On the stability of dynamic processes in economic theory*

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Abstract

The notion of stability in the sense of Lyapunov is applied to economic dynamic processes of the Champsaur-Drèze-Henry type.

Our purpose in this note is to fill a small gap in the literature concerning dynamic processes in economic theory, of the type presented by Champsaur et al. (1977). Indeed, these authors do not discuss stability in the sense[1] of Lyapunov (1907). However, a recent result of Maschler and Peleg (1976) on this kind of stability (presented in a discrete model) can easily be applied to both continuous and discrete processes used in economics. We shall present this result for a very general class of such processes and conclude with references to a few economic applications.

For our purpose a (set valued) *dynamic system* is simply a pair $\langle X, \phi \rangle$, where X is a compact subset of R^m and ϕ a correspondence from X to its nonempty subsets. Let T be a subset of $[0, \infty)$ containing 0 and x_0 an element of X. Then a *ϕ-process starting at x_0* is a pair of functions: $x(\cdot) : T \to X, \dot{x}(\cdot) : T \to \mathbb{R}^m$, such that: $x(0) = x_0$ and, $\forall\, t \in T$, $\dot{x}(t) \in \phi(x(t))$. If $T = \{0, 1, 2, \ldots, \}$ and $\dot{x}(t) = x(t{+}1)$ then the process $\langle x(\cdot), \dot{x}(\cdot) \rangle$ is called *discrete*. If $T = [0, \infty)$, if $x(\cdot)$ is absolutely continuous on any interval $[0, \tau]$ in T, and $\dot{x}(t) = dx(t)/dt$ for almost every t in T, then the process $\langle x(\cdot), \dot{x}(\cdot) \rangle$ is called *continuous*.[2] In the first case the dynamic system is said to be discrete and in the second case to be continuous. In both cases, for any $x_0 \in X$, the pair $\langle x(\cdot), x_0 \rangle$ is called a *ϕ-trajectory starting at x_0*.

Given a dynamic system $\langle X, \phi \rangle$, a nonempty subset Q of X is said to be *stable* (in the sense of Lyapunov) *with respect to ϕ*, if the following condition holds: For every neighborhood W of Q there exists a neighborhood V of Q such that if $x_0 \in V$ and $\langle x(\cdot), x_0 \rangle$ is a ϕ-trajectory starting at x_0, then $x(t) \in W$ for every $t \in T$. It is clear that any closed stable set Q is *invariant* in the sense that every trajectory starting in Q remains in Q.

Let $u : \mathbb{R}^m \to \mathbb{R}^n$ be a continuous vector-valued function $u = (u_1, \ldots, u_n)$. The dynamic system $\langle X, \phi \rangle$ is:

Monotone in u if $\forall\, x_0 \in X$, $\forall\, \langle x(\cdot), x_0 \rangle$, a ϕ-trajectory starting at x_0, $\forall\, \{t, t'\} \subset T$, if $t' > t$, then $u(x(t')) \geq u(x(t))$.

Neutral in u if $\forall\, x_0 \in X$, $\forall\, i \in \{1, \ldots, n\}$, $\exists \langle x(\cdot), x_0 \rangle$, a ϕ-trajectory starting at x_0, such that $\forall\, j \neq i$, $j \in \{1, \ldots, n\}$, $\forall\, t \in T$, $u_j(x(t)) = u_j(x_0)$.

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1. As pointed out by Negishi (1962), this is the same as Samuelson's stability of the second kind Samuelson (1947). The term "stability in the sense of Lyapunov" is used by Arrow and Hahn (1971). Heal (1968) and Hori (1975) also use this concept of stability.
2. See Champsaur et al. (1977, Section 5).

Efficient in u if $\forall x_0 \in X$, $\forall\langle x(\cdot), x_0\rangle$, a ϕ-trajectory starting at x_0, $\lim_{t\to\infty} x(t) \neq \emptyset$ and, $\forall \overline{x} \in \lim_{t\to\infty} x(t)$, $\nexists y \in X$ such that $u(y) \gneqq u(\overline{x})$.[3]

A result similar to the following was proved by Maschler and Peleg (1976) for discrete processes. The argument in their proof is easily adapted to continuous processes and is reproduced in the Appendix.

Proposition 1 *If the dynamic process $\langle X, \phi\rangle$ is monotone in a continuous function $u = (u_1, \ldots, u_n)$ from X to $\mathbb{R}^n$, then, for every subset Y of X, the set $Q(Y) = \{x \in X \mid \exists y \in Y$ such that $u(x) \geq u(y)\}$ is stable.*[4]

The condition $Y = Q(Y)$ is sufficient but not necessary for the stability of Y. To see this, consider the continuous dynamic system $\langle X, \phi\rangle$ where $X = [0, 1] \subset \mathbb{R}^1$ and $\phi(x) = 1 - 2x$. This system is monotone in the function $u(x) = x - x^2$ on $[0, 1]$. Then for any $y \in [0, \frac{1}{2}]$, the set $[y, \frac{1}{2}]$ is stable, but $Q(y) = [y, 1 - y]$. Note however that the range of u is the same over the set $[y, \frac{1}{2}]$ and over the set $[y, 1 - y]$; see Figure 1.

