Price-quantity competition with varying toughness*

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Abstract

For an industry producing a single homogeneous good, we define and characterize the concept of oligopolistic equilibrium, allowing for a parameterized continuum of regimes with varying competitive toughness. This parameterization will appear to be equivalent to the one used in the empirical literature. The Cournot and the competitive outcomes coincide, respectively, with the softest and the toughest oligopolistic equilibrium outcome. The concept offers an alternative to the conjectural variations approach with better foundations. It can be viewed as a canonical description of oligopolistic behavior which can receive different theoretical justifications and provide a convenient tool for modeling purposes. Two illustrative cases (linear and isoelastic demands) are developed and the possibility of endogenizing (strategically) the choice of competitive toughness by the firms is examined.

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1 Introduction

A fundamental problem of oligopoly theory is equilibrium indeterminacy. This indeterminacy is not only of the kind associated with a given equilibrium concept and the possibility of multiple solutions. It is mainly related to the choice of the equilibrium concept itself. This is most simply demonstrated in static models by the choice between quantity competition (usually assimilated to the approach of Cournot, 1838) and price competition (linked to Bertrand's 1883 critique of Cournot). However, this second kind of indeterminacy only reflects the variety of observed competition regimes varying in their degree of toughness and implying different firm conducts.

In order to formalize this variety of oligopoly regimes, we shall adopt an approach that was pioneered by Shubik (1959), where firms do not privilege one strategic variable but behave strategically in price-quantity pairs. This approach will lead us to a comprehensive and canonical concept of oligopolistic equilibrium providing a unified formulation of the whole spectrum of enforceable non-cooperative equilibria. The proposed concept remains Cournotian and considers oligopolistic competition as a generalization of monopoly. Two types of competition are at stake: the struggle of the whole industry for market size, and the struggle of each individual firm for its market share. With each type of competition will be associated a constraint. The set of solutions (corresponding to different competition regimes) can be parameterized according to the relative values of the Lagrange multipliers associated with these two constraints. This parameterization can be interpretd in terms of competitive toughness. At one extreme, one finds the Cournot solution as the softest oligopolistic equilibrium, and, at the other extreme, one finds the competitive equilibrium when competitive toughness is maximal. As mentioned by Shubik (1959), the price equilibrium (corresponding to Bertrand competition) is a particular price-quantity equilibrium, which coincides with the competitive equilibrium when all firms are producing at equilibrium. All other enforceable outcomes are intermediate to these two extremes.

This parameterization will appear to be equivalent to the one used in the empirical literature, building econometric models that incorporate general equations where each firm conduct in setting price or quantity is represented by a parameter, itself viewed as an index of competitiveness. This so-called "conduct parameter method" has been at the basis of the *new empirical industrial organization* and has generated a large number of empirical studies (for a synthesis see Bresnahan, 1989, and, for more recent references, Corts, 1999). It is related to the conjectural variations approach since both methods encompass the same theories of oligopoly. But, as stressed by Bresnahan, "the phrase 'conjectural variations' has to be understood in two ways: it means something different in the theoretical literature than the object which has been estimated in the empirical papers" (Bresnahan 1989, p. 1019). Moreover, in the theoretical literature, conjectural variations have been criticized for their lack of theoretical foundations, at least in static models.[1]

The goal of the canonical representation of oligopolistic competition that we propose here is to provide a convenient tool to the theorist (it amounts to use a generalization of Cournot equilibrium), which is more game-theoretically founded than the conjectural variations theory, and which, in parallel to the conduct parameter method used by the econometrician, still nests a continuum of theories of oligopolies. This theoretical tool will be derived and justified from different approaches, such as the "supply-function," the "facilitating practices" or the "min-pricing scheme" approaches.[2] Each of these approaches may be more relevant for specific industries. For example, introducing meeting-the-competition clauses (as a facilitating practice) cane appropriate for retail (e.g. Sears catalogue) and intermediate product markets, or using supply-function models can be fruitful, for example, in studying the electric power generation market.[3] Our canonical representation is meant for all and readily amenable to empirical testing.

In Section 2 we define and characterize the concept of oligopolistic equilibrium for an industry producing a single homogeneous good. The resulting parameterization is derived and compared to the empirical conduct parameterization. In Section 3, we examine some alternative theoretical justifications of the concept. In Section 4, we analyze the two standard cases of a linear and of an isoelastic demand and, in Section 5, we examine the possibility of endogenizing (strategically) the choice of competitive toughness by the firms. We briefly conclude in Section 6.

[1] For references and a discussion, see Martin (2002, pp. 50–51).

[2] See, respectively, Grossman (1981), Salop (1986), and d'Aspremont et al. (1991).

[3] See, for example, Green and Newbery (1992).

2 Oligopolistic equilibrium

We take an industry consisting of n firms ($n > 1$), each firm i producing the same good with a technology described by an increasing cost function C_i, which is continuously differentiable on $(0, \infty)$ and such that $C_i(0) = 0$. The demand D for the good is a function of market price P, with a finite continuous derivative $D'(P) < 0$ over all the domain where it is positive and such that $\lim_{P \to \overline{P}} D(P) = 0$, for some $\overline{P} \in (0, \infty]$. More assumptions on costs and demand will be introduced along the way.

2.1 Definition and characterization

In our approach, neither the price nor the quantity will be privileged as a strategic variable.[4] Each firm i is supposed to choose, simultaneously with its competitors, a price-quantity pair $(p_i, q_i) \in \mathbb{R}_+^2$. We introduce the concept of oligopolistic equilibrium by combining the Cournot and the Bertrand approaches. The following firm conduct can be assumed. As in the standard interpretation of Cournot competition, q_i is the quantity of output that firm i decides to supply. This means that the *production in advance* assumption used by Maskin (1986) holds. However, we will not consider Bertrand-Edgeworth competition but pure Bertrand competition as defined, for instance, in Vives (1999, p. 117): p_i is the *list price* at which firm i is ready to supply *all* demand, whenever this price is lowest. If more than one firm list the lowest price a sharing rule has to be applied to allocate the excess demand. This is given by functions $s_1, s_2, \cdots, s_n$ defined on $\mathbb{R}_+^{2n}$ such that:

$$\begin{array}{lllll} s_i(p,q) > 0 & \text{if } i \in \arg\min\limits_j & \text{and} & D\left(\min\limits_j \{p_j\}\right) > \sum\limits_j q_j, & \\ s_i(p,q) = 0 & \text{otherwise,} & \text{and} & \sum\limits_i s_i(p,q) = D\left(\min\limits_j \{p_j\}\right) - \sum\limits_j q_j. & \end{array} \tag{1}$$

[4]See Shubik (1959) and Shubik and Levitan (1980). Friedman (1988), who investigates differentiated-products oligopoly, mentions a homogeneous-goods duopoly model of Alger (1979). Closest to the following definition is Maskin (1986).

Because of each firm commitment to serve all demand, no consumer will accept to pay more than the lowest price $\min\{p_1, \cdots, p_n\}$. Moreover, because of production in advance, any firm is willing to sell all its output at the *discount price* $P = \min\{p_1, \cdots, p_n, D^{-1}(q_1 + \cdots + q_n)\}$. Therefore firm i payoff can be defined as:

$$\begin{aligned}\Pi_i^{CB}(p_i, p_{-i}, q_i, q_{-i}) \equiv \min\left\{p_i, p_{-i}, D^{-1}\left(q_i + \sum_{j\neq i} q_j\right)\right\}[q_i + s_i(p_i, p_{-i}, q_i, q_{-i})] \\ -C_i(q_i + s_i(p_i, p_{-i}, q_i, q_{-i})).\end{aligned} \tag{2}$$

We thus obtain a *Cournot-Bertrand oligopoly game* in quantities and prices. An *oligopolistic equilibrium* is a $2n$-tuple (p^*, q^*) which is a *Nash equilibrium* of this oligopoly game that satisfies in addition a *credibility condition:* at equilibrium, no firm should be forced to sell more than it would voluntarily sell, i.e. $s(p^*, q^*) = 0$ or, equivalently,

$$\sum_j q_j^* = D\left(\min_j\{p_j^*\}\right). \tag{3}$$

This credibility condition eliminates equilibria where the commitment to serve all demand is binding for some firm. Notice that this definition does not exclude the case where at equilibrium a firm i chooses the quantity $q_i^* = 0$, that is, to be *inactive*.

The following lemma delivers the main tool of this paper. It gives a formal and canonical characterization of the concept of oligopolistic equilibrium which is more convenient to use for modeling purposes and leads, as we will see, to the standard econometric parameterization of the set of equilibria.

