The Mathematical Continuum: From Intuition to Logic*

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> *.... the conceptual world of mathematics is so foreign to what the intuitive continuum presents to us that the demand for coincidence between the two must be dismissed as absurd. Nevertheless, those abstract schemata which supply us with mathematics must also underlie the exact science of domains of objects in which continua play a role.*
> Hermann Weyl, Das Continuum, 1918.

It is a difficult task for a mathematician to talk at the end of a Husserl colloquium that has been so strongly oriented towards philosophy. The richness and relevance of Husserl's work in so many domains between philosophy and scientific knowledge make it difficult to find time, among the varied and interesting themes of this colloquium, to discuss the problems of the foundations of mathematics — one of Husserl's main preoccupations. But Dag Follesdal, when discussing Kurt Gödel's and Hermann Weyl's conceptions of the continuum, did mention two important aspects of Husserl's influence on the thinking of mathematicians.

I will approach the discussion of the continuum from the perspective of trying to obtain a foundation for 'mathematical knowledge' as part of our way of interpreting and reconstructing the world, and not just as a 'purely logical', (meta)mathematical investigation of Mathematics. Nonetheless, some references to technical work in Pure Mathematics and Mathematical Logic will be inevitable.

The starting point for this article are comments about the continuum made by Hermann Weyl in the book Das Kontinuum [Weyl, 1918]. Weyl, a mathematician of great stature, was strongly influenced by Husserl in his numerous foundational and philosophical reflections. In particular the 'phenomenology' of the continuum is at the heart of the most interesting, and modern, observations in [Weyl,1918]. Other important references for these notes will be the articles by René Thom, Jean Petitot and Jacques Bouveresse in the book Le Labyrinthe du Continu, as well as the reflections of Wittgenstein (in different places, to be cited in the text) and in [Chatelet, 1993].

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1 The Intuition

Our intuition about the continuum is built from common or stable elements, from invariants which emerge from a plurality of acts of experience: the perception of time, of movement, of a line extended, of a trace of a pencil...

Time. Weyl considers "time as a fundamental continuum", and the "phenomenal time" of Husserl and Bergson as "conscious experience" of the present which coexists with "memory of the instant gone". Its measure is based on the comparison of temporal segments [Weyl, 1918; p.109–111]. St. Augustine in his Confessions, which Weyl unfortunately does not cite, has the same point of view. Time is a primary notion, independent of movement. The measuring of time happens in the *memory*, because on remembering we compare the temporal segment of a short syllable, "which is not there anymore", to a long one [St. Augustine, 401; Lib. XI][1]. Contrary to Aristotle's opinion in Physics, for St. Augustine movement is not a primary notion: it is time that permits us to describe it as velocity (in modern terms, as a function of space and time).

Many times St. Augustine and Weyl describe the intuition of time as a continuous flux: "river ... experience in transformation". For Weyl phenomenological time is a duration without points, made out of parts that link together, that superimpose over each other, because "this is now, but meanwhile now is no more" [Weyl, 1918; p. 111].

Movement. We can 'see' the continuum in the movement of an object. For Aristotle time presupposes movement: the regular movement of celestial bodies, of the Sun around the Earth, gives us the measure and even the concept of time: the continuity of movement describes that of time. Galileo, Newton and Einstein construct the same hierarchy: from the movement of bodies to time. The continuum which we derive reminds us again of flux, the passage from the power to the act: but the direct vision of movement has no need to appeal to memory. Weyl then proposes an interesting distinction: the continuous line which is there, the "tracks of the tramway" (an image also dear to René Thom [Thom, 1990]) and the curve, a potential path, "which a pedestrian walks on... the trajectory of a point in movement". When this point "finds itself in a determined position, it coincides with a determined point of the plane, without being itself this point of the plane". "In movement, the continuum of points on a trajectory recovers in a continuous monotone fashion the continuum of instants" [Weyl, 1918; ch.II par. 8]. But this is merely a simple superposition: for Weyl the temporal continuum does not have points, the instants are merely "transitions", the present is only possible due to the simultaneous perception of the past and of the future.

The String extended. A thread, a string extended (another of Thom's images), is another experience of the continuum. The chemist or the physicist will tell us that this line is composed of fibers, of molecules. But, if we observe it and discuss it from the point of view of its continuity, it is its macroscopic reality which informs and contributes to our intuition of the continuum. The string extended cannot have jumps nor holes: its tension cannot support the lack of a small part, not even a point.

The Pencil on a sheet. This is the most common experience of the continuum: no one entertains discourse or conscious reflection of the continuum before having drawn lines on pieces of paper thousands of times. The experience is neat: a set of black points transforms the curve into a line, in the sense of Weyl. The points are collected in the trace, which makes their individuality disappear. These points become evident again, as isolated points, when two lines cross each other.

Cauchy in his first demonstration of the Theorem of the Mean Value (see par.2 below) does not go further than the intuition of the continuum that comes from strings and curves traced by a pencil and their crossings.

Viewing the traces of pencil over paper suggests from where our intellectual experience of points — isolated and without dimensions — could have come: from the crossing of two lines. The points are not part of our intuition of the continuum, clearly at least not from the temporal continuum, as Weyl tells us, but also not part of the spatial continuum, as Wittgenstein explains. For Wittgenstein, a curve is a *law*, it is not made out of points; "the intersection point of two lines is not the common element of two classes of points, but the *intersection of two laws*" [Wittgenstein,1964; quoted by Bouveresse, 1992]. If the line or the curve of the movement has only one dimension, that given by the law that describes it, then we are forced to conceive of a point, as the crossing of two lines, as devoid of dimension. This is also suggested by two extended lines that just touch each other, by pencil traces that cross each other on a paper sheet. *The point without dimension is a conceptual construction, a necessary consequence of a line as a one dimensional law.* It is a posterior construction, specific to Set Theory, which 'puts together' the points to reconstruct the line. From this construction comes the set-theoretical inversion of priorities – the continuum as a set of points — an inversion rejected, for different reasons, by Weyl, Wittgenstein and Thom.

2 The Mathematics

One of the most important theorems about the mathematical continuum is intuitively obvious: if on a plan a continuous line has one of its extremities in one side of a right line and the other on the other side of the same right line, then the continuous line cuts through the right line.

<u>Theorem of the Mean Value</u> *If the function $f(x)$ is continuous with respect to the variable x between a and b, and if we call c an intermediary value between $f(a)$ and $f(b)$, then we can alway satisfy the equation $f(x) = c$, for at least one value of x between a and b.*

<u>Proof</u> (Cauchy, 1821) It is enough to see that the curve which has equation $y = f(x)$ will meet one or more times the line $y = c$, inside the interval between a and b; now, it is evident that this will be what will happen when the hypotheses are met. QED

This proof is not a proof. It is not that the reasoning is faulty, it is the definitions that are missing: Cauchy does not have (yet) a rigourous notion of continuity, nor of a curve (Weierstrass). He appeals to the evidence of threads and traces of pencil. Fortunately the theorem in Analysis is true, we can demonstrate it rigourously. Poinsot, in a course in the Ecole Polytechnique in 1815, believed he had demonstrated, in a similar fashion, that every continuous curve is differentiable everywhere, on the left or on the right. The counterexample is well-known[2].

Actually, at the beginning of the XIX century, the 'intuition' about the continuum in Mathematics needed to be made precise. The Ether of Physics was also in the scientific spirit of everyone, with the homogeneity of a perfect continuum. There was a choice to be made: in one side Leibnitz infinitesimals, on the other the limits, the continuity in terms of 'for all ϵ, there exists a δ' (Cauchy, Weierstrass).

What is the invariant, the stable among the many experiences of the world that refer to the continum? Certainly a invariance of scale: all the little bits of time, of a line, even of a string ... keep the same properties that of a longer one (with the perception of continuity of an extended string, we don't see the atoms). In general, the magnifying glass does not change our intuition of the continuum. Or more formally all homothethies preserve the structure of the continuum. Then, the absence of jumps and of holes: no stop to jump further (the jumps), no abyss in which Zeno's arrows can be lost (the lacunas or absence of individual points).