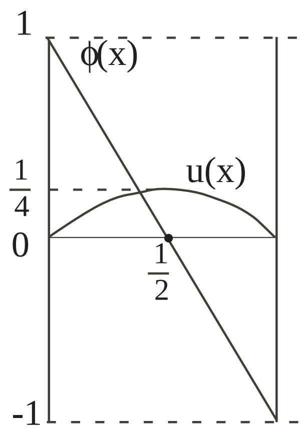

Figure 1

This last property has some degree of generality. Indeed, as proved in the Appendix, we have:

Proposition 2 *If the continuous dynamic system $\langle X, \phi\rangle$ is monotone, neutral and efficient in the continuous function u, then every closed invariant set Y verifies,*[5,6] $u(Y) = u(Q(Y))$.

For discrete dynamic systems, this result does not hold in general as the following example demonstrates. Let X and $u = (u_1, u_2)$ be such that $u(X)$ can be represented as in Figure 2. Let, for any $x \in X$, $\phi(x) = \{y \in X \mid u_1(y) = u_1(x)$ or $u_2(y) = u_2(x)$ and $\nexists z \in X$ such that $u(z) > u(y)\}$. The conditions of Proposition 2 are met, but, for any $x_0 \in X$, $Y = \{x_0\} \cup \phi(x_0)$ is

3. $\lim_{t\to\infty} x(t) =_{\text{def}} \{y \in \mathbb{R}^m \mid \exists\, t_1, \ldots, t_\nu \ldots, \lim_{\nu\to\infty} x(t_\nu) = y\}$. $\forall\, u, v \in \mathbb{R}^n$, $u \gneqq v$ means, $\forall\, i \in \{1, \ldots, n\}$, $u_i \geq v_i$ with strict inequality for at least one i.
4. In other terms, any Q such that, $\forall\, y \in Q$, $\{x \in X \mid u(x) \geq u(y)\} \subset Q$ is stable.
5. For any function u on x and any set $Z \subset X$, $u(Z)$ is the image of Z under u, i.e., $u(Z) = \{u(x) \mid x \in Z\}$.
6. A simple example of a continuous dynamic system which is monotone, neutral and efficient but where some closed invariant subset is not stable, is the following. Let $X = [0, 4] \subset \mathbb{R}^1$ and $\phi(x) = -2[(x-1)(x-3)^2 + (x-3)(x-1)^2]$. This system is monotone and efficient in the function $u(x) = \min\{(3/4), x - (x^2/4)\}$. Indeed $u(x)$ is maximal for any x in the interval $[1, 3]$; also $\phi(x) > 0$ for $x < 1$ and $\phi(x) < 0$ for $x > 3$. Finally it is easy to see that the set $\{2\}$ is invariant and not stable since: $\phi(x) = 0$ for $x = 2$, $\phi(x) < 0$ for $1 < x < 2$, and $\phi(x) > 0$ for $2 < x < 3$.

closed invariant and clearly $u(Y) \neq u(Q(Y))$; see Figure 2 where $u(Y) = \{u(x_0), a, b\}$, $u(Q(Y))$ is the shaded area.

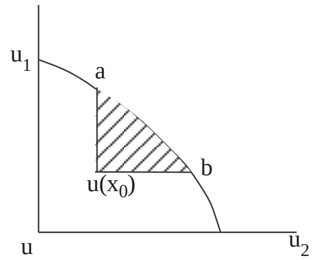

Figure 2

Note however that the conclusion of the proposition holds true for discrete dynamic systems with the same characteristics, if Y is required to be a closed invariant subset of X such that $u(Y)$ is convex.[7]

These propositions apply to several dynamic processes in economic theory. For example, MDP-processes, as introduced in Drèze and de la Vallée Poussin (1971), and Malinvaud (1970–1971), and studied in Champsaur et al. (1977), are monotone and efficient in the utilities u_i of the participating agents. When the sharing rule for distributing the surplus is regarded as part of the process itself, they are also neutral. (Note that if we consider only constant sharing rules, our definition of neutrality is weaker than the one used in Champsaur, 1976). In this context, Proposition 1 implies in particular that if $\overline{x}$ is Pareto optimal, then $Q(\overline{x}) = \{x \in X \mid u(x) = u(\overline{x})\}$ is stable. Under additional assumptions – like strict quasi-concavity of the u_i's and convexity of X–$Q(\overline{x})$ is a singleton and any subset of Pareto optima is stable. Other examples are Edgeworth's process as formulated by Uzawa (see Negishi, 1962) which is monotone in individual utilities and Heal's process (see the references in footnote 1) which is monotone in the objective function of the planning authority.