Lemma 1 *A 2n-tuple (p^*, q^*) is an oligopolistic equilibrium if and only if, for each firm i, (p_i^*, q_i^*) is solution to the program,*

$$\max_{(p_i,q_i)\in\mathbb{R}_+^2}\left\{p_i q_i - C_i(q_i) : p_i \leq \min_{j\neq i}\{p_j^*\} \text{ and } p_i \leq D^{-1}\left(q_i + \sum_{j\neq i} q_j^*\right)\right\}, \tag{4}$$

and satisfies

$$\sum_j q_j^* = D\left(\min_j\{p_j^*\}\right). \tag{5}$$

Proof: Let us start by proving sufficiency. Suppose, that, for each i, (p_i^*, q_i^*) is solution to (4), satisfying (5), but that, for some i, and some $(p_i, q_i) \in \mathbb{R}_+^2$,

$$\begin{aligned} &\min\left\{p_i, p_{-i}^*, D^{-1}\left(q_i + \sum_{j\neq i} q_j^*\right)\right\}[q_i + s_i(p_i, p_{-i}^*, q_i, q_{-i}^*)] - C_i(q_i + s_i(p_i, p_{-i}^*, q_i, q_{-i}^*)) \\ > \ &\min\left\{p_i^*, p_{-i}^*, D^{-1}\left(q_i^* + \sum_{j\neq i} q_j^*\right)\right\} q_i^* - C_i(q_i^*). \end{aligned}$$

If $D^{-1}(q_i + \sum_{j\neq i} q_j^*) \leq \min\{p_i, p_{-i}^*\}$, then $s_i(p_i, p_{-i}^*, q_i, q_{-i}^*) = 0$ and the pair (p_i', q_i), with $p_i' = D^{-1}(q_i + \sum_{j\neq i} q_j^*)$, satisfies the two constraints of (4), giving higher profit to firm i, which is a contradiction. If $D^{-1}(q_i + \sum_{j\neq i} q_j^*) > \min\{p_i, p_{-i}^*\}$, then the pair (p_i', q_i'), with $p_i' = \min\{p_i, p_{-i}^*\}$ and $q_i' = q_i + s_i(p_i, p_{-i}^*, q_i, q_{-i}^*)$, again satisfies the two constraints of (4), with a higher profit for firm i, leading to a contradiction. It remains to observe that the credibility condition (3) is automatically satisfied since it coincides with (5).

To prove necessity, suppose that, for some i, and some $(p_i, q_i) \in \mathbb{R}_+^2$, s.t. $p_i \leq \min_{j\neq i}\{p_j^*\}$ and $p_i \leq D^{-1}(q_i + \sum_{j\neq i} q_j^*)$, $p_i q_i - C_i(q_i) > P^* q_i^* - C_i(q_i^*)$, with $P^* = \min_j\{p_j^*\} = D^{-1}(\sum_j q_j^*)$, and (p^*, q^*) an oligopolistic equilibrium. WLOG we may suppose that at least one of the two constraints holds as an equality. If $p_i = D^{-1}(q_i + \sum_{j\neq i} q_j^*)$, demand is still served with the price-quantity pair (p_i, q_i), thus generating a profitable deviation in the Cournot-Bertrand oligopoly game. If $p_i = \min_{j\neq i}\{p_j^*\} < D^{-1}(q_i + \sum_{j\neq i} q_j^*)$, any price-quantity pair (p_i', q_i), with $p_i' > p_i$, leads to the payoff$p_i q_i - C_i(q_i)$, since $p_i = \min_{j\neq i}\{p_j^*\} < \min\{p_i', D^{-1}(q_i + /sum_{j\neq i} q_j^*)\}$ and $s(p_i', p_{-i}^*, q_i, q_{-i}^*) = 0$, so that it is a profitable deviation in the oligopoly game. Finally, observe that (5) coincides with the credibility condition (3). ■

From now on, on the basis of this lemma, we shall call *oligopolistic equilibrium any* $2n$-tuple (p^*, q^*) satisfying (5) and such that, for each i, (p_i^*, q_i^*) is solution to (4). It has been derived as an equilibrium of the Cournot-Bertrand oligopoly game, but will receive other justifications below. In fact, it can also receive a direct interpretation. Indeed the profit maximization program (4) includes the two types of competition in which a firm is involved: the one of the whole industry for market size (represented by the aggregate demand constraint of the Cournot kind), and the one of each individual firm for its market share (represented by the constrained imposed by the minimal price).

In order to further justify this characterization of an equilibrium, we start by comparing the set of oligopolistic equilibria with standard oligopolistic outcomes. We let (P^C, q^C) denote a Cournot outcome, that is, a quantity vector q^C satisfying:

$$q_i^C \in \arg \max_{q_i \in [0,\infty)} \left\{ D^{-1} \left(q_i + \sum_{j \neq i} q_j^C \right) q_i - C_i(q_i) \right\}, \quad i = 1, \cdots, n, \tag{6}$$

and a price $P^C = D^{-1}(\sum_j q_j^C)$. Alternatively, a competitive (Walrasian) outcome is denoted (P^W, q^W) and is such that

$$q_i^W \in \arg \max_{q_i \in [0,\infty)} \{P^W q_i - C_i(q_i)\}, \quad i = 1, \cdots, n, \tag{7}$$

with P^W satisfying $\sum_j q_j^W = D(P^W)$. As far as Bertrand competition is concerned, we only consider, for simplicity, the case where firms have constant marginal costs[5]: $C_i(q_i) = c_i q_i$, $i = 1, \cdots, n$. Then, a Bertrand outcome (P^B, q^B) with $P^B = \min_i\{p_i^B\}$ is characterized by

$$p_i^B \in \arg \max_{p_i \in [0,\infty)} (p_i - c_i) d_i(p_i, p_{-i}^B) \tag{8}$$

$$= \frac{D(\min\{p_i, p_{-i}^B\})}{\# \arg \min\{p_i, p_{-i}^B\}}, \quad \text{if } p_i = \min\{p_i, p_{-i}^B\}, \quad \text{with } d_i(p_i, p_{-i}^B)$$

$$= 0, \quad \text{otherwise.} \tag{9}$$

Finally, a collusive outcome (P^m, q^m) corresponds to

$$(P^m, q^m) \in \arg \max_{(P,q) \in \mathbb{R}_+^{n+1}} \left\{ P \sum_i q_i - \sum_i C_i(q_i) : \sum_i q_i \leq D(P) \right\}. \tag{10}$$

[5]We do not exclude the case where constant marginal costs differ from one firm to another. However, in that case, to avoid non-existence, it can be assumed that profits are measured in (small) discrete units of account (e.g. in cents). We then compute the corresponding equilibrium prices and take their limit as the size ε of the unit of account tends to zero. In the duopoly case, for example, taking $c \in (c_1, c_2]$ and a positive integer m_ε to be such that $c_1 \leq \varepsilon m_\varepsilon \leq c \leq \varepsilon(m_\varepsilon + 1)$, the prices $p_1 = \varepsilon m_\varepsilon$ and $p_2 = \varepsilon(m_\varepsilon + 1)$ form an equilibrium (where only firm 1 produces), and varying c we get all "Bertrand equilibrium" outcomes. See, for example, Mas-Colell et al. (1995), Exercise 12.C.4. Another way to get all Bertrand equilibria is to have firm 2 randomizing in a small enough interval above firm 1 price (see Blume, 2003).