There we have the formidable invention of Cantor and Dedekind. It will make people forget Leibnitz's ideas until the invention of Nonstandard Analysis, a century later, because of its conceptual simplicity, its precision, its constructivity. Take the totally ordered set of the integers, **N**: 0 the rationals Q as fractions of integers. The set Q is also totally ordered and has already some interesting properties for the continuum : it is in effect a dense order (between any two rationals, there is always a third rational), hence invariant by homothethy and without jumps. But Q has an uncountable number of holes or lacunas. Add on all the limits, in the sense of Cauchy, or, what turns out to be the same, define a real number as the set of rationals that are smaller than itself (a Dedekind cut). This is the set-theoretical construction of Cantor-Dedekind which is the standard formalisation of the continuum, that of the real line R of Analysis. It satisfies the invariance of scale, it has no jumps or lacunas. A curve in space will be continuous, if it is described by a law, which does not introduce jumps nor lacunas and is parametrised by this line[3].

2.1 The Impredicative Definition of the Real Line

There is still a problem with the construction we have sketched: if a real number is the limit of all the rationals that precede it, we are using and we are preparing the ground for 'circular definitions'.

Firstly, there is always an infinity of (positive) rationals smaller than whichever (positive) real: hence we need to use, when defining it, the collection N of all the integers, in its totality. And the classical definition of this totality has the following structure: N is the smallest set that contains zero and which if it contains n it contains $n + 1$. Said in a different way, N is the intersection of all sets that contain zero and that are closed under the successor operation. But N has also this property: to define it using the phrase 'all the sets that..' we quantify over a collection that contains N itself. The *defeniens* uses the *defeniendum*.

Secondly, once the real line has been constructed, whenever we define, for example, least upper bounds or greatest lower bounds, we do it once again using the quantifications which can make reference at what is being defined (the collection of the upper bounds or of the lower bounds includes the 'definiendum', the smallest or greatest bound, which is also an upper or a lower bound).

Poincaré and Weyl, who were well aware of these problems in Analysis, gave a rigouous definition of 'impredicative notion' in mathematics [4]. Poincaré observes that these definitions are not always contradictory, but they always present the dangers of circularity. Lebesgue, in 1902, built the General Theory of Integration over an essentially impredicative definition (the Lebesgue measure). The question was hotly discussed at the beginning of the century, in particular under the impulse of Russell.

We will hint to the consequences of this discussion in Weyl's books.

Does this circularity separate the Cantor-Dedekind construction and hence Analysis, from the 'intuition of the continuum'? Clearly not. Already in Aristotle we notice a circularity in the discourse on the continuum: the continuum is presented as one "totality already formed, which, on its own, gives meaning to its components " [Panza,1992][5].

The same way the present time of St. Augustine and Weyl is circular: none of its parts (past, present, future) has meaning without mutual reference to each other; time itself *is* the simultaneous perception of the past, the present and of the future. The present time that it is not there anymore, it is past, or that it is not there yet, it is future, and that we only understand when inserted in the whole of time or within a segment of time. The same is true about the continuity of the string or the line, which is not conceived of points, but globally, or at least through a 'enlarged locality'. *The impredicativity of Analysis proposes a possible formalisation of this intuitive circularity, in particular of phenomenological time; it is one of its expressive richness, another point of contact between intuition and mathematics.*

This way the division between time and Mathematical Analysis, which disturbs Weyl (the absence of points in the phenomenological time in comparison with the points which form the real line) is in part, but only in part, reduced: the real points can be, a posteriori, isolated, but their definition and their Analysis, à la Cantor-Dedekind, requires 'a global look' at the continuum, the same way the intuition of the present requires that of the past and of the future. In <u>Das Kontinuum</u> Weyl is worried, as most mathematicians at the beginning of the century, about the necessity of rigour in the mathematical definitions: too many paradoxes have disrupted the foundational work, the definitions tinged with circularity are suspected. For this reason, he tries a novel approach, which avoids impredicativity, as it is based on a predicative approach of Mathematical Analysis. This attempt will not affect his concrete work in Mathematics (see [Chand.,1987]) nor his futher foundational reflections (see the next footnote). Weyl is probably missing, in that temporary restriction of his mathematical tools, a common element between Analysis and the intuition of (temporal) continuum, of which he particularly cares. However, given his mathematical talent, the few pages he sketched on this point will be considered a paradigm by other logicians that, later on, will continue to prefer the stratified certitudes to the expressive circularities of impredicativity (see [Feferman,1988]). But the challenge of his book is primarily his insatisfaction with the mathematical analysis of the temporal continuum and in his critique of the artificial unity of the space-time, a very important (and very criticable) acquisition of the Mathematical Physics of his time[6]. Time, due to its irreversibility, to the nature of its continuum, is very different from space, as many thinkers, from St. Augustin to Weyl, have made the effort to tell us.

3 Between Intuition and Mathematics

Cantor and Dedekind have proposed a precise mathematical formalisation of the intuitive continuum, with at least three points of contact with our intuitive demands: the invariance of scale, the absence of jumps and of holes. This formalisation is based on very clear 'construction principles' : the sequence of natural numbers, quotients, limits of convergent sequences. Because iteration gives us the integers (we will come back to this point) and quotients give us the rationals; a convergence criterium for a sequence given by a rule gives us a *method to construct* the reals. A convergence criterium for a sequence, even if the sequence is not known a priori, indicates, without ambiguity, by retracing the interval, what we define as 'the real limit' of this sequence[7].

The theoretical import of this construction is massive and its conceptual force rekindles our vision of the world. Because *not only Mathematics and its structures, it is our knowledge that is not stratified.* Once a language and a expressive geometry intervene with the description of the world, they enrich it with forms, which acquire an objective autonomy. This is the basis and the result of the intersubjectivity, it emerges from the world, it is full of history and because of this, it is not absolute nor arbitrary. But above all *this language, this geometry will influence our original intuition*, for a dynamic game is then played. This game goes from our intuitions to their formalisations and when it returns to the the intuition, it modifies it. A 'classical' mathematician does not see a trace of a pencil, without seeing the continuum of $\underline{\mathrm{R}}$, which parametrizes the trace as a curve. He will talk about the continuity of this trace, of space, of time, of movement, directly in terms of his analytical language. Also the trace over the sheet, the comtemplation of movement are instruments for his own reflection, 'eyes for the mind' for the construction that he is trying to master, Analysis. And, before any proofs, he starts to use his intuition over the mental spaces of Analysis and Geometry, trying to understand them as he understands the string, as if they were realities of the same level[8]. From this comes the usual platonic ontology of most mathematicians. It is a formidable help to formulating conjectures and even proofs: Cauchy has 'seen' the right Theorem of the Mean Value. René Thom also has 'lived' for a long time amongst the continuous and differentiable varieties. His deep immersion into this conceptual space, his mathematical genius, have allowed him to 'see', first, and classify the singularities (the catastrophes), an exceptional mathematical (and cognitive) performance. For him, as for many mathematicians of the continuum, "the Continuum precedes ontologically the discrete", for the latter is merely an "accident coming out of the continuum background","a broken line" ... "the archetypical continuum is a space that has the property of a perfect qualitative homogeneity", hence it gives us a vision, more than a logico-mathematical construction [Thom, 1992]. Actually Thom goes further "any demonstration is a revelation of a novel structure, where the elements solidify the intuition and where the reasoning reconstructs the progressive genesis" [Thom,1990;p.560]. An intuition, non emergent from the world, but observation of the universe of Mathematics where the "form of existence is without doubt different from the concrete and material existence of the world, but nevertheless subtly and deeply linked to the objective existence." For this reason "the mathematician must have the courage of his inner convictions; he will affirm that the a mathematical structures have an existence independent of the mind that has conceived them; ... the platonist hypothesis ... is ...the most natural and philosophically the most economical" [Thom,1990;p.560]. Dana Scott more prudent said to this author: "it does no harm".

