Note however that if the $\lambda_i$'s were taken as arbitrary positive constants in (24), the *direction* of the vector $\dot{x}(\overline{x})$ defined by (30) and (31) would not be modified; only its *length* (which is constant when $a(\overline{x}) = 1, \forall\,(\overline{x})$) would be modified by the variable factor $a(\overline{x})$.

The economic significance of the MDP dynamic system is further brought to light by considering it in an alternative (although related) way, as we did above, in (11)–(12), for the M70 process. Indeed the process may be viewed as one that determines at each $\overline{x} \in \mathring{X}$ the feasible direction that solves the following "local" problem, derived from (24)–(26):

$$\max_{\dot{x}\in\mathbb{R}^{n(m-1)}} \dot{\overline{W}} \overset{\text{def}}{=} \sum_{i\in N}\sum_{h\in M_-} \left.\frac{\partial \overline{W}}{\partial x_{ih}}\right|_{\overline{x}} \dot{x}_{ih} \tag{33}$$

subject to

$$\sum_{i\in N} \dot{x}_{ih} = 0 \quad \forall\, h \in M_-, \tag{34}$$

$$\|\dot{x}\| = 1. \tag{35}$$

In view of (28), the maximand (33) reduces to

$$\dot{\overline{W}} = \sum_{i \in N} \sum_{h \in M_-} a(\overline{x}) \pi_{ih}(\overline{x}_i) \dot{x}_{ih},$$

and an easy computation shows that the solution of this problem yields for each component $\dot{x}_{ih}$, $i \in N$ and $h \in M_-$, an expression proportional to (30). Using then (32) for determining $a(\overline{x})$ through (29), the value of the maximand at this solution is proportional to:

$$\sum_{i \in N} \sum_{h \in M_-} \left( \pi_{ih} - \frac{1}{n} \sum_{j \in N} \pi_{jh} \right)^2.$$

This magnitude, which is expressed in units of commodity $h = 1$, is simply the well known "surplus" that characterizes all processes of the "MDP" type. The gradient direction determined by such a process is thus the one which, by maximizing the rate of increase of (24), as specified by (33)–(35), in fact maximizes the amount of commodity 1 that can be generated at each point $x$ of the solution, under the condition of preserving both feasibility and the utility level $\overline{v}_i$ of each agent.

Other intrinsic properties of the MDP dynamic process (13)–(14) are rather well known (see, e.g., Malinvaud 1972, Chapter 8, or Drèze and de la Vallée Poussin 1971, Section 1). For the sake of completeness in our comparison, we just restate them here, in the same terms as our previous lemmas and propositions:

**Lemma 3** *For $\delta_i > 0 \; \forall \, i \in N$, the MDP process (13)–(14) is individually monotone.*

**Proof.** See, e.g., Malinvaud (1972, p. 192).

**Proposition 3** *For the MDP process (13)–(14) and for every $x^0 \in \mathring{X}$, there exists a unique solution $x(\cdot) : [0, \infty) \to \mathring{X}$, which is such that* $\lim_{t \to \infty} x(t)$ *exists and is a Pareto efficient allocation.*

**Proof.** Analogous to the proof of Proposition 2.

## 6. Concluding remarks

The preceding developments call naturally for some comparative assessment of the three processes involved. Leaving aside computational issues, such as those relating, for example, to the convergence rate, we shall consider instead two points which are perhaps more directly relevant for the social scientist.

The essential difference between the WMP process and the M70 process lies in the property of *individual* monotonicity of the latter. As our approach has made apparent, this results from a simple, but quite specific, modification of the former, viz. a normalization of the gradient vector of each utility function. The social weights represented by coefficients $\lambda_i$ are in a sense made endogenous, and linked to preference characteristics of each agent.

In the spirit of the usual planning interpretation of the above processes, individual monotonicity may be seen as a desirable property because it enhances the acceptability of the "plan" by the agents of the economy. By the same token, however, it reduces the distributional choices available to the planning authority. Actually, in the M70 process there is no such choice left whatsoever, since the directions of utility increase are unique for each $i$; on the other hand, the MDP process offers more variety from this point of view, thanks to the vector parameters $\delta_i$: by his "neutrality" theorem, Champsaur (1976) has exhibited the range of such choices.

There is however another aspect whereby both the M70 and the MDP processes distinguish themselves from the WMP process, that we might call *operationality*. The economic meaning of any dynamic process is much conditioned by the observability of the variables and parameters that govern its behavior. From this point of view, the marginal utilities whereby the WMP process is defined offers a rather poor basis. On the contrary, the marginal rates of substitution used in the MDP process are independent of any direct measurement of utility: they are theoretically observable, and in practice at least susceptible of being expressed (or "revealed") in unambiguous numerical terms. The MDP process therefore could be considered for implementation. This is also true[9] of the M70 process, in view of (7).

Finally, it should be pointed out that for both the M70 and the MDP processes presented here, their (unique) solutions verify the definition of an *exchange curve* as defined by Smale (1976).

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9. We owe this point to Paul Champsaur.