Incentive Games with Incomplete Information: An Application to a Public Input Model*

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1. Introduction

The purpose of this communication is to extend some results of d'Aspremont and Gérard-Varet (1975) concerning an incentive game with incomplete information to the public input model of Groves and Loeb (1975). This model involves a group of firms using a public input which is made available to the group by a central agency, knowing only the "revealed" revenue functions of the firms. Groves and Loeb formulate the associated incentive problem in the framework of a non-cooperative n-person game with complete information in which the strategy space of each player (each firm) is the set of plausible revenue functions which he may reveal as his "true" revenue function. The payoff of each player is the profit he realizes after sharing the cost of the public input according to some given rule. The main result of Groves and Loeb (1975) is to exhibit a rule for which the true revenue functions of the players form a Nash Equilibrium. This is the notion of incentive compatibility introduced by Hurwicz (1972). It should be noted however, that this notion is rather demanding since it requires that, a priori, each firm considers each possible revenue function of each other firm as if it was the true function of this other firm. But each firm may have "beliefs" concerning the likelihood of the true revenue function of each other firm. For this reason the problem... is reformulated here in the framework of a game with incomplete information and Harsanyi (1967-68)'s Bayesian Equilibrium concept is substituted to the Nash Equilibrium concept. The main benefit of this new approach is the possibility of finding an incentive compatible rule which allows the central agency to balance its budget in all instances. This last property could not be ensured by Groves and Loeb's approach.

2. The model

Consider a group of n firms (n players) using some public input, the quantity K of which is fixed by a central agency and is, by definition, available to every firm regardless of its use by the others. To each firm i, $i = 1, 2, \ldots, n$, is associated a revenue function $\pi_i(K, \alpha_i)$ of the level K of public input, where α_i is some multidimensional parameter which is assumed to belong to some bounded open subset A_i of some Euclidean m-dimensional space. We make the following assumptions on each function π_i:

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(i) π_i is twice continuously differentiable on $\mathbb{R} \times \mathbb{R}^m$.

(ii) For every $\alpha_i \in A_i$ the second derivative of $\pi_i(\cdot, \alpha_i)$ with respect to K is negative on $\mathbb{R}$.

(iii) For every $\alpha_i \in A_i$, the first derivative of $\pi_i(\cdot, \alpha_i)$ with respect to K, denoted $\pi_i'(\cdot, \alpha_i)$, is such that $\pi_i'(0, \alpha_i) < 0$ and $\lim_{K\to\infty} \pi_i'(K, \alpha_i) \leq 0$.

(iv) For every $K \in \mathbb{R}$ and every $\alpha_i \in A_i$ the vector of partial derivatives

$$\left(\frac{\partial}{\partial \alpha_{i,1}}\pi_i'(K, \alpha_i), \frac{\partial}{\partial \alpha_{i,2}}\pi_i'(K, \alpha_i), \ldots, \frac{\partial}{\partial \alpha_{i,m}}\pi_i'(K, \alpha_i)\right)$$

is defined and nonnull.

The incomplete information is introduced by assuming that each firm i does not know the true parameter of the other firms but has a subjective joint probability distribution μ_i on the space

$$A_{-i} \stackrel{\text{def}}{=} A_1 \times A_2 \times \ldots \times A_{i-1} \times A_{i+1} \times \ldots \times A_n$$

of all possible true values of these parameters. The central agency is supposed to know[1] all these probability distributions.

To each n-tuple $\alpha \in A \stackrel{\text{def}}{=} A_1 \times A_2 \times \ldots \times A_n$ of true parameters we may associate the following game $\Gamma(\alpha)$ in normal form. Each player has a strategy set $A_i(\alpha_i)$ consisting of all the possible parameter values he can "reveal" to the central agency. For every i and every α we assume that $A_i(\alpha) = A_i$. Also, the payoff of each firm is determined following two rules:

Collective efficiency rule: based on the declarations $a \in A$, the central agency chooses a level of public input $K(a)$ which maximizes the total collective profit $[\sum_i \pi_i(a_i, K) - p; K]$ where $p > 0$ is the given price for the public input. By assumptions (i)–(iv), $K(a)$ is uniquely determined and positive for every $a \in A$.

Cost-sharing rule: each firm i is supposed to participate both to the declared revenues of the others and to the cost of the level of input $K(a)$ by adding to its revenue function the following amount:

$$\sum_{j\neq i} \pi_j(a_j, K(a)) - p \cdot K(a) - C_i(a_i, a_{-i}),$$

where C_i is some differentiable real-valued function on $A_i \times A_{-i}$ a priori chosen by the central agency. The vector $C \stackrel{\text{def}}{=} (C_1, C_2, \ldots, C_n)$ will be called a *distribution rule*. Hence for each $\alpha \in A$ and each i, we have the following payoff function:

$$P_i(a; \alpha_i) \stackrel{\text{def}}{=} \pi_i(\alpha_i, K(a)) + \sum_{j\neq i} \pi(a_j(K(a)) - p \cdot K(a) - C_i(a_i, a_{-i}).$$

We have thus defined a non-cooperative game with incomplete information $\Gamma(C)$, which depends on the chosen distribution rule and which is the set of all possible games $\Gamma(\alpha), \alpha \in A$, together with

1. In a moral general framework as studied in d'Aspremont and Gérard-Varet (1975) the subjective probability distributions are themselves parametrized. The firm may then reveal false values of these parameters. However, all our present results are not preserved for this general case.