The advantages of the platonic hypothesis in the 'linguistic synthesis' for the everyday communication amongst mathematicians are enormous, due to the efficacy of the objective signification that it can give to the language and to the crucial 'scribbles in the blackboard'. But the foundational and philosophical drawbacks that it entails are also very important, for all transcendent ontology disguises the historical and cognitive process, the project of intellectual construction, of which Mathematics is rich, and in particular the 'proof principles' and the 'construction principles' which are at the basis of its nature.

3.1 Other Constructions of the Continuum

Discussing the continuum we have tried to describe how the mathematical intuition is built in our relation with the world, by "these acts of experience ... within which we live as human beings" [Weyl,1918;p113]. On the basis of these life experiences, we propose descriptions and deductions, we make wagers, not arbitrary, but full of history and of intersubjectivity, of invariance within a plurality of experiences. Those wagers, organised in mathematical theories, are our linguistic (Algebra, Analysis) and spatial formalisations (Geometry). The 'transcendental objectivity' (in the Husserlian sense) but not transcendent, which emerges by these intellectual constructions and which modifies itself and enriches itself in history, will give (mathematical) forms to the world: forms that are not 'already there' and which will also modify and enrich our original intuitions.

These proposals, these constructions, which aim to an objectivity not absolute anymore, but strong, full of intellectual and cognitive paths, of theorems, of intersubjective communication, are not unique. In the case of the continuum, Leibniz had proposed another construction, in an way too incomplete to resist the very robust construction of Cauchy, Weierstrass, Cantor and Dedekind. It was necessary to wait for the Mathematical Logic of this century, so that an alternative proposal became a new Mathematical Analysis, Non-Standard Analysis[9]. The non-standard analyst describes the continuum differently: despite a number of conservative extension results for the new theories with respect to Standard Analysis (they prove the same theorems, within the standard fragment of the language), his real numbers like 'halos of integers' are a different thing altogether and it is possible to demonstrate new theorems. The ordered set of non-standard numbers, the new real line, loses, for example, the invariance of scale (Hartong), one of the strong invariants of our different views of the continuum, see [Barreau&Hartong,1989]. The non-standard analysist hence view the geometrical space, the physical world in effect, in a different way; this change of theory and of intuition of the mathematical continuum seems to offer new insights in Mathematical Physics (see [Cutland,1988] and the articles by Lobry, Lutz and Reeb in [Salanskis&Sinanceur,1992]; [Salanskis,1991] proposes an epistemological analysis of the non-standard continuum).

Thom himself does not believe that the standard analysis, at the heart of his work, gives a definitive representation of the continuum: "it seems to me premature to impose to the continuum a structure as rigid as R as an additive group, and I would prefer to give it more flexibility" [Thom,1992]. Dissatisfied by the arithmetical (and logical) generativity of the (non-)standard continuum, exactly like Weyl was by the treatment of the "continuous flux" and of the phenomenal time as a set of real points, he will suggest new ideas, trying "to identify the reals as the numbers in foliation rotation over the torus". One obtains this way "more interesting classes of irrationals" and one can study the "mysteries" of the point and of the "perfect continuum", of which "we can only say that it is an unpronounceable mystique" [Thom,1992]. His mathematical audacity sketches here a new conceptual construction, which goes beyond the invariants that have guided the continuum conceived by Leibniz, Cantor-Dedekind or the one from non-standard analysis. This conception is built from his mathematical work experience, which is comparable, for this author, for its force and its evidence, to the experience of the world.

But the intuition that is constructed in the praxis of Mathematics is different from that which emerges directly from our relation with the physical world, even if they do get mixed up in our 'working mathematicians' minds. The first one, in what concerns standard analysis for example, is based on the Cantor-Dedekind construction and the work derived from that in more than a century. If Cauchy in his 'proof' of the Theorem of the Mean Value had made reference to well-defined notions of curve and continuity, if he could have appealed to the rigorous mathematical intuition of the standard reals, built over the correct definitions given some decades later, then his proof would have been a proof. He would have used the 'informal rigor' of the practice of mathematics. In a somewhat different understanding of this notion from Kreisel's, *the informal rigor is based on observations 'from above and from a distance' of definitions and constructions that we know to be potentially rigorous and then by the development of an informal deduction*: the rigor stays more in the precision of the notions than of the deductions. This method is so typical of work in Mathematics, so much based on 'intuition', because *it is built on the history and the practice of Mathematics.* This mathematical intuition, and the informal rigor which is grounded on it, is not the one of the 'man in the street' (even less the one of the paleolithical man): all the training in Mathematics, from the student to the researcher, is to acquire this informal rigor, difficult balance between intuition and formal rigor, which permits a demonstration and its comprehensible expression.

The identification of these two kinds of intuition, the one of the trained mathematician and the other developed only in everyday life, into one single 'pure intuition', is the origin of the difficulties to developing a cognitive analysis, not purely psychological, not purely logical of Mathematics. For the analysis of mathematical intuition, which is not given, which is not an absolute, but it is built in the interplay of acts of experience, language, design and formalization, is actually part of the analysis of Mathematics as a form of knowledge. Moreover, the confusion between different levels or kinds of intuition, from the common sense one to the one in the experienced mathematician, beyond history, gives a comparable or identical level to the objectivity from the physical world and to the objectivity from Mathematics: in both cases the intuition of evidence will be the same, as well as the one of invariants and stabilities. One intuition 'pure and unique' forces us to believe in the unicity of the theory possible; it makes difficult a comparative analysis of different theories, or of wagers of representation, which are proposed to treat mathematically the world and our intuitions of it and which are full of history and of questioning, as the intuition of the continuum.

4 From Mathematics to Logic

Take the subsets, the parts, of the set N of the integers, $P(N)$. If two subsets A and B are strictly included into each other they differ by a finite or infinite subset, but, we would say, in 'a discrete way', by successive jumps: it is integers, well separated ones, that A is lacking to get to B. I hope the reader can 'see' this in his head, using his mathematical intuition. But, this is not really the case: $P(N)$ contains chains (totally ordered sets) with the same type of order as R (i.e. the kind of order of the continuum: dense, without jumps or holes). The proof is easy: Q is countable, choose a bijective enumeration of Q by N and associate to each real number the integers which enumerate its Dedekind cut. Then you have a bijection (an order isomorphism) between R and a chain inside $P(N)$. Our construction principles have given us very rich structures, R and $P(N)$, so rich that they escape the intuitive naïve observation. Actually these structures do not exist: the property that we just 'saw' is not there, it is not *explained* as we explain a property of the world, *we have demonstrated it*, as we have built these structures, as conceptual constructions. The well-trained analyst can short-circuit this proof and see immediately the continuous chain, for the Dedekind cuts are as concrete for him as this table (to paraphrase Gödel). In any case, to construct the chain in $P(N)$, we have made some 'choices'. We have presented Q as a set of pairs (fractions) of the integers N. Each rational corresponds actually to an infinity of equivalent fractions; hence to give a bijective enumeration of Q we must enumerate $N \times N$ (easy) and *choose* a representant for each set of equivalent fractions. This choice is effective, for these equivalence classes are decidable – and the Theory of Recursively Enumerable Sets (and Recursive Functions) realises the Axiom of Choice[10]. This axiom, this principle, is a construction, or allows a construction, that of the "set of choices", composed of one element for each set in the collection considered (see the note). Hence it is a construction principle for using a specific mathematical structure, it allows the construction of new structures. But it is also a principle of proof: once presented 'in abstracto' (that is, at formal level, with no intended domain of interpretation, as if it held for all collection of sets, without any hypothesis on decidability nor on order that allowed the choice of the 'first element' of each set) it becomes a purely mathematical or logical stake: further than the finite (or decidable or ordered) it completely cuts itself off from the practices of life and it acquires a level of abstraction that makes it independent of the 'poor', 'without structure' formalisations of mathematics (the *formal set theories*, see paragraph 5). Nonetheless the trained mathematician uses it everyday, without fear of error, knowing without knowing that he's using a powerful proof principle, which only the specific structure of certain constructions makes applicable. And he confuses his cognitive performance, the *vision* of the conceptual structures of his daily intellectual practices, with a mystical ontology.