Appendix

Proof of Proposition 1 Choose $\overline{x} \in X$ and let $Q = Q(\{\overline{x}\})$. Let W be a neighborhood of Q, W^0 an open subset of W such that $W^0 \supset Q$; denote the complement of W^0 by $-W^0$. For every $y \in -W^0$, the continuity of u allows us to construct, for some $i \in \{1, \ldots, n\}$ and some natural number r, the open set

$$W_{i,r} = \left\{ x \in X \mid u_i(x) < u_i(\overline{x}) - \frac{1}{r} \right\},$$

such that $y \in W_{i,r}$. Since $-W^0$ is compact, it has a finite covering $W_{i_1 r_1}, W_{i_2 r_2} \ldots W_{i_k, r_k}$. Also, by the continuity of u, the following set V is a neighborhood of Q (since $-W_{i_k r_k}$ is a neighborhood of Q),

$$V = \bigcap_{k=1}^{K} (-W_{i_k r_k}) = -\left(\bigcup_{k=1}^{K} W_{i_k r_k} \right) \subset W^0 \subset W.$$

But the monotonicity of $\langle X, \phi \rangle$ implies: $\forall x_0 \in V$, $\forall \langle x(\cdot), x_0 \rangle$, a ϕ-trajectory starting at x_0,

$$\forall \{t, t'\} \subset T,\ t' > t, u_{i_k}(x(t')) \geq u_{i_k}(x(t)) \geq u_{i_k}(\overline{x}) - \frac{1}{r_k},\ 1 \leq k \leq K.$$

7. Indeed one can use an argument very similar to the proof of Proposition 2.

Therefore, for every $t \in T$, $x(t) \in V \subset W$. Hence Q is stable. The result follows from noticing that for any $Y \subseteq X$, $Q(Y) = \cup_{y \in Y} Q(\{y\})$, and from the fact that any union of stable sets is stable. ■

Proof of Proposition 2 Let $P = \{x \in X \mid \nexists y \in X \text{ such that } u(y) \gneqq u(x)\}$. Take any closed invariant subset Y of X. Let $Q = Q(Y)$ and assume $u(Q) \setminus u(Y) \neq \emptyset$. Choose $\hat{u} \in u(Q) \setminus u(Y)$ and suppose first that $\hat{u} \notin u(P)$. Since by definition of Q, there is $y_0 \in Y$ such that $\hat{u} \gneqq u(y_0)$ we may take $i \in \{1, \ldots, n\}$ such that $\hat{u}_i > u_i(y_0)$. By assumption there is a ϕ-trajectory $\langle x_1(\cdot), y_0 \rangle$ starting at y_0, converging to P and such that: $\forall\, j \neq i$, $\forall\, t \in [0, \infty)$, $u_j(x_1(t)) = u_j(y_0)$. Also, for any $x \in \lim_{t \to \infty} x_1(t) \subset P$, $u_i(x) > \hat{u}_i$. Hence by the continuity of $u_i \circ x$ on $[0, \infty)$, there is some $t_1 \in (0, \infty)$ such that $u_i(x_1(t_1)) = \hat{u}_i$. In addition, $\forall\, t \in [0, t_1], x_1(t) \in Y$. Indeed if, a contrario, there was some $t' \in (0, t_1]$ such that $x_1(t') \notin Y$, then it would contradict the assumption that Y is invariant. Now, let $y_1 = x_1(t_1)$. If $\hat{u} \gneqq u(y_1)$ we may repeat the procedure by choosing some other $i \in \{1, \ldots, n\}$ such that $\hat{u}_i > u_i(y_1)$ and denoting $\langle x_2(\cdot), y_1 \rangle$ the ϕ-trajectory starting at y_1 such that $u_j(x_2(t)) = u_j(y_1)$, $\forall\, j \neq i$, and such that, for some $t_2 \in (0, \infty)$, we get $u_i(x_2(t_2)) = \hat{u}_i$. Again it is clear that $\forall\, t \in [0, t_2], x_2(t) \in Y$. Letting $y_2 = x_2(t_2)$ we may start again. Eventually at some stage k, we shall get $\hat{u} = u(y_k)$ with $y_k \in Y$, a contradiction to the assumption that $\hat{u} \in u(Q) \setminus u(Y)$.

So suppose now that $\hat{u} \in u(P)$. By the preceding argument we know that there is $y \in Y$ and $i \in \{1, \ldots, n\}$ such that, $\forall\, j \neq i$, $u_j(y) = \hat{u}_j$ and $\hat{u}_i > u_i(y)$. Again there is a ϕ-trajectory $\langle x(\cdot) y \rangle$ starting at y and covering to P such that $\forall\, t \in [0, \infty), \forall\, j \neq i$, $u_j(x(t)) = u_j(y)$ and $x(t) \in Y$. Hence for any $x \in \lim_{t \to \infty} x(t), u(x) = \hat{u}$ and $u(x) \in u(Y)$, since $u(Y)$ is closed. ■

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