the n probability distributions μ_i, $i = 1, 2, \ldots, n$. We want now to introduce the notion of Bayesian Equilibrium for this game. For this we need to introduce the normalized strategies of each player and their expected payoff function conditional to their own true parameter. A *normalized strategy* for player i is a function, assumed measurable and denoted a^*, associating to each possible value of his true parameter $\alpha_i \in A_i$ a strategy $a_i(\alpha_i) \in A_i$. Also, for player i and any true parameter $\alpha_i \in A_i$, his *expected payoff conditional on* α_i is:

$$E_i(a_i, a^*_{-i}; \alpha_i) \stackrel{\text{def}}{=} \int_{A_{-i}} P_i(a^*_1(\alpha_1), \ldots, a^*_{i-1}(\alpha_{i-1}), a_i, a^*_{i+1}(\alpha_{i+1}), \ldots, a^*_n(\alpha_n); \alpha_i) d\mu_i(\alpha_{-i})$$

with $a_i \in A_i$ and $a^*_{-i} \stackrel{\text{def}}{=} A^*_1 \times A^*_2 \times \ldots \times A^*_{i-1} \times A^*_{i+1} \times \ldots \times A^*_n$.

By the definition of a^* and the continuity of each P_i on A (by assumptions (i)–(iv)) we see that E_i is a well defined function on $A_i \times A^*_{-i}$.

Finally a n-tuple $\overline{a}^* \in A^* \stackrel{\text{def}}{=} A^*_1 \times A^*_2 \times \ldots \times A^*_n$ of normalized strategies form a *Bayesian Equilibrium* (BE) if and only if $\overline{a}(\alpha_i)$ maximizes $E_i(a_i, a^*_{-i}; \alpha_i)$ for every $\alpha_i \in A_i$ and every i.

3. Distribution rules and their properties

In this section we define a new notion of incentive compatibility as a property of distribution rules, and then characterize the class of distribution rules having this property. A distribution rule C is said to be an *incentive compatible distribution rule* (ICDR) if and only if, in the associated game $\Gamma(C)$, the n-tuple of normalized strategies $\hat{a}^*$ defined by the condition that for every $\alpha_i \in A_i, \hat{a}^*_i(\alpha_i) = \alpha_i$, $i = 1, 2, \ldots, n$, is a BE in $\Gamma(C)$. Our first proposition is a characterization of all ICDR's by the following property.
A distribution rule C is a *subjectively discretionary distribution rule* (SDDR) if for every $a_i \in A_i$ and $a'_i \in A_i$,

$$\int_{A_{-i}} C_i(a_i, \alpha_{-i}) d\mu_i(\alpha_{-i}) = \int_{A_{-i}} C_i(a'_i, \alpha_{-i} d\mu(\alpha_{-i}), i = 1, 2, \ldots, n.$$

The term “subjectively discretionary” reflects the fact that the expected value of C_i must be constant on A_i for every i.

Proposition 1 *The class of all ICDR's coincides with the class of all SDDR's.*

For brevity we skip the proof of this proposition since, with assumptions (i)-(iv), it should be a mere transposition of the proof of Proposition 2 in d'Aspremont and Gérard-Varet (1975). Groves and Loeb (1975) exhibit a smaller class of distribution rules having a stronger incentive compatibility property. However, with any such distribution rule, one cannot in general ensure that the budget of the agency remains balanced. We shall say that a distribution rule C is a *balanced distribution rule* (BDR) if and only if, for every $a \in A$:

$$\sum_i C_i(a_i, a_{-i}) = \sum_i \left[\sum_{j \neq i} \pi_j(a_j, K(a)) - p \cdot K(a) \right] + p \cdot K(a).$$

Finally, we get

Proposition 2 *There exists a distribution rule which is both a SDDR and a BDR.*

Proof First we may define for every i a real-valued function g_i by letting for every $a_i \in A_i$

$$g_i(a_i) \stackrel{\text{def}}{=} \int_{A_{-i}} \left[\sum_k \left\{ \sum_{j \neq k} \pi_j(a_j, K(a)) - p \cdot K(a) \right\} + p \cdot K(a) \right] d\mu_i(a_{-i}).$$

The required distribution rule C may be constructed as follows: for $i = 1, 2, \ldots, n$ and $a \in A$ let $n_i = i + 1 (\text{mod } n)$

$$C_i(a_i, a_{-i}) \stackrel{\text{def}}{=} \frac{1}{n} \left[\sum_k \left\{ \sum_{j \neq k} \pi_j(a_j(K(a)) - p \cdot K(a) \right\} + p \cdot K(a) - g_i(a_i) + g_{n+1-i}(a_{n+1-i}) \right].$$

By an argument given in d'Aspremont and Gérard-Varet (1975), pp. 17–18, it may be shown that each C_i is differentiable on A_i. Also it is easy to verify that such a C is a BDR and that

$$\int_{A_{-i}} C_i(a_i, \alpha_{-i}) d\mu_i(\alpha_{-i}) = \frac{1}{n} \int_{A_{-i}} g_{n+1-i}(\alpha_{n+1-i}) d\mu_i(\alpha_{-i})$$

which is constant in a_i. Hence C is a SDDR and the result follows. ■

Since by Proposition 1, any SDDR is an ICDR, this result means that it is possible to find a distribution rule which is incentive compatible and allows the agency to keep its budget balanced.

References

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