Let us try again: cardinality is in first approximation the number of elements of a set. Cantor has shown, by a simple diagonal construction that R has more elements than N, the integers. The reader clearly sees the real line and the 'integer points' well isolated and regularly spaced. The rationals Q are dense and hence give an approximation for each real number. But they are as numerous as the integers. Is it true that if a subset of R is larger than N or Q then it has necessarily the cardinality of R? What does say the observation about this object universe? What says the pure intuition? Nothing. But still the reals are there, God at least must know them all, with their subsets.

To answer these questions it is necessary to make precise the 'frame of the set theoretical construction', to make precise our 'basic principles': if we consider the reals inside the universe of construtibles of Gödel we say yes, if we consider the reals inside the set theoretical universe of Cohen we say no. We do not know which framework God prefers. The question, from Cantor to Frege, Gödel, Cohen and D.Scott has been a key issue in Mathematical logic: it is the challenge of the Continuum Hypothesis (HC). We will refer again to it in section 5.2; but before that we must discuss 'iterations' and 'horizons'.

5 Construction Principles and Proof Principles

One of the 'theoretical situations' that gives 'certainty' or 'structural solidity' in the work of mathematicians and logicians is the joining of different methods, which converge to the same construction. When very different ideas with technical and cultural origins very much apart can be translated into each other, possibly up to isomorphisms, we are sure that we have in our hands a significant construction. For these connections, with different degrees of proximity, sometimes just embeddings without isomorphisms, are to be found in all interesting domains of Mathematics. That is the unity of Mathematics: these bridges, these translations, this to-and-from, these intellectual percourses through rough tracks, sometimes in parallel, which may arrive by shortcuts to well-known valleys. The audacious explorers (constructors!) will be rejoined by others, which proposed totally different paths, with (sometimes) independent goals.

The relationship between Intuitionistic Logic and Theory of Categories (by means of the Theory of Types) gives one of the more interesting and elegant examples of this kind of correspondence. A few remarks on this subject will allow us to clarify the notions of 'proof principle and construction principle', to mention a categorical semantics of impredicative definitions and of the notion of 'variation', which are the heart of the analysis of the continuum.

After that, we will briefly go back to the Axiom of Choice and the Continuum Hypothesis as logical axioms and as mathematical constructions.

5.1 Conjunction, Quantification and Products

In Intuitionistic Logic[11] we say to have a 'proof' of a conjunction $A \wedge B$ (in a unique, canonical way), if we have a proof of A, a proof of B *and* the possibility of reconstructing from a proof of $A \wedge B$ a proof of A and one of B (we have projections). Let A and B be two sets, two spaces, any two mathematical structures: what we have just defined is simply the cartesian product $A \times B$ of A and B with its projections, which associate to each element, or proof of $A \times B$, one element or proof of A and one of B. More precisely it is the Category Theory version of the cartesian product, the product invented by the geometers, which thanks to its categorical generalisation gives us also the product of two topological spaces, two partial orders, of any two mathematical structures ... in their categories (of structures). We have already gone to constructions, having started with proofs: $A \times B$ is the *categorical* (in fact geometric) *semantics* (interpretation) of the intuitionist conjunction $A \wedge B$.

In Mathematics, in Algebra, in Geometry when we have a construction, we usually have another, its dual, for free. Category Theory says that it is enough to reverse all the arrows, that is the direction of all morphisms or functions between objects, to obtain the dual of a given construction. In the case of the product, we reverse the direction of the projections. This way we obtain the categorical coproduct, which can be constructed in several categories. This corresponds to the notion of intuitionistic disjunction: the famous intuitionistic disjunction $A \vee B$, of which we have a proof if and only if we have a proof of A or a proof of B and we know of which one the proof is. In particular, for this notion of disjunction, $A \vee \neg A$ (A or not A) is not demonstrable: to prove it, it is necessary to have a proof of A or a proof of $\neg A$, hence the 'excluded third option' is not valid. More formally, write $\underline{S} \vdash C$ to mean '$\underline{S}$ demonstrates C'; then, in full generality,

$$S \vdash A \vee B \text{ if and only if } S \vdash A \text{ or } A \vdash B$$

and hence the theoretic 'or' ($\vee$) corresponds to the metatheoretic 'or'. In a classical system this beautiful intuitionist symmetry theory/metatheory is lost, for the implication from left to right is false. Hence this intuitionistic 'or' is not so odd: it is simply the dual of a very familiar geometric construction, the cartesian product (and it transfers into the theory the metatheoretical disjunction).

We can also show that the intuitionist implication can be interpreted as the exponential objects in the categories closed under this construction. The exponential object thus defined represents the set of morphisms or functions between two objects of a category and, in the intuitionistic systems, a proof of $A \rightarrow B$ is a morphism, a function – a term or calculation in Type Theory – from A to B.

But Mathematics needs variables. The syntactic entity represented by $x, y, \ldots$, which is an <u>individual variable</u> in Mathematical Logic, is a projection in Category Theory. When it appears within a formula, this generalises to the notion of <u>fibration</u>, a categorical way of talking about *variation*. Thus the universal quantification $\forall x \in B.A(x)$ (for all x in B we have $A(x)$) corresponds to a <u>fibred product</u> (or pullback), a notion well-established in Geometry, a kind of 'generalised cartesian product': actually universal quantification generalises conjunction, for $A(x)$ must be true at the same time for any x in B. This is an infinite conjunction or a limit: very informally it corresponds to $A(b) \wedge A(b') \wedge \ldots$ for all elements $b, b', \ldots$ in B.

How do we understand the existentencial quantification $\exists x \in B.A(x)$ (there exists a x in B such that $A(x)$ holds), always in the first-order case (that is when the variables are individual ones) ? The seminal observation of Lawvere is that this is nothing but the dual of the product above, with respect ot the operation of substitution (formally $\forall x$ and $\exists x$ correspond, respectively, to right and left adjoints to the substitution functor). Thus, once more, syntactic principles from Logic, indeed Frege's I order universal quantification, nicely corresponds to actual constructions in geometry.

Matters get more complicated when we consider variables over propositions or sets (we will write them with capital letters $X, Y, \ldots$). Why this extra work? When discussing the continuum from the logical point of view this is inevitable: the real numbers of Analysis are sets of integers, the numerical codes of (equivalence classes of) Cauchy sequences of rationals. For this reason the Arithmetic of second order, with variables ranging over propositions is considered the logical counterpart of Cantor-Dedekind's Analysis. Here comes the difficulty: the variables do not vary over a set or predicate, as in $\forall x \in B.A(x)$, but instead they vary inside the collection *Prop* of all the sets or propositions, including $\forall X \in Prop.A(X)$, the proposition that we are trying to define, for $\forall X \in Prop.A(X)$ is in *Prop*. Danger, danger: impredicativity got us. No problem: we will sort things out in two different ways. Through a normalisation theorem (we will see this in paragraph 5.3) and through a construction, that does not depend on the logic, and which has its origin in Geometry (the Grothendieck topos and the ideas of Lawvere). Inside these geometrical categories we can give a structural meaning, as a closure property of certain categories, to this stake that worries many logicians (but very few mathematicians and computer scientists). Briefly the variation will happen now over a category and not simply over an object of a category, as in the first order case, for we need to give meaning to the variables over propositions and each proposition is an object; thus it is necessary that this category be closed under products indexed over itself. All this gives a new structure for the variation and a strong closure property. The circularity of the impredicative definitions becomes then a theorem, the closure of certain categories under generalised products, whose origin is geometrical (see [Asperti&Longo, 1991]).

The only difficulty is that the construction cannot be done inside a classical Set Theory ([Reynolds, 1984]), instead one needs an intuitionistic environment ([Pitts, 1987],[Longo, Moggi, 1991]). Once again, but this is complicated, the geometrical symmetry between $\forall X$ and $\exists X$ can be represented as left and right adjunctions, with respect to a functor that also generalises the cartesian product, the diagonal functor.