We can now state the following:

Proposition 2 *Any Cournot outcome* (P^C, q^C)*, competitive outcome* (P^W, q^W) *and (when marginal costs are constant) any Bertrand outcome* (P^B, q^B) *is an oligopolistic equilibrium outcome, but a collusive outcome* (P^m, q^m) *is not (unless it is also a Cournot outcome).*

Proof: If (P^r, q^r) (with $r = C, W, B$) were not an oligopolistic equilibrium outcome, some i would be able by (4), through some choice $(p_i, q_i) \in \mathbb{R}_+^2$ s.t. $p_i \leq \min\{P^r, D^{-1}(q_i + \sum_{j \neq i} q_j^r)\}$, to get a profit $p_i q_i - C_i(q_i) > P^r q_i^r - C_i(q_i^r)$. This implies $P^C q_i^C - C_i(q_i^C) < D^{-1}(q_i + \sum_{j \neq i} q_j^C) q_i - C_i(q_i)$ in the Cournot case, contradicting the assumption that q^C is a Cournot equilibrium. In the competitive case, we get $P^W q_i^W - C_i(q_i^W) < P^W q_i - C_i(q_i)$, resulting again in a contradiction. An in the Bertrand case, the same argument holds directly if $\min\{p_i, P^B\} = P^B$. Otherwise, with $p_i wPB$, the deviation (p_i, q_i) would a fortiori be feasible in the Bertrand game, where i would be constrained by the aggregate demand (instead of the residual demand, according to the second constraint in (4), that is $q_i \leq D(p_i) - \sum_{j \neq i} q_i^B$). Finally, if (P^m, q^m) is a collusive but not a Cournot outcome, it must be such that for some i, some $q_i \in \mathbb{R}_+$ and $P = D^{-1}(q_i + \sum_{j \neq i} q_j^m)$,

$$
\begin{aligned}
& Pq_i - C_i(q_i) + P\sum_{j \neq i} q_j^m - \sum_{j \neq i} C_j(q_j^m) \\
\leq \; & P^m \sum_j q_j^m - \sum_j C_j(q_j^m) w P q_i - C_i(q_i) + P^m \sum_{j \neq i} q_j^m - \sum_{j \neq i} C_j(q_j^m)
\end{aligned}
$$

implying $P < P^m$. Therefore, (P, q_i) is an admissible deviation for firm i in the oligopoly game, which entails profit $Pq_i - C_i(q_i)$, so that (P^m, q^m) cannot be an oligopolistic equilibrium outcome. ■

Notice that coincidence of Cournot and collusive outcomes, impossible under differentiability of the cost and demand functions (except if there is only one active firm[6], cannot be generally excluded. If, for instance, demand is non-differentiable at price P^*, one may have (under cost

[6] The "collusive outcome" involving several active firms should be distinguished from the "monopoly equilibrium" where one firm is producing the optimal monopoly output and all other firms do not produce. For an analysis of the monopoly equilibrium see Amir and Lambson (2000).

symmetry and $n \geq 2$) the first order conditions:

$$\begin{aligned} 1/n[-\epsilon^+ D(P^*)] &< 1/[-\epsilon^+ D(P^*)] \leq 1 - C'(D(P^*)/n)/P^* \\ &\leq 1/n[-\epsilon^- D(P^*)] < 1/[-\epsilon^- D(P^*)] \end{aligned}$$

with $P^* = P^C = P^m$, and with $\epsilon^- D(P^*)$ and $\epsilon^+ D(P^*)$ the left-hand and right-hand demand elasticities at P^*, respectively.

2.2 Parameterization of equilibria

We shall now investigate the first order condition of the oligopolist's program defined by (4). For each firm i, we have to distinguish the case where, at the solution, it is active ($q_i^* > 0$) and the case where it is not[7] ($q_i^* = 0$). All equilibrium prices, by contrast, should be strictly positive. Introducing Kuhn-Tucker multipliers $(\lambda_i, \nu_i) \in \mathbb{R}_+^2 \setminus \{0\}$ associated with the first and second constraints in (4), respectively, general first order conditions require, by the positivity of p_i^* and the non-negativity of q_i^*, that $q_i^* - \lambda_i - \nu_i = 0$, and $p_i^* - C_i'(q_i^*) + \nu_i / D'(\min_j \{p_j^*\}) \leq 0$ with $(p_i^* - C_i'(q_i^*) + \nu_i / D'(\min_j \{p_j^*\}))q_i^* = 0$. Therefore, if firm i is active, we get

$$\begin{bmatrix} q_i^* \\ p_i^* - C_i'(q_i^*) \end{bmatrix} = \lambda_i \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \nu_i \begin{bmatrix} 1 \\ -1/D'(\min_j \{p_j^*\}) \end{bmatrix}, \tag{11}$$

whereas, if firm i is inactive, we let $\lambda_i = \nu_i = 0$.

The multiplier λ_i is associated with the min-pricing constraint and the multiplier ν_i with the demand constraint. They can be interpreted as the shadow costs firm i would accept to bear in order to ease the pressure coming from its competitors inside and outside the industry, respectively. For an active firm, we define the normalized parameter $\theta_i \equiv \lambda_i / (\lambda_i + \nu_i) \in [0, 1]$ (for an inactive firm i, θ_i is undefined). This parameter may be viewed as an index of the *competitive toughness* displayed by firm i within the industry, at equilibrium (p^*, q^*). The corresponding degree of monopoly of each firm i in the set of the n^* active firms (with $p_i^* = \min_j \{p_j^*\}$ and

[7] Shubik (1959) suggests to call such a firm a "firm)in-being" (by analogy to the famous term "fleet in being," introduced by Lord Torrington and used by Kipling).

$q_i^* > 0$) can then be expressed as a function of θ_i:

$$\frac{p_i^* - C_i'(q_i^*)}{p_i^*} = (1-\theta_i)\frac{q_i^*/\sum_j q_j^*}{-\epsilon D(\min_j\{p_j^*\})} \equiv \mu_i\left(\theta_i, \min_j\{p_j^*\}, q^*\right), \tag{12}$$

where ϵ is the elasticity operator (i.e., $\epsilon f(x) \equiv (df(x)/dx)(x/f(x))$, for a differentiable function $f(x)$).

Oligopolistic equilibria may hence be characterized by first order conditions (12), together with equality (5), giving a system of $n^* + 1$ equations

$$\mu_i(\theta_i, P^*, q^*) = \frac{P^* - C_i'(q_i^*)}{P^*} \quad (i = 1, \cdots, n^*) \text{ and } \sum_j q_j^* = D(P^*) \tag{13}$$

in $n^* + 1$ unknowns (P^*, q^*), parameterized by $\theta = (\theta_1, \cdots, \theta_{n^*})$ in the set $\Theta^* \subset [0,1]^{n^*}$ of the parameter values which entail a solution to the system (13) and ensure sufficiency of the first order conditions. We thus obtain endogenously a parameterization by θ of the set of oligopolistic equilibria.

We shall not examine here the question of existence and the sufficiency of first order conditions. Local curvature requirements should be somewhat softened by the presence of a kink in the boundary of the admissible set. However it should be stressed that *sufficient global conditions for the existence of an oligopolistic equilibrium are necessarily weaker than those ensuring existence of standard solutions, such as Cournot or competitive equilibria*, since these solutions, as we have seen above, are enforceable as particular instances of oligopolistic equilibria. In any particular case the difficulty is to determine the set Θ^*. Illustrative examples are given below.

The following proposition shows that potential oligopolistic equilibria coincide at one extreme, when competitive toughness is minimal ($\theta = (0, \cdots, 0)$), with the Cournot solution and, at the other extreme, when competitive toughness is maximal ($\theta = (1, \cdots, 1)$), with the competitive equilibrium. All other oligopolistic equilibria correspond to intermediate values of θ, including (when marginal costs are constant) those associated with Bertrand competition.

We can now state the following

Proposition 3 *Assuming that $\theta = (0, \cdots, 0)$ (respectively $\theta = (1, \cdots, 1)$) belongs to Θ^*, and that, for any i, $D^{-1}(q_i + \sum_{j \neq i} q_j)q_i - C_i(q_i)$ is quasi-concave (respectively $C_i(q_i)$ is convex) in*

q_i, any oligopolistic equilibrium corresponding to $\theta = (0, \cdots, 0)$ (respectively $\theta = (1, \cdots, 1)$) leads to a Cournot (respectively a competitive) outcome.

Proof: Recall that in the case of Cournot equilibrium the first order conditions (which are sufficient by assumption) are:

$$\frac{P^C - C_i'(q_i^C)}{P^C} = \frac{q_i^C / \sum_j q_j^C}{-\epsilon D(P^C)} \equiv \mu_i(0, P^C, q^C) \text{ for all } i,$$

coinciding, by (12), with the first order conditions for an oligopolistic equilibrium with $\theta = (0, \cdots, 0) \in \Theta^*$. With $\theta = (1, \cdots, 1) \in \Theta^*$, we see from (12) that each firm equalizes marginal cost to price, so that we get the condition (assumed to be sufficient) for a competitive equilibrium. The result follows. ■

In relation to this result, it is important to note that the degree of monopoly equations (12) are in fact the rewriting of the behavioral equations used in the empirical literature to estimate firm and industry conduct (how firms et prices and quantities). These equations are typically parameterized, for each firm, by an "index of the competitiveness of oligopoly conduct" (Bresnahan, 1989, p. 1016) which is the exact complement of our parameter θ_i (i.e. equal to $(1-\theta_i)$). This parameterization is used by econometricians to nest existing theories of oligopoly.