Here we have a game of principles of proof and principles of construction that have very different origins and motivations. We understand ones through the others and this way we obtain one of these conceptual chains that are the kernel of the mathematical construction.

We have used implicitly in this sketch of a mathematical semantics of proofs, some constructions that take us back to infinity and the continuum. We have touched the continuum in two ways: the semantics of the notion of variation or change, which is one of the elements of the phenomenon of the continuum, and the impredicative definitions. But there is more than this: there are also passages to the limit, which are implicit in the categorical constructions of the product. The universal quantification $\forall x \in B.A(x)$ is simply an infinite conjunction, a limit. We then go back to infinity, to limits and to the continuum in Mathematics and Logic.

5.2 Limits and Closings of the Horizon

Despite the supporting references to systems of Intuitionstic Logic, the reader should not suppose that the author is a 'devoted intuitionist' as we can still find them (and of great scientifique value) in Northern Europe. The notion of conceptual construction discussed here is the one which emerges from the practice of Mathematics and it is more general than the one of Brouwer or as formalised by Heyting. The interest for Intuitionism is first *mathematical*: these systems have a correspondence in Geometry (Topos Theory) which is hard to find for other logical systems. But the interest in Intuitionism is also *methodological*, because of the emphasis it puts on the notion of construction[12]. But we should not make a limiting religion of our extraordinary creative possibilities, when it comes to mathematical constructions. Infinity for example has been part of our practices of language and of our perception of space for a long time, too long for us to try and expel it from our mathematical practice or from the logical theoretizations.

Consider the sequence $1, 2, 3, \ldots$ that we can iterate without any reason to stop. Its closure, on the horizon, which we call ω, is it not as clear and certain as the finite iteration? Nowadays with computers that do iteration so well, we can observe what happens after iteration more easily than in the past: the finitist engagement in Logic this century is in the origin of (the development of) these formidable digital machines that have changed our daily life[13]. This finitist effort should remain with the machines! We can continue, as mathematicians have done forever, using this construction, this going to the limit, without fear of losing our "unshakeable certainties". And we can state with no problem:

$$\omega + 1, \omega + 2, \ldots, \omega + \omega = \omega \times 2$$

But now the playing is easy, the construction evident:

$$\omega \times 2, \omega \times 3, \ldots, \omega\omega = \omega^2$$

Why not carry on? The rule is there:

$$\omega^2, \omega^3, \ldots, \omega^\omega$$

So long as we have a persisting iteration, we, human beings, we get bored. This is one of the differences between us and the computers: boredom. Computers don't get bored: iteration is their strongest point. We, once we have understood, once we have detected a regularity, we look further afield, we see the horizon, ω or even ω^ω, as we see the image of poplars in [Chatelet, 1993;ch.2.2]: we enclose into one single look the range that repeats itself in the direction of the horizon and we project it over an actual infinity. This is a human experience which is gradually made explicit in concepts through the centuries; maybe it has its origins in the Oriental religions, as Weyl would have it[14]; in any case, this experience has developed because of and within the mathematical practice, where religious commitment and platonist ontology can mix up and justify a conceptual construction, as with Galileo, Newton or Cantor. But what happens if we continue the iteration of the exponentials? We have ω to the power ω to the power ω ... on the limit, in the horizon this will be simply ω to the power ω, ω times. This ordinal we call ϵ_0, it gives the smallest solution to the equation $x = \omega^x$. Do we need a transfinite ontology to describe and use this construction? No, a simple principle of going to the limit, to the horizon, suffices, if we have an explicit iteration (as in this case) or a criterium of convergence (as in the case of Cauchy sequences). The ordinal ω is not in the world, it is not a convention, nor merely a symbol: *it synthetises a principle of construction*, a "disciplined gesture" to paraphrase Chatelet, rich of history. Its rigorous use in Mathematics had given it a meaning, has inserted it inside operative contexts, has shown us its different points of view, briefly has found it a place within the conceptual network we call Mathematics. This gesture reiterated gives us ...$\omega \times 2$,... ω^2,...ω^ω... ϵ_0. And whatever follows[15].

The utilisation of ϵ_0 in proofs has huge consequences. To begin with, Gentzen showed, in 1936, the consistency of Arithmetic, by induction up to ϵ_0, hence using methods beyond the finite ones, which are below ω[16]. Next, this "skeletons of infinity" can be found in the minimal construction of a model of Set Theory: Gödel's constructibles, which takes us back to the continuum. Gödel's idea in 1938 was briefly as follows: starting from the empty set, repeat by induction up to ϵ_0 the constructions formalised by Set Theory in their language and noting else[17]. The real numbers built inside this mathematical structure do satisfy the Axiom of Choice (and the Continuum Hypothesis), for reasons of minimality that we can guess: the sets of real numbers have minimal cardinality (see [Jech,1973], [Devlin,1973]). Cohen in 1966 proposed another construction for Set Theory: he adds generic or arbitrary elements, whose properties are "forced" bit by bit, during the construction of the model, in a way that does *not realise* the Continuum Hypothesis (or the Axiom of Choice).

We normally say that these two major results show the independence of the Continuum Hypothesis (and of the Axiom of Choice) from the formal Set Theories (Zermelo-Frankel, etc..). But this is not the most interesting aspect: *the meaning of these theorems is in their proofs.* They consist of mathematical (set-theoretical) constructions inside which certain properties are realised and through this they give us precise information about the nature of these properties (in particular about the structure of the continuum and the cardinality of subsets of the Cantorian reals: they depend on the construction made). The fact that these properties are independent from the Formal Set Theories concerned (the independence) says nothing about the continuum, but simply underlines the poverty of these formalisations, which are independent of any structure and which were born exactly to answer the questions about the continuum and about choice. Frequently formalism forgets the constructive and structural nature of Mathematics: Gödel and Cohen's constructions remind us of this.

5.3 The Infinite in the Trees

Trees in Mathematics have their root on top: a unique node, which branches downwards. A tree is finitely branching if each node has a finite number of nodes below it; a branch is a sequence of consecutive nodes, a path without jumps that starts at the root and develops to the bottom. Consider now the following principle, known as König's Lemma (KL): "in an finitely branching, infinite tree, there is an infinite branch".

The reader certainly understands, 'sees' this geometrical property of trees: if the infinite tree cannot grow infinitely horizontally (since it is finitely branching) it must grow infinitely vertically. This is an easy observation about the construction of trees, by an 'insight' onto the plane or structure of trees. However, we cannot, in general, effectively produce (construct by a calculable process) the infinite branch, even if the nodes are labelled and the tree is effectively produced (recursively enumerable). More precisely: one cannot give an algorithmic rule, write a program that generates the infinite branch, for the computer will have to go down paths for exploration and returns, erasing and reconstructing its memory in a non-effective way. Hence this principle, even if evident, goes beyond usual effectiveness; it is not intuitionistically acceptable[18].

Yet this principle has several applications. One is implicit in the categorical analysis of the impredicative definitions, mentioned in 5.1: a somewhat related principle, the Uniformity Principle (UP, see the latest note above), is used in the construction of the categories closed under products indexed over themselves ([Rosolini, 1986], [Hyland, 1988], [Longo&Moggi, 1991]; see [Longo, 1987] for a partly informal exposition). The principle hence contributes to giving structural semantics to the syntax of impredicativity: as we have mentioned in paragraph 5.1, we can 'understand' the impredicative definitions as closure properties of certain categories. Moreover, Tait-Girard proof of the normalisation theorem for impredicative Type Theory (see [Girard&al, 1989]) uses König's Lemma and one "comprehension" axiom over Sets of the following form

$$\exists X \in Sets. \forall x.(x \in X \iff A(x)).$$

The naive platonists, which accept this axiom, and the limitative constructivists of different schools, that reject it, all attribute to it an ontological content, on the basis of a "prejudice (in fact a medieval one) according to which the *same* logic holds for Mathematics and the real world – this implies, as a consequence, that an existential quantification must refer to singular individual entities really existing as separated, independent and transcendent entities" [Petitot, 1992]. This mistake that Petitot describes very well, is based on forgetting the role of proofs in Mathematics; it is sufficient to observe closely the argument for "strong normalisation" in Type Theory, in [Girard&al, 1989; par. 14] for example, to see that this axiom is simply a *principle of proof*: it 'just" permits to *replace* one variable over propositions (or types) for a given collection of terms, defined during the proof. Where is the ontological miracle?

A major consequence of the Strong Normalisation Theorem for Girard's system, and also for other systems starting with Gödel's 1958 system, is a demonstration of the consistency of Arithmetic of first and second order, and hence of Formal Analysis (see [Girard&al, 1989])[19].

In summary, non-effective insights or conceptual constructions are part of the mathematical practice and the metamathematical theoretization, with no need to refer to 'ontological' principles. The consequences of an 'existentially quantified' assertion (a comprehension axiom, say) are *logical consequences* of a possible (or assumed) constructions. The insight into trees may be as certain as an effective procedure.

As a matter of fact, even much stronger properties of trees than the previously described compactness property, (KL), may be acceptable. This is too complex a matter to be described in short, but it may be worth hinting that also the so called "determinacy for Δ_0 trees" bases its reliability on an insight into the planar structure of trees. Consider a well-founded tree (roughly, a tree with no infinite branch) and let two players play the following game. Player one moves downwards from the root by choosing one node; player two moves further by choosing a node below. The first player that cannot move anymore (he is on a leaf) has lost. Fact: there is always a winning strategy for one of the players. The proof goes by an 'easy', but powerful induction: it is trivial for 'one node trees'; given a tree, assume it for all the trees obtained by erasing the root, then prove it for the whole tree (obvious). Surprisingly enough, one may derive from this fact the consistency of Arithmetic (and even more): the expressiveness (and difficulties) depend on (the careful - impredicative - definition of) the rich structure of well-founded trees and the use of induction on them (see [Moschovakis,1980], for example). The latter turns out to be convincing, even certain, by the insight into the planar structure of trees.

6 The Logical Independence

The first great result of incompleteness, or of independence with respect to an interesting formal system, is Gödel's result, in 1931. In particular, Gödel's "first incompleteness theorem" shows that formal Arithmetic, which can code all effective processes, contains undecidable propositions, if it is consistent[20]. The second incompleteness theorem says that Arithmetic does not show its own consistency. More precisely, the second theorem shows that Gödel's undecidable proposition is equivalent to consistency and hence the theorem shows that the proposition is true, in the standard model, if we suppose the consistency of Arithmetic.

Later, to show the consistency from Gentzen to Girard, it was necessary to come out of the effective finitism and make use of stronger principles of proof, as hinted above.

We have also mentioned two other major results of independence, as consequence of Gödel and Cohen's constructions: the Continuum Hypothesis and the Axiom of Choice are not demonstrable nor refutable within Formal Set Theories.

By this, are there mathematical truths that we cannot reach through 'demonstration'? How would this be implied by the results of incompleteness or independence, if we just mentioned the existence of constructions which *demonstrate* the consistency of Arithmetic, of the Continuum Hypothesis and of the Axiom of Choice? *There are no propositions that are 'true and not demonstrable' in Mathematics.* True and demonstrable with respect to what, with respect to which construction and which proof principles ? One must make this precise.

There is in the usage of the this phrase, 'true but not provable', a 'slipping of meaning', very relevant and typical of naive Set Theory. We only have a precise notion of 'truth of a proposition' with respect to given mathematical structures (there are in fact several notions: Tarski's, Kripke's, Brouwer-Heyting-Kolmogorov's...). But we believe naively that there exists a set of true propositions. And hence the mystical reasoning: we move from the *notion of truth* to the *collection*, which exists in God's mind and which contains, one by one, the true propositions, in a well-ordered fashion. Which one, then, between the Continuum Hypothesis (CH) and its negation $\neg CH$ belongs to this collection?

In Mathematics when we talk about the truth of a proposition, it is necessary to say what we mean by this (that is, with respect to which notion of truth and with respect to which structures) and moreover it is necessary to show the truth with respect to this structure, to this notion. That was what Gödel did with the proposition "this proposition is not provable", which he showed to be codifiable and undecidable in formal Arithmetic. He also showed that this proposition is valid in the standard model (under the hypothesis of consistency, as a consequence of the second theorem of incompleteness). But Gödel did not say that the consistency of Arithmetic, undemonstrable in Arithmetic, is "true": for that it would be necessary to use Gentzen's proof, based on stronger principles. Gödel (and Cohen) will give us structures where the Continuum Hypothesis and the Axiom of Choice are true (or false) and they *proved it.*

Hence, what is this phenomenon of incompleteness, so important for the treatment of the continuum in Logic?

When discussing the intuitionist conjunctions and disjunctions, we saw a perfect correspondence between proof principles and categorical constructions. But this is not always this perfect. *The incompleteness of a formal theory, with respect to a precise structure, appears when we have a rift, a gap, between proof principles and construction principles.* Formal axioms, abstract principles, syntax for the manipulation of symbols and proofs in one side, constructions, in general geometrical or structural ones, in the other. The mathematical and logical difficulty lies in 'putting the finger on' the gap by providing theorems, making precise the proof principles and the construction principles utilised. We usually begin by 'destilling' the latter from our practice of Mathematics. This preliminar operation is particularly difficult, for the mathematical praxis includes, as Chatelet says, the "metaphors", the "allusive strategems" as "devices deliberately producers of ambiguity, which induce experiences of deep thinking, having as their kernel the relation and the operativity"...of "gestures that unveil a structure and reveal in ourselves other gestures" ... "equilibres en porte à faux qui ne se rompent qu'en emportant un espace plus ample" [Chatelet,1993; p. 32-37]. *These metaphors, these strategems are not the 'psycological sugar' of the discovery, they contribute to the practical construction and get specified over time, with effort, into new principles of constructions, new structures.* The design of conceptual invariants (geometrical, algebraic..) is also part of this process, invariants which are the synthesis of a plurality of experiences in the interior and the exterior of mathematics. A (small) part of the work of the mathematician consists of making explicit the principles of construction which he uses to define his structures. A (large) part of the work of the logician consists in the choice and the formalisation of principles of proof, observing and organising the mathematical constructions, to make up systems of axioms, rules of inference ...[21]. The analysis of links with what preceedes and grounds the work of both, at the cognitive level, or which can be found at the exterior of Mathematics still needs to be done.

Actually once the principles of proof and construction are well-described, there is not always a clear demarcation between them; think for example of the Principle of Uniformity or of König's Lemma, or the Axiom of Choice.. which are always between proof and construction. Amonsgt the ones we have seen, perhaps only the axioms of comprehension do not look like principles of construction and are pure 'proof techniques'. But also the rules and the formal axioms of the Arithmetic of Peano-Dedekind or the logical systems of first order of Frege and Hilbert, the Set Theories, are very clearly principles of proof, derived from mathematical constructions (Number Theory, Analysis, ..). *It is in the difficult to detect, but possible gaps, between formal proofs and mathematical constructions, that incompleteness theorems can be found.*

The incompleteness theorems of Gödel (the first: under the hypothesis of consistency, there is an undemonstrable proposition; the second: the consistency is undemonstrable) and the consistency proofs from Gentzen to Girard show that in the construction of the integers and their properties we use, or we can use, if we we accept them, strong principles, beyond formal arithmetic: we show hence that the consistency is a *true* property over the integers (and hence we show the *truth* of the undecidable proposition given by the first theorem of Gödel, which is non-demonstrable in Arithmetic and equivalent to consistency).

The constructions of Gödel and Cohen prove the same thing about CH and AC: they show that they are true (or false in Cohen's case) on certain structures, contructed using certain principles, but that they are non-demonstrable using simply the axioms and rules, the proof principles, of Formal Theories of Sets[22]. In other words, all these results (and many others: Paris-Harrington, Kruskal-Friedman ...) by proving the truth or validity of certain propositions over *possible* mathematical structures (universes of sets or of numbers) or by proving their unprovability within given formal systems (described by *possible* proof-principles), 'simply' display the gap between mathematical constructions and formal theories.

Thus, *one should never say in Philosophy of Mathematics the phrase 'there are true but non-demonstrable propositions', for this phrase makes no sense in Mathematics.* A working mathematician (not on Sundays, for then he does the usual naive platonic philosophy) asks immediately ' *Non-demonstrable* with respect to which system (to which proof principles)? *True* in which structure (using which construction principles and notion of truth)?'

This century, the formalism, in Logic and Proof Theory, which going further, has found in finitism and formalism its origins, have without doubt helped to answer these questions. But why logical formalism, a philosophical indirect springing of the mathematical practice, should be the ultimate source of our certainties, of our analyses of proof and of construction in Mathematics?

The conceptual networks, inside which the mathematical constructions are embedded, do not give us the ultimate certainties, but insert each construction within other forms of knowledge. These give it a meaning, several meanings, whose connections and compatibilities, form the net, relatively solid, of our relation with the world. It is the practical unity of Mathematics and its emergence from the world which contitutes its foundations: this frame and the balances of theories, which translate each other, interpret each other, give root to each of its nodes in our forms of knowledge.

The analysis of proofs, Proof Theory, is one of its instruments. The different structural semantics will provide others. But it is necessary to insert Mathematics in the triangular relation history–individual–world, by reconstructing the cognitive and historic percourses which are at the origin of the mathematical invention.

Our effort towards the comprehension of the world is like a walk over quicksand: when we throw the net of our knowledge, of which mathematics is but a small part, this net will permit us to advance a few steps, just by its extension. The challenge of naturalization, as cognitive analysis, and as analysis of the historical and collective construction of concepts (mathematical ones in particular) consists in finding a few supporting points for this net.

7 Three Levels and the Richness of the Continuuum

In his essay about the Continuum Jean Petitot [Petitot,1992] underlines an essential 'bimodality' of the continuum. First a form of giving, a pure intuition, emerging from the world, purely phenomenal and without autonomous objectivity, for which it is necessary to elaborate a (mathematical) theory of its psycho-physical genesis, as part of cognitive analyses of pure intuitions. Then a mathematical reconstruction (non-univocal) of the intuition, which acquires a value and an objective reality, but which, as any transcendental totality, cannot be a complete determination of the intuitive giving.

In this article instead we have underlined, initially, the non-unicity of the intuition of the continuum. Then we have developed an analysis which emphasized three levels: the intuition one, the construction principles one, and the proof principles one.

On the first level, the richness of the world and of points of view from which to observe it, compatible points of view, non isolated, but built from a dialogue with evolution and history, suggest a plurality of intuitive approaches and ground mathematics in our relation to the world. In part we find these points of view in the different mathematical constructions of the continuum, which constitute the second level. These constructions enrich and modify the original intuitions, which are not that simple when the mathematical praxis adds to them its depth. But thanks to Logic there is a third level, where the analysis of the proof (as well as the Formal Set Theories, their axioms, their rules of inference) plays an essential role. Clearly, the incompleteness results lay in between the second and the third level, as a precise form of indetermination of mathematical constructions by formal theories.

Speaking transcendentally, the mathematical objectivity does not find its origin in the unicity of the intuitive giving, nor in the categoricity (or unicity) of the psycho-physical genesis of that one, but instead in the common, historical and cognitive (hence intersubjective) process of the conceptual construction. That is, *in the mathematical construction the value and the objective realities are not to be found in the mathematical entities (the integers, the real numbers, ω or ϵ_0 for example) but in the process of constitution of these so-called entities, as conceptual constructions:* the iterations, the passages to the limit, the closures of horizons, the constitutions of invariants[23]. In the case of the continuum, the mathematical objectivity is also in the richness of interaction of three levels we mentioned: intuition, Mathematics, Logic. This interaction is not a vicious circle, but a virtuous one, extraordinary example of the dynamicity of our forms of knowledge: Logic, which only extracts formal rules from the constructive practices of Mathematics, offers, thanks to the incompleteness theorems and Non-standard Analysis, for example, new mathematical structures, which suggest a new intuition about the continuum. A further starting point, through games of dynamic reflections, for other constructions and formalisations.

When we return to the *Theories*, after all these metatheoretical considerations, we could say that we then better appreciate one of the aspects of the expressive force of Mathematics. Mathematics knows how to bring back to its interior this reflexivity of the world and of the forms of knowledge: the impredicative analysis of the continuum, the impredicative logical systems are possible examples. But we should also cite the analysis of resonance within dynamical systems and many other mathematical theories where any stratification would distance us from the world.

8 Discrete and Continuum in Metamathematics

Number Theory, at least the elementary one, but also Logic and the set theoretical practices, after Cantor, Dedekind and Frege, give ontological priority to discrete notions and derive the mathematical continuum from the integers, the way we have briefly described. By contrast Leibniz and Thom consider the continuum as the original giving, central to all mathematical construction, while the discrete is only represented as a singularity, as a catastrophe. Physics is also thorn between these two tendences. In one hand it makes sense to say that every process should be continuous, or even two times differentiable. On the other hand we could affirm that the world is discrete: think about the atoms, the elementary particles, the quanta. Even within quantuum mechanics, where we might expect to find only discrete representations, we find the two schools (see, for example, the debate in [Salanskis&Sinaceur, 1992]). What about the delta functions of Dirac? These smooth and infinitely differentiable curves, that on the limit become pointed and mathematically disagreable: a true 'catastrophe'. But it is Mathematical Logic, the way it was specified this century, that utilises exclusively symbolic languages and the discrete inferences as ultimate foundations of Mathematics, even in the analysis of the mathematics of the continuum.

It is possible that this priority attached to the integers have solid cognitive motivations and found its origin in an apparent 'isomorphism' of discrete registers of experience: when the sensations present themselves isolated one from another, the discrete sensibles (tact, vision, hearing) appear isomorphic. A source perhaps of a certain 'canonicity' of the operation of counting in the human experience. It is also necessary to recognise that Mathematics, Logic in particular, reflect very well this 'intuitive categoricity' (formally: unicity up to isomorphism) of the discrete. If we take the different constructions of the set-theoretical universes (Gödel, Cohen..) their representations of the integers are isomorphic. Hence they are not affected by the Continuum Hypothesis nor by the Axiom of Choice nor by their negation. That is, we have an effective procedure to eliminate these principles from any proof, in Number Theory, that uses it. Moreover, we can show in Set Theory that any non-standard model of formal Arithmetic contains an initial segment isomorphic to the (standard) familiar sequence of integers: the true place of finite counting. Similarly in Category Theory, every topos contains an object of natural numbers with the same characteristics. Nothing comparable happens to the continuum: as we have seen, the experience of the world, of space and of time, does not impose a canonical continuum to our intuition. Also in Mathematics the non-standard line, for example, is locally and globally non-isomorphic to the standard one.

There is no mystical ontology of the natural number here: *it is conceivable that the sensible and discrete set of natural numbers and the practice of counting constitute an experience that imposes itself with much more evidence and of unicity, amongst our acts of life, than the intuitions of the continuum. Perhaps this experience is pre-human (see the monkeys in [Hauser&al,1996]). Where from comes this canonicity, this almost unicity, of the representation of natural numbers in Mathematics: a beautiful correspondence between mathematics and the world, a strong sign of the way mathematics emerges out of our praxis.*

In any case, beyond history, we could pose the problem of knowing if it is this practical priority of the discrete which forces the canonicity of the representation of the standard natural numbers in Logic and in Mathematics (against the plurality of representations of the continuum). Or perhaps if it is the (arbitrary) foundational choice made in Logic of considering only, of defining itself even, as discrete resasoning, which forced upon us this canonicity of the discrete. That is, by dealing with countable sequences of axioms and finitary algebraic rules (at least discrete ones), Logic may have, a posteriori, convinced us of the central position of the discrete, simply by giving us 'categorical' logical systems, as far as finite counting is concerned (as Thom seems to suggest). In other words, we could then believe that:
- Either *the finite, the discrete, common foundation of Formalism and Intuitionism*, which in their different versions and combinations constitute the two main currents in Proof Theory (and Logic in general) *justifies itself by the historic (and evolutive) weight of counting*;
- Or it was the recent (but at least as old as Boole's "Laws of Thought") choice of founding Mathematics only over linguistic rules, discrete sequences of symbols, has made us *forget*, by the expressive force of the metamathematics it generated, the direct role of Geometry and of the continuum in the mathematical reasoning (this could be seen as part of what René Thom calls "le delire logique", [Thom, 1990]).

Amongst the Greeks, but also amongst the modern geometers, one demonstrates theorems by direct inspection of figures, by their continuous movements (rotation, sliding, superposition..) Think about the direct proof of Pythagora's theorem by decomposition and sliding of squares and rectangles. Build like Euclid a square of twice the area of a given square: draw the four lines parallel to the diagonals by the vertices; indicating by your finger, in silence, without languages nor symbols, the four triangles duplicated, give a demonstration complete and precise, rich of a rigour that is not algebraic-linguistic. I claim that this "compelling role" of Geometry is crucial in todays' work on trees in Proof Theory (the planar structure of infinite trees is crucial in König's Lemma or in Determinacy, as hinted above).

It was from the Analytic Geometry of Descartes that the powerful tool of the algebric language tended to take over, for its generality, the direct role of Geometry in proofs. This generality is sometimes fictitious, for there is no modern discourse about differentiable varieties, even when very complex and general ones, without a sketch on the blackboard. An appeal to intuition, no doubt, but which interacts with the proof, or which can even constitute a proof. *The formal skeleton of the finitary logical languages is only a fragment of the mathematical reasoning: an essential frame which has given us solid proof principles (and very powerful symbolic machines, the computers) but whose incompleteness with respect to the mathematical constructions are very important (the theorems of incompleteness or independence).*

Perhaps if, instead of looking for the finitist certainties our founding fathers had developed the analysis of proofs as movements of images, geometries which decompose and superimpose themselves, as the Greeks did, or if they had treated them as continuous movements between metaphors and conceptual images, giving emphasis to the plastic character of reasoning, "the allusions that nurture themselves from the indetermination for forcing a higher determination" [Chatelet,1993] we would have now foundational systems that would force a single representation of the continuum, despite its plurality of experiences (which we wouldn't notice anymore), and on the contrary a plurality of forms of counting. A major consequence of this bifurcation in history could have been a very different notion of Computing Machine, possibly analogue and continuous, for todays digital computers are the direct sons of calculability, as representation of deduction, à la Hilbert, Gödel, Herbrand,Turing, Church ... and hence as a discrete and effective procedure. Another consequence could have been a different notion of rigour, a notion that has become so strong and clear this century thanks to formalism. In any case, why the mathematical rigour should be but formal, interminable sequences of symbols without meaning and mechanical rules? Now that we have understood well what is formal-linguistic rigour and that we have machines much 'more rigourous' than us, we, humans, can serenely try to extract the principles of proof from the geometrical constructions, from drawings, from the intuitions of the continuum. Even if we believe that the role of counting in our perception and history (since the ... apes) has forced the mathematical canonicity of the discrete integers, in particular through the research of the finitist certainties at the turn of century, we could nowadays go further. Actually we have attained a very good level of formal mathematical rigour and the paradoxes are distant. *Hence we can now reconstruct the meaning and the practice of demonstrations and widen our notion of rigour, by encompassing also diagrams, metaphors, images. These constitute themselves in the continuous movements of reasoning, which goes from one concept to the next by slidings, analogical upturnings, tenuous figurative links which extend themselves with continuity.* This is not about opposing a new Proof Theory to the old one, but about enriching Logic and Proof Theory, making it come out of the formalist cage which generated the so-called "logico-computational hypothesis" for the human intelligence: "Intelligence ... is effectively defined as that which can be manifested by the communication of discrete symbols" ([Hodges, 1992]). Hence a picture by Piero della Francesca or the Greek construction of a square with twice the area does not contain "direct" intelligence, even less foundational interest for Mathematics: intelligence develops only after its traduction in finite algebraic languages, if necessary pixel by pixel, over discrete cartesian coordinates. By contrast, the construction and the synthetic handling of images, as a continuum, as an impredicative 'whole' linked to its parties, is the heart of an analysis, which is possible today, of the intelligence and of mathematical proof.

We found traces of this in the work of logicians which have insisted on the role of Geometry. For example, in the denotational (or categorical, see paragraph 5.1) semantics of Lawvere-Scott for intuitionistic systems (or for programming) the geometry, the continuity, give signification to the lists of symbols 'without meaning', for "Geometry is more compelling", as Dana Scott suggested once. Or also in the geometry of proof nets by J.Y. Girard, where the symmetries and the direct manipulation of images (networks over the plan) come into the play of logical derivations, in an essential way. Moreover geometry is central in the recent mathematical development of husserlian analysis of knowledge, as in Petitot's work[24]. It is perhaps 'vision' that is more compelling, as some neurophysiologists claim.

But this widening of Proof Theory should not be just a new game of mathematical rules, as this would only give us a new mathematical discipline. Wittgenstein had forseen this happening with the hilbertian metamathematics [Shanker, 1988] and it has in fact happened. Metamathematics became a new and beautiful kind of mathematics, where the principal results have been indirect: a precise notion of formal rigour and ... Computer Science, but not the explicit foundation of the mathematical practice, as was Hilbert's dream. We can not 'found' mathematics (its "rules of the game" as Wittgenstein says) over a mathematical discipline, a logical-mathematical system also made up of mathematical "rules of the game". *There cannot be an internal foundation, purely formal and mathematical, of Mathematics: the incompleteness theorems are not accidents, they underline the gap between the metamathematical principles of proof (once transformed into a mathematics of formal rules) and the rigourous practice of mathematical constructions.* It is then necessary to increase the variety of tools for the foundation of mathematics, first by the direct constructions of Geometry (which is being done), then with other forms of knowledge; that is, retaking the metaphor of knowledge as a network (end of Sect. 6) *it is necessary to insert the partial network of Mathematics in the wider one of the other forms of knowledge.* The project to aim at should take mathematics out of its 'auto-foundational' game (metamathematics as a form of mathematics) and look for its cognitive origins in our relation to the regularities of the world, in the connections to different conceptual constructions, in the mental invariants that we build while living and historical beings.

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Notes

1. "There are three forms of time:... the present time of things gone, which is our memory; the present time of present things, which is our vision; the present time of future things, which is our wait" [St. Augustine,401;Lib XI, ch.XX]. Evidently St. Augustine is not talking about intuition: time, its measuring, Mathematics and the knowledge of God himself reside in *memory.*

2. An early counterexample is due to Weierstrass. One variant of his example is the function $f(x) = \Sigma \frac{sin(2^n x)}{2^n}$. One was surprised until Poincaré by the fact that this function is nowhere differentiable (and with good reasons...).

3. It is interesting to note how usually we talk, in Mathematics and in Logic, about the 'reduction' (à la Cantor-Dedekind) of the real numbers to the integers, as if the reals were already 'there', as if the 'informal practice' of the mathematical continuum (see Cauchy's demonstration of 1821) made reference to an external objectivity, that we must understand by reduction (the same way we reduce some chemical realities to Physics). This is comprehensible in the naif platonic practice of Mathematics, but it it is less so for the formalist/definitionist vision of mathematics still prevailing in Logic.

4. In Set Theory, in writing $\forall y$ for 'for all y', a set b is defined impredicatively if, typically, it is given in the form

$$b = \{x | \forall y \in A \ P(x, y)\}$$

where b can be an element of A (the same set or collection of sets A which appears in the definition of **b**). Briefly in an impredicative theory there is no stratification of the mathematical universe and it is acceptable to define one element b using a predicate/set A which can contain b. Informally we can not