These parameters are also directly related to the *conjectural variation approach* (Bowley, 1924). They can be taken as continuous-valued and used to estimate conjectural derivatives. In spite of its lack of theoretical foundations,[8] the conjectural variation approach is meant to fill the gap left open by the theory. In this approach, each firm i when choosing its quantity is supposed to make a specified type of conjecture concerning the reaction of the other firms to any of its deviations. These conjectures are introduced directly into the first order conditions. They are not part of the description of the oligopoly game. Following the presentation in Dixit (1986), a sufficient specification[9] consists in introducing conjectural derivatives $r_i = \sum_{j \neq i} \partial q_j / \partial q_i$ for each i. The corresponding first order conditions are:

$$\frac{P^* - C_i'(q_i^*)}{P^*} = (1 + r_i) \frac{q_i^* / \sum_j q_j^*}{-\epsilon D(P^*)}, \tag{14}$$

[8] It has also been criticized as leading to a mismeasurement of market power (Corts, 1999).

[9] Dixit considers the more general case where r_i is a function of both q_i and $\sum_{j \neq i} q_j$.

giving the same characterization as (12) with $r_i = -\theta_i$. In other words, comparing first order conditions, the set of oligopolistic equilibrium outcomes appears as the selected subset of outcomes obtained by conjectural variations restricted to the compensating (non-collusive), i.e. r_i to be in the interval[10] $[-1, 0]$, for every i. The concept of oligopolistic equilibrium thus provides some game-theoretic foundation to the concept of conjectural variations, since the conjectural variation terms (within the relevant class) can be identified with the parameterization of the equilibria of a fully specified game.

3 Other game-theoretic approaches

The formal concept of oligopolistic equilibrium has been defined as the characterization of the Nash equilibrium of a Cournot-Bertrand oligopoly game. However, we shall now see that there are other theoretical ways (and these are illustrative of many others) to arrive at the same formal characterization. Each of these different theories might be more adapted to some specific industries and some particular situations. This is why the concept of oligopolistic equilibrium as defined by Lemma 1 has to be seen as a canonical representation and a convenient theoretical tool.

3.1 The min-pricing scheme approach

One can define an alternative oligopoly game where firms are not committed to serve all demand. We suppose instead that each firm includes a *meeting competition clause* (or *price-match guarantee*) in its sales contracts, guaranteeing its customers that they are not paying more than what they would to a competitor, so that each customer acts as if facing the single market price $P^{\min}(p) = \min_j\{p_j\}$, where $P^{\min}$ is *the min-pricing scheme* (d'Aspremont eat al., 1991). Combining this guarantee with the assumption that each firm i brings q_i to the market, we again have that it is willing to sell its output at the discount price $P = \min\{P^{\min}(p), D^{-1}(q_1 + \cdots + q_n)\}$.

[10] Matching variations ($r_i > 0$) are excluded, and in particular those leading to the collusive solution.

We thus get payoff functions given, for firm i, by

$$\Pi_i^{pm}(p_i, p_{-i}, q_i, q_{-i}) \equiv \min \left\{ P^{\min}(p_i, p_{-i}, D^{-1}\left(q_i + \sum_{j \neq i} q_j \right) \right\} q_i - C_i(q_i). \tag{15}$$

We thus get an alternative game, a *price-matching oligopoly game* in prices and quantities, using these payoffs. The corresponding oligopolistic equilibrium (p^*, q^*), a $P^{\min}$-*equilibrium*, is a Nash equilibrium satisfying in addition the *no-rationing restriction*

$$\sum_j q_j^* = D(P^{\min}(p^*)), \tag{16}$$

eliminating equilibria where customers would be willing to buy more at the equilibrium price $P^{\min}(p^*)$.

Proposition 4 *A 2n-tuple (p^*, q^*) is a $P^{\min}$-equilibrium if and only if it is an oligopolistic equilibrium, i.e. for each firm i, (p_i^*, q_i^*) is solution to the program (4) and the market balance restriction (5) holds.*

Proof: Suppose first that (p^*, q^*) is a $P^{\min}$-equilibrium (so that $\sum_j q_j^* = D(\min_j\{p_j^*\})$), but that, for some i, and some $(p_i, q_i) \in \mathbb{R}_+^2$, $p_i q_i - C_i(q_i) > p_i^* q_i^* - C_i(q_i^*)$, with $p_i \leq \min\{p_{-i}^* D^{-1}(q_i + \sum_{j \neq i} q_j^*)\}$. The pair (p_i', q_i), with $p_i' = \min\{p_i, p_{-i}^*, D^{-1}(q_i + \sum_{j \neq i} q_j^*)\}$, is then a deviation w.r.t. the $P^{\min}$-equilibrium, a contradiction.

To prove the other direction, suppose now (p^*, q^*) is an oligopolistic equilibrium (so that again) $\sum_j q_j^* = D(\min_j\{p_j^*\})$), but that, for some i, some $(p_i, q_i) \in \mathbb{R}_+^2$, and $p_i' \equiv \min\{p_i, p_{-i}^*, D^{-1}(q_i + \sum_{j \neq i} q_j^*)\}$, we have $p_i' q_i - C_i(q_i) > p_i^* q_i^* - C_i(q_i^*) \geq 0$. Then (p_i', q_i) satisfies the two constraints in the program (4) and gives higher profit to firm i, again a contradiction. ■

Hence, the min-pricing scheme approach is another way to get oligopolistic equilibria and a relevant one to investigate the large number of markets where the price-match guarantee is offered.

3.2 The supply function approach

Another approach assumes that firms strategies are *supply functions* (Grossman, 1981, and Hart, 1982).[11] A supply function equilibrium is a Nash equilibrium of a game where each firm i strategies are functions S_i associating with every price p_i in $[0, \infty)$ a quantity $q_i = S_i(p_i)$. The functions S_i may be restricted to some admissible set $\mathcal{S}$. For any n-tuple S of supply functions in $\mathcal{S}^n$, the price $\hat{P}(S)$ is said to be well defined if it is non-negative and uniquely solves the equation

$$\sum_{j=1}^{n} S_j\left(P\right) = D\left(P\right). \tag{17}$$

The corresponding payoffs are defined by:

$$\begin{aligned} &\Pi_i^{\mathcal{S}}(S_i, S_{-i} = \hat{P}(S_i, S_{-i})S_i(\hat{P}(S_i, S_{-i})) - C_i(S_i(\hat{P}(S_i, S_{-i}))), && \text{if } \hat{P} \text{ is well defined,} \\ &\Pi_i^{\mathcal{S}}(S_i, S_{-i} = 0, && \text{otherwise.} \end{aligned} \tag{18}$$

Observe that at an equilibrium S^*, for any firm i, maximizing $\Pi_i^{\mathcal{S}}(S_i, S_{-i}^*)$ amounts to select P^* in

$$\arg \max_{P \in \mathbb{R}_+} \{P\, D_i^*(P, S_{-i}^*) - C_i(D_i^*(P, S_{-i}^*))\}, \tag{19}$$

with $D_i^*(P, S_{-i}^*) = \max\{D\left(P\right) - \sum_{j \neq i} S_j^*\left(P\right), 0\}$ the residual demand function. Indeed, firm i could as well choose any supply function S_i for which $S_i\left(P\right) = D_i^*(P, S_{-i}^*)$ has the unique solution P^*.

The multiplicity of supply function equilibria is well known. In order to compare this concept with our own, we shall restrict strategies to the set $\mathcal{S}_+$ of *non-decreasing supply functions.*[12] We then have the following characterization:

Proposition 5 *For any supply function equilibrium $S^* \in \mathcal{S}_+^n$, there is an oligopolistic equilibrium (p^*, q^*) such that $q^* = S^*(\hat{P}(S^*))$ and, for any j, $p_j^* = \hat{P}(S^*)$. Conversely, for any oligopolistic equilibrium (p^*, q^*), there is a supply function equilibrium $S^* \in \mathcal{S}_+^n$ such that $S^*(\min_j\{p_j^*\}) = q^*$.*

[11] See also Singh and Vives (1984).

[12] As Delgado and Moreno (2004) do. However they assume in addition that firms are identical.

Proof: Let $S^* \in \mathcal{S}_+^n$ be a supply function equilibrium. We show that (p^*, q^*) with $q^* = S^*(\hat{P}(S^*))$ and, for any j, $p_j^* = \hat{P}(S^*)$, is an oligopolistic equilibrium. Using, for any i, the fact that the residual demand $D_i^*(P, S_{-i}^*)$ is decreasing in P and that the profit $p_i q_i - C_i(q_i)$ is increasing in p_i we get, by (19), that (p_i^*, q_i^*) maximizes $p_i q_i - C_i(q_i)$ on

$$\Delta_i \equiv \{(p_i, q_i) \in \mathbb{R}_+^2 \mid q_i \leq D_i^*(p_i, S_{-i}^*\}.$$

By Lemma 1, for $(^*, q^*)$ to be an oligopolistic equilibrium, (p_i^*, q_i^*) should maximize $p_i q_i - C_i(q_i)$ on

$$\hat{\Delta}_i \equiv \left\{ (p_i, q_i) \in \mathbb{R}_+^2 \mid p_i \leq \min_{j \neq i}(p_j^*), q_i \leq \max\left\{ D(p_i) - \sum_{j \neq i} q_j^*, 0 \right\} \right\},$$

for every i. Since, by construction, $(p^*, q^*) \in \hat{\Delta}_i$ and $\hat{\Delta}_i \subset \Delta_i$, this is clearly true.

To prove the converse, let us suppose that (p^*, q^*) is an oligopolistic equilibrium. We may construct an associated supply function equilibrium by imposing to every firm i an admissible supply function S_i simply characterized by a price-quantity pair (p_i, q_i), and such that $S_i(P) = q_i$ if $P \leq p_i$, and $S_i(P) = \infty$ otherwise. Clearly, the solution to (19) cannot be larger than $\min_{j \neq i}\{p_j^*\}$, hence any profitable deviation by some firm i from S_i^* must involve a price below $\min_{j \neq i}\{p_j^*\}$ and a quantity below $D(p_i) - \sum_{j \neq i} q_j^*$, and thus constitute a deviation with respect to the oligopolistic equilibrium. The result follows. ■

We see that, with the restrictions imposed on the admissible class of supply functions, the outcomes corresponding to the two sets of equilibrium outcomes coincide, including the Cournot and the competitive outcomes.

To establish more clearly the relation between the two concepts, we may consider the first order condition to firm i program (19) at equilibrium, while restricting to differentiable supply function in SS_+. Denoting $q_i^* = D_i^*(P^*, S_{-i}^*)$, we get:

$$q_i^* + \left(D'(P^*) - \sum_{j \neq i} S_j^{*'}(P^*) \right) (P^* - C_i'(q_i^*)) = 0, \tag{20}$$

which is equivalent to

$$\frac{P^* - C_i'(q_i^*)}{P^*} = \frac{q_i^* / \sum_k q_k^*}{-\epsilon D(P^*) + \sum_{j \neq i}(q_j^* / \sum_k q_k^*) \in S_j^*(P^*)} = \mu_i(\theta_i, P^*, q^*), \tag{21}$$

with μ_i as defined in (12), taking

$$\theta_i = 1 - \frac{-\epsilon D(P^*)}{-\epsilon D(P^*) + \sum_{j \neq i} (q_j^* / \sum_k q_k^*) \in S_j^*(P^*)}. \tag{22}$$

Not surprisingly, the Cournot solution corresponds to an elasticity $\epsilon S_j^*(P^*)$ of the supply functions equal to 0 for all j, and the competitive solution to $\epsilon S_j^*(P^*) = \infty$ for at least two j's. Notice also that varying the elasticities of the supply functions in the relevant class allows us to cover the whole range of admissible values of the θ_i. The competitive toughness of firm i, as measured by θ_i at the oligopolistic equilibrium, is seen to correspond positively to a measure of the "reactivity of the other firms" (with respect to prices) as anticipated by firm i at the supply function equilibrium. Although the elasticity of a supply function chosen by any firm is indifferent from the point of view of the firm itself (for which only the price-quantity pair actually implemented matters), it is crucial in shaping the anticipations of its competitors.

4 Two illustrative cases

In order to illustrate the concept of oligopolistic equilibrium, and visualize the set of equilibrium outcomes as parameterized by $\theta \in \Theta^*$, we refer to two standard cases of homogeneous duopoly, the cases of linear and isoelastic demand with linear cost functions. Since technological lead is an important feature in many applied models (for example those analyzing the relationship between innovation and competition), he will admit asymmetric costs and assume that each firm i has a constant marginal cost c_i (by convention, $c_1 \leq c_2$).

4.1 Linear demand

In the first case, we suppose demand $D(P)$ to be linear, equal to $a - P$, so that Marshallian elasticity is $P/D(P)$. We assume $ac_2 > c_2 - c_1$, so that the monopoly price $P^M = (a + c_1)/2$ that would be set by firm 1 is larger than the cost c_2 of firm 2. The case where only the efficient firm is active at equilibrium is simple. The set of equilibria is then described by all prices between the competitive price $P^W = c_1$ and the highest Bertrand price[13] $P^B = c_2$. When both firms

[13] See footnote 6.

are active, from condition (12), equilibrium (p^*, q^*) with $p_1^* = p_2^* = P^*$ is seen to verify for each firm i

$$\frac{P^* - c_i}{P^*} = (1 - \theta_i)\frac{q_i^*}{q_1^* + q_2^*}\frac{D(P^*)}{P^*} = (1 - \theta_i)\frac{q_i^*}{P^*}, \tag{23}$$

giving a market share for firm 1

$$\frac{q_1^*}{q_1^* + q_2^*} = \frac{1}{1 - \theta_1}\frac{P^* - c_1}{a - P^*} = 1 - \frac{1}{1 - \theta_2}P^* - c_2 a - P^*, \tag{24}$$

with

$$P^* = \frac{(1 - \theta_1)(1 - \theta_2)a + (1 - \theta_2)c_1 + (1 - \theta_1)c_2}{(1 - \theta_1)(1 - \theta_2) + (1 - \theta_2) + (1 - \theta_1)} \tag{25}$$

The equilibrium price P^* is lower bounded by the highest Bertrand price $P^B = c_2$, and upper bounded by the Cournot price $P^C = (a+c_1+c_2)/3$, corresponding to $\theta_2 = 1$ and to $\theta_1 = \theta_2 = 0$, respectively. These expressions are valid only for values of θ in $[0, 1 - (c_2 - c_1)/(a - c_2)] \times [0, 1]$. Indeed, for values of $\theta_1 > 1 - (c_2 - c_1)/(a - c_2)$, firm 2 becomes inactive and we are back to the first case.

To illustrate, we can take the values for the costs $c_1 = 2/3$ and $c_2 = 1$ and, choosing $a = 10/3$, so that $P^* \in [1, 5/3]$, we obtain the following representation of the set of equilibrium outcomes with the two firms active, in the market price-market share space $(P^*, q_1^*/(q_1^* + q_2^*))$. This set is given by the region bounded by the vertical axis and the two thick curves, the lower one corresponding to $\theta_1 = 0$, the upper one to $\theta_2 = 0$. The increasing curves (including the thick one) are three elements of the family of curves stemming from firm 1 first order condition (with $\theta_1 = 1/3, 1/6, 0$, respectively, from left to right). Similarly, the decreasing concave curves are three elements, corresponding to the same values of θ_2, of the family of curves stemming from firm 2 first order condition.

The decreasing convex gray curve in Figure 1 corresponds to the uniform case $\theta_1 = \theta_2 = \overline{\theta}$. It links the Cournot outcome (determined at the point of intersection of the two thick curves, with $\overline{\theta} = 0$) to the Bertrand outcome (associated with the highest possible market share for the efficient firm and the highest value of $\overline{\theta}$, $1 - (c_2 - c_1)/(a - c_2) = 6/7$). We observe that, as $\overline{\theta}$ increases, the market expands and splits more and more asymmetrically in favor of the efficient firm.

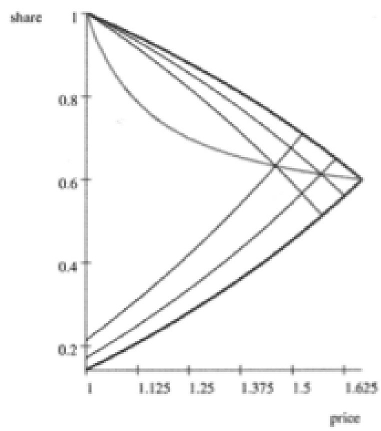

Figure 1

4.2 Isoelastic demand

The demand function is now taken to be $D(P) = P^{-\sigma}$, with $1/\sigma \in (1 - c_1/c_2, 2)$, so that existence of a Cournot equilibrium is ensured, and firm 1 monopoly price $P^M = c_1/(1 - 1/\sigma)$ is larger than the cost c_2 of firm 2. From condition (12) we see that, at equilibrium (p^*, q^*) with $p_1^* = p_2^* = P^*$, the degree of monopoly of firm i is

$$\frac{P^* - c_i}{P^*} = \frac{1 - \theta_i}{\sigma} \frac{q_i^*}{q_1^* + q_2^*}, \tag{26}$$

giving the market share form firm 1

$$\frac{q_1^*}{q_1^* + q_2^*} = \frac{\sigma}{1 - \theta_1} \frac{P^* - c_1}{P^*} = 1 - \frac{\sigma}{1 - \theta_2} \frac{P^* - c_2}{P^*}, \tag{27}$$

with

$$P^* = \frac{(1 - \theta_2)c_1 + (1 - \theta_1)c_2}{1 - \theta_2) + (1 - \theta_1) - (1 - \theta_1)(1 - \theta_2)/\sigma}. \tag{28}$$

The equilibrium price P^* is lower bounded by the highest Bertrand price $P^B = c_2$, and upper bounded by the Cournot price $P^C = (c_1 + c_2)/(2 - 1/\sigma)$. It is decreasing in each θ_i. These expressions are valid only for values of θ in $[0, 1 - \sigma(1 - c_1/c_2)] \times [0, 1]$. For values of θ_1 in $[1 - \sigma(1 - c_1/c_2), 1]$, firm 2 becomes inactive.

Figure 2 shows, for parameter values $\sigma = 1, c_1 = 2/3$ and $c_2 = 1$, the set of equilibrium outcomes with both firms active, in the market price-market share space $(P^*, q_1^*/(q_1^* + q_2^*))$ as the region bounded by the vertical axis and the two thick curves. The solid curves are associated with constant values of competitive toughness, corresponding to the same values of θ_i above: 1/3, 1/6, 0, respectively, from left to right. These curves are increasing for firm 1 and decreasing for firm 2. The gray curve is associated with varying equal degrees of competitive toughness for both firms. Figure 2 is strikingly similar to Figure 1, and leads to the same comments.

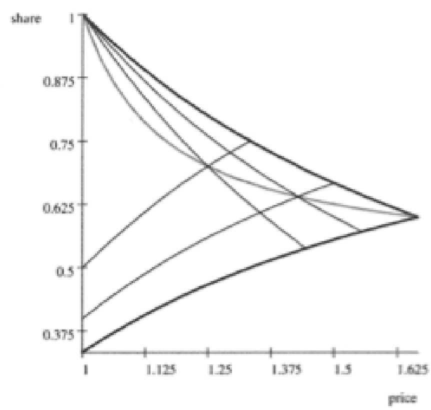

Figure 2

Some significant properties concerning the effects of competitive toughness on profits can be deduced from both examples. In the uniform case ($\theta_1 = \theta_2 = \overline{\theta}$):[14]

- the inefficient firm looses as competitive toughness increases (both its market share and the price decrease);
- if the cost advantage of the efficient firm is large enough, its profit increases as competitive toughness increases (a higher market share more than compensates a lower price);

[14] These properties correspond to properties (b), (c) and (d) in Definition 2.2 of Boone (2001), where the "intensity of competition" parameter is supposed to be the same for all firms. Cournot and Bertrand regimes, but no the intermediate cases, are considered in an example. In Boone (2000), the parameter measuring "competitive pressure" is differentiated but his examples concentrate on comparative statics when the fundamentals vary.

- if costs are equal for both firms (implying equal market shares), profits decrease as competitive toughness increases.

In the case where competitive toughness is asymmetric, effects are more intricate. If the value of θ_i is ket unchanged and θ_j, $j \neq i$, increases, the price as well as firm i market share fall, implying a lower profit for firm i. However, an increase in the value of θ_i while keeping θ_j, $j \neq i$, constant, is associated with a larger market share for firm i but with a lower price, so that an increase in its profit is not guaranteed.

5 Competitive toughness under firm control

In the three justifications of an oligopolistic equilibrium, which have been given and are based respectively on the Cournot-Bertrand, the min-pricing and the supply-function oligopoly games, the parameters measuring the competitive toughness of firms were part of the characterization (through first order conditions) of the set of potential equilibria. For these theories, competitive toughness is an endogenous parameter out of the direct control of individual firms. Apart from an empirical estimation, and in order to get a theoretical specification of these parameters, either a more complete description of the context or an equilibrium selection argument is needed. Examples of such possibilities are respectively given by Kreps and Scheinkman (1983) two-stage game, with capacity recommitment followed by price competition in a symmetric duopoly, and Delgado and Moreno (2004), who look for a coalition-proof supply function equilibrium. In both cases the Cournot outcome is obtained, corresponding to $\theta \equiv 0$ in the present canonical model. But one cannot exclude the possibility of oligopoly games where the toughness parameters can be exogenously fixed and could even be under firm control. In support of this claim, we give two different illustrations.

5.1 Exogenous competitive parameters

Let us consider two very different interpretations of the competitive toughness parameters, taken as exogenous. In one interpretation more competition results from a more socially-oriented attitude, in the other it results from a more aggressive conduct.

5.1.1 Game 1

The first interpretation is based on a modified oligopoly game where each firm i, simply choosing a quantity as in the Cournot model, has a modified objective function where the parameter $\theta_i \in [0, 1]$ explicitly appears and can be interpreted as a *coefficient of collective concern* giving some positive weight to the total surplus. This coefficient is similar to the "coefficient of cooperation" used in the literature (e.g. Cyert and De Groot, 1973, Martin, 2002) and inspired by Edgeworth's "coefficient of effective sympathy." But here it applies not only to the sum of the other firms profits but also to the consumers' surplus. The profit of firm i is defined as

$$\begin{aligned}\Pi_i^{cc}(q_i, q_{-i}, \theta_i) &= (1-\theta_i)\left[q_i D^{-1}\left(q_i + \sum_{j\neq i} q_j\right) - C_i(q_i)\right] \\ &+ \theta_i\left[\int_0^{q_i+\sum_{j\neq i} q_j} D^{-1}(Q)\,d\,Q - \sum_j C_j(q_j)\right].\end{aligned}$$

When $\theta_i = 0, \Pi_i^{cc}(q_i, q_{-i}, 0)$ reduces to the Cournot profit function. When every $\theta_i = 1$, $\Pi_i^{cc}(q_i, q_{-i}, 1)$ amounts to the total surplus (total profit plus consumer surplus). Considering an equilibrium q^* of this modified game, the first order conditions can be computed for n^* active firms $i = 1, 2, \cdots, n^*$:

$$0 = \frac{\partial \Pi_i^{cc}(q^*, \theta_i)}{\partial q_i} = D^{-1}\left(\sum_j q_j^*\right)\left[1 - \frac{(1-\theta_i)q_i^*/\sum_j q_j^*}{-\epsilon D(D^{-1}(\sum_j q_j^*))}\right] - C_i'(q^* - i),$$

and are equivalent to the system defined by (12) and (13), leading to the same set of potential solutions.

5.1.2 Game 2

To introduce the second interpretation, we define a game of "tempered" Bertrand-Edgeworth competition. To simplify, we suppose two firms with linear costs, $C_i(q_i) = c_i q_i$, with $c_i > 0$ ($i \in \{1, 2\}$). The parameter $\theta_i \in [0, 1)$, fixed in advance, is interpreted as the probability that firm i will have an *aggressive* rather than a *compromising* conduct. We suppose that the game has two stages and we look for a sub game perfect equilibrium. At the first stage, each firm i quotes the maximum price $p_i \in (0, \overline{P})$ at which it commits to sell up to quantity $q_i \in [0, \infty)$, to be produced in advance. At the second stage, the conduct of each firm i is decided according to the probability θ_i. Under a *compromising conduct*, adopted with positive probability $1 - \theta_i$, firm i supplies at quoted price p_i a quantity $x_i \in [0, q_i]$ which should not exceed the residual demand, namely $\max[D(p_i) - q_j, 0]$. By contrast, under an *aggressive conduct*, adopted with probability θ_i, it undercuts its rival by an arbitrarily low amount $\varepsilon > 0$ (which is taken as an exogenous parameter), and supplies at price $\psi_i = \min\{p_i, p_j - \varepsilon\}$ a quantity $x_i \in [0, q_i]$ no larger than what it can actually sell: $D(\psi_i)$ with a compromising competitor (or with an aggressive competitor if $\psi_i < \psi_j$ and $\max[D(\psi_i - q_j, 0]$ otherwise. Clearly, since supplying x_i entails no cost, one of these bounds will always be binding at second stage equilibrium. Also, we assume that, in case of indifference, a firm always chooses to produce the maximal quantity it can sell. However, an equilibrium may require the consumers tone rationed and we assume "efficient rationing."

Now consider the expected profit of firm i at the first stage, anticipating the second-stage equilibria, and involving the four states of the world (with two aggressive or two compromising firms, or with one aggressive and the other compromising) and their corresponding probabilities. Take, as a first case, the expected profit Π_i^{BE} of firm i as a function of $(p, q, \theta) \in (0, \overline{P})^2 \times [0, \infty)^2 \times [0, 1)^2$ when $p_i \leq p_j$,

$$\begin{aligned} \Pi_i^{BE}(p, q, \theta) &= (1 - \theta_i) p_i \min\{q_i, \max[D(p_i) - q_j, 0]\} \\ &+ \theta_i (1 - \theta_j) \psi_i \min\{q_i, D(\psi_i)\} \\ &+ \theta_i \theta_j \psi_i \min\{q_i, \max[D(\psi_i) - q_j, 0]\} - c_i q_i. \end{aligned} \tag{29}$$

Notice that the first term in this expression, corresponding to the case i is compromising, involves only the probability $(1 - \theta_i)$ since i's payoff is the same whether j is aggressive or compromising. When $p_i > p_j$ (implying $\psi_i = p_j - \varepsilon < \psi_j$) the expression for the expected profit of firm i becomes independent of probability θ_j:

$$\begin{aligned} \Pi_i^{BE}(p, q, \theta) &= (1 - \theta_i) p_i \min\{q_i, \max[D(p_i) - q_j, 0]\} \\ &+ \theta_i (p_j - \varepsilon) \min\{q_i, D(p_j - \varepsilon)\} - c_i q_i. \end{aligned} \tag{30}$$

Observe that, for $\theta_1 = \theta_2 = 0$, the program of each firm becomes:

$$\max_{(p_i,q_i)} p_i q_i - c_i q_i \text{ s.t. } p_i \leq D^{-1}(q_i + q_j),$$

so that we get again the Cournot solution. For other values of θ we have the following fact:

Proposition 6 *At an equilibrium $(p^*, q^*) \in (0, \overline{P})^2 \times (0, \infty)^2$ of the tempered Bertrand-Edgeworth duopoly, with $\theta \in [0,1)^2$ and positive supplies of both firms in all states occurring with positive probabilities, the following conditions necessarily hold:*

(i) $p_1^* = p_2^* = D^{-1}(q_1^* + q_2^*)$ and, for $i = 1, 2$,

(ii) $q_i^* \in \arg\max_{q_i \in [0, D(p_j^* - \varepsilon) - q_j^*]} \pi(q_i, \theta_i)$, with

$$\pi_i(q_i, \theta_i) \equiv [(1-\theta_i)D^{-1}(q_i + q_j^*) + \theta_i(p_j^* - \varepsilon) - c_i]q_i.$$

Proof: Under condition (i), necessity of condition (ii) is easily proved. Suppose that $\pi_i(q_i, \theta_i) > \pi(q_i^*, \theta_i)$ for some $q_i \in [0, D(p_j^* - \varepsilon) - q_j^*]$. Then the pair (p_i, q_i) with $p_i = D^1(q_i + q_j^*) \in [p_j^* - \varepsilon, \overline{P})$ is a profitable deviation for firm i, since $\Pi_i^{BE}(p_i, p_j^*, q_i, q_j^*, \theta) = \pi_i(q_i, \theta_i) > \pi_i(q_i^*, \theta_i) = \Pi_i^{BE}(p^*, q^*, \theta)$ by Eqs. (29) and (30).

It remains to establish necessity of condition (i). Let us begin by considering the case of an equilibrium (p^*, q^*) such that $p_1^* = p_2^* = P^*$, and show that $q_1^* = q_2^* = D(P^*)$ in this case. If $q_1^* + q_2^* < D(P^*)$, then by Eq. (29),

$$\Pi_i^{BE}(p^*, q^*, \theta) = [(1-\theta_i)P^* + \theta_i(P^* - \varepsilon) - c_i]q_i^*, \;\; i = 1, 2,$$

and we see from Eq. (30) that some firm i would increase its profit by (slightly) increasing its price. If $q_1^* + q_2^* > D(P^*)$ then, again by Eq. (29), $\theta_i(P^* - \varepsilon) \geq c_i$ and $P^* \in \arg\max_{p_i}\{p_i[D(p_i) - q_j^*]\}$ for $i = 1, 2$. An increase by firm i of its price to p_i larger than but arbitrarily close to P^* would only have a second order effect on the revenue component $p_i[D(p_i) - q_j^*]$ at its maximum $p_i^* = P^*$, but would determine a discontinuous positive increase on the complementary revenue component (due to the shift from (29) to (30)), through a jump from $(P^* - \varepsilon)[(1 -$

$\theta_j)\min\{q_i^*, D(P^*-\varepsilon)\} + \theta_j \min\{q_i^*, D(P^*-\varepsilon) - q_j^*\}]$ to $(P^*-\varepsilon)\min\{q_i^*, D(P^*-\varepsilon)\}$. Thus, $q_1^* + q_2^* = D(P^*)$ if $p_1^* = p^* - 2 = P^*$.

Now consider an equilibrium (p^*, q^*) such that, say, $p_i^* < p_j^*$. The are two possible cases.

Case 1. If $\theta_j(p_i^* - \varepsilon) < c_j$, then $q_j^* = D(p_j^*) - q_i^*$ by Eq. (30), since $q_j^* < D(p_j^*) - q_i^*$ would trigger an upward price deviation by firm j, and $q_j^* < D(p_j^*) - q_i^*$ a downward quantity deviation by the same firm. But then $p_i^* < p_j^* = D^{-1}(q_i^* + q_j^*)$ implies $q_i^* < D(p_i^*) - q_j^*$ and (by Eq. (29)) firm i would gain by increasing its price. Case 1 is thus excluded.

Case 2. If $\theta_j(p_i^* - \varepsilon) \geq c_j$, firm j optimal choice should satisfy $q_j^* = D(p_i^* - \varepsilon) > D(\psi_i^*)$.[15] Thus $D(p_i^*) - q_j^* \leq D(\psi_i^*) - q_j^* < 0$ and, by Eq. (29), firm i supplies $x_i^* = 0$ in at least one state with positive probability (both firms compromising with probability $(1-\theta_i)(1-\theta_j) > 0$), contradicting the assumption of the proposition and leading to exclusion of case 2.

Therefore, in both cases we get a contradiction, and condition (i) follows. ■

By computing the first order condition for maximization of $\pi_i(q_i, \theta_i)$ at an equilibrium (p^*, q^*) with $\theta \in [0,1)^2$ and positive supplies of both firms in all states that occur with positive probabilities, we obtain, using $p_1^* = p_2^* = P^* = D^{-1}(q_1^* + q_2^*)$ and $q_i^* \in (0, D(P^*-\varepsilon) - q_j^*)$, for $i = 1, 2$,

$$\frac{P^* - c_i}{P^*} = (1-\theta_i)\frac{q_i^*/(q_1^* + q_2^*)}{-\epsilon D(P^*)} + \theta_i \frac{\varepsilon}{P^*}. \tag{31}$$

As ε tends to zero, this condition eventually coincides with condition(12), implying the same set of (potential) equilibria. The case where $\theta_i = 1$ for some i can be treated as limit cases.

5.2 Choosing θ strategically

Although the interpretation of θ is very different in the two examples, it is exogenously fixed in both and the set of resulting potential equilibrium (with both firms active) are the same. Moreover, in both examples, one can consider the situation where each firm i would choose its θ_i strategically in a preliminary stage. We shall not investigate extensively this question here,

[15] In case $\theta_j(p_i^* - \varepsilon) = c_j$, firm j is indifferent between any $q_j \imath [D(p_j^*) - q_i^*, D(p_i^* - \varepsilon)]$, but is supposed to choose $D(p_i^* - \varepsilon)$.

but we may at least show what answer we get in the two illustrative cases of linear and isoelastic demands.

In both case, with linear costs, the profit of firm i is equal to the product of the degree of monopoly, the market share and the aggregate expenditure level. When increasing its competitive toughness, a firm increases its market share and may or may not increase the aggregate expenditure, but anyway must endure a loss in its degree of monopoly. In the linear demand case, using (23) and (24), the equilibrium profit of firm i combines these three components as follows:

$$\Pi_i(P^*, P^*, q_i^*, q_j^*) = (1-\theta_i)\left(\frac{q_i^*}{q_1^* + q_2^*}\right)^2 (a - P^*)^2 = \frac{(P^* - c_i)^2}{1-\theta_i}. \tag{32}$$

The equilibrium profit of firm i is thus an increasing function of θ_i directly, and a decreasing function of θ_i through the price P^*. From (25), it can be computed to be

$$\Pi_i^*(\theta) = (1-\theta_i)\left(\frac{(c_j - c_i) + (1-\theta_j)(a-c_i)}{(1-\theta_i) + (1-\theta_j) + (1-\theta_i)(1-\theta_j)}\right)^2, \tag{33}$$

for $\theta \in [0, 1-(c_2 - c_1)/(a - c_2)] \times [0,1]$ (with $c_2 - c_1 < a - c_2$). Thus, at the first stage, the payoffs are given by $(\Pi_1^*(\theta), \Pi_2^*(\theta))$ and the strategy spaces are $[0, 1-(c_2-c_1)/(a-c_2)]$ and $[0,1]$ respectively for firm 1 and firm 2. In order to determine the best reply of firm i in θ_i, it is useful to derive the sign of the partial derivative of $\Pi_i^*(\theta_i, \theta_j)$ with respect to θ_i:

$$\text{sign}\{\theta_i \Pi_i^*(\theta)\} = \text{sign}\{1 - (2-\theta_j)\theta_i\}. \tag{34}$$

From this formula, the quasi-concavity of $\Pi_i^*(\theta_i, \theta_j)$ in θ_i is straightforwardly established since the sign can only switch from positive (at $\theta_i = 0$) to negative. Also, when the two firms have equal unit costs, the sign can only be zero for both firms if $\theta_i = \theta_j = 1$, which leads to the Bertrand equilibrium at the second stage. Otherwise, the efficient firm 1 will choose at equilibrium the lowest competitive toughness that eliminates its rival, namely $\theta_1 = 1-(c_2-c_1)/(a-c_2)$, leading to the price $P^B = c_2$. Therefore, in the linear demand case, the only oligopolistic equilibrium outcome, resulting from the sub-game perfect equilibrium of the two-stage game, is the Bertrand outcome with the highest possible equilibrium price.[16]

[16] If individual competitive toughness is interpreted as a degree of aggressively, then this conclusion is similar

In the isoelastic demand case, combining the degree of monopoly (26), the market share (27) and the aggregate expenditure $P^{1-\sigma}$ gives the following expression for the profit function at equilibrium:

$$\Pi_i(P^*, P^*, q_i^*, q_j^*) = \frac{1-\theta_i}{\sigma}\left(\frac{q_i^*}{q_1^*+q_2^*}\right)^2 P^{*1-\sigma} = \frac{\sigma}{1-\theta_i}\left(\frac{P^*-c_i}{P^*}\right)^2 P^{*1-\sigma}. \tag{35}$$

From (28), the equilibrium profit of firm i can be written as a function of θ:

$$\begin{aligned} \Pi_i^*(\theta &= \sigma(1-\theta_i)\left(\frac{c_j - c_i + (1-\theta_j)c_i/\sigma}{(1-\theta_i)c_j + (1-\theta_j)c_i}\right)^2 \\ &\times \left(\frac{(1-\theta_i)c_j + (1-\theta_j)c_i}{(1-\theta_i)+(1-\theta_j)-(1-\theta_i)(1-\theta_j)/\sigma}\right)^{1-\sigma}, \end{aligned} \tag{36}$$

for $\theta \in [0, 1-\sigma(1-c_1/c_2)] \times [0,1]$ (with $c_1 \leq c_2$ and $1/\sigma \in (1-c_1/c_2, 2)$). This again determines both the payoffs and the strategy spaces of both firms. As above, in order to analyze the best reply of firm i in θ_i, it is useful to evaluate the sign of the partial derivative of $\Pi_i^*(\theta_i, \theta_j)$ with respect to θ_i, $\text{sign}\{\partial_i \Pi_i^*(\theta)\}$, which can be reduced to the sign of the following expression

$$\begin{aligned} &(\sigma-1)(1-\theta_i)(1-\theta_j)[c_j - (1-(1-\theta_j)/\sigma)c_i] \\ +\ &[(1-\theta_i)+(1-\theta_j)-(1-\theta_i)(1-\theta_j)/\sigma][(1-\theta_i)c_j - (1-\theta_j)c_i]. \end{aligned} \tag{37}$$

This sign is clearly positive for any $\theta_i \in [0,1)$ if $\theta_j = 1$. So taking $\theta = (1-\sigma(1-c_1/c_2), 1)$ together with the Bertrand equilibrium outcome (with price $P^B = c_2$ and firm 2 inactive when $c_1 < c_2$) describes a sub-game perfect equilibrium of the two-stage game for any σ (with $1/\sigma \in (1-c_1/c_2, 2)$).

But other sub-game perfect equilibria exist for some values of the parameters. If $\sigma = 1$, there exists a sub-game perfect equilibrium, with both firms active in the associated sub-game, and with equilibrium strategies $\theta \in [0, c_1/c_2) \times [0,]$ such that $(1-\theta_1)/(1-\theta_2) = c_1/c_2$, so that $\partial_i \Pi_i^*(\theta) = 0$, for $i = 1, 2$ (because of the upper bound imposed on $\theta_1, c_1/c_2 > 1/2$ is required). In the symmetric case (i.e. $c_1/c_2 = 1$) for instance, $\theta = 0$ together with the Cournot equilibrium outcome is a sub-game perfect equilibrium of the two stage game.

to the one obtained by Boone (2004) showing that a more aggressive outcome (here, the elimination of the less efficient firm) should be expected when firms differ in their efficiency levels.

If $\sigma \neq 1$, from the two first order conditions for an interior maximum $\partial_1 \Pi_1^*(\theta) = \partial_2 \Pi_2^*(\theta) = 0$, we get

$$c_j - (1 - (1 - \theta_j)/\sigma)c_i = -c_i + (1 - (1 - \theta_i)/\sigma)c_j,$$

Leading to

$$(1 - \theta_j)c_i = -(1 - \theta_i)c_j,$$

which is only possible for $\theta = (1, 1)$. But $\theta_1 = 1$ is outside firm 1 admissible strategy space except when the two firms have identical costs (a case already treated). This excludes equilibria with interior θ. Taking $\theta^* = (0, 0)$ we can however easily check that $\partial_i \Pi_i^*(\theta^*) \leq 0$, for $i = 1, 2$, under the condition that $c_1/c_2 \geq (1 + \sigma - 1/\sigma)/\sigma$, hence leading to the Cournot equilibrium outcome. The condition is satisfied for any cost configuration if $1 + \sigma - 1/\sigma \leq 0$ (that is, $\sigma \leq (\sqrt{5} - 1)/2 = 0.618$). Otherwise, the condition forbids a too large efficiency gap between the two firms, and the more so the closer σ becomes to one.

The existence of a subgame perfect equilibrium leading to the Cournot outcome under enough complementarity (a low enough σ) is easily explained by the fact that, in that case, among the three terms in the firm payoff – the market share, the degree of monopoly and the aggregate expenditure level – the last two are now decreasing in the firm competitive toughness (and the more so the higher the level of complementarity).

6 Conclusion

In this paper we have seen that a great variety of oligopolistic regimes can be analyzed in a single static model where all regimes that are potentially enforceable as non-cooperative equilibria can be parameterized in terms of individual competitive toughness. Oligopolistic situations can vary in many respects: their sectoral characteristics, the efficiency distribution of firms, the norms of conduct in the industry, the timing of decisions, the consumer rationing scheme, the contracting possibilities, etc. The specification of all these features may generate a complex model in which the competition regime is well determined. An example already mentioned is Kreps and Scheinkman (1983). But our claim is that, in many oligopolistic situations, the equilibrium analysis can be performed in a single reduced canonical model. This canonical model then provides a convenient tool to investigate some relevant issues for competition and economic policy.[17] Moreover it is ready-made for empirical testing, since its parameterization happens to be isomorphic to the one used in econometric models adopting the "conduct parameter method."

Of course, many issues remain to be treated. The extension of the concept to differentiated good industries has already been undertaken (d'Aspremont and Dos Santos Ferreira, 2005, and d'Aspremont et al, 2007). Furthermore, the existence problem requires more development, in particular to explore the role of the kink in the producer's feasibility frontier in relieving curvature conditions. In studying the existence problem though, there is now a crucial change in focus, since the main question is not the existence of equilibrium in a particular regime but the determination of the set of enforceable competition regimes, a set that changes with the oligopolistic situation. The study of the variation of this set in relation to the context is a new and important issue on the agenda.

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[17] See, for example, d'Aspremont et al. (2004) where this approach is used to study the relationship between competitive toughness and the incentives to R& D-investment and growth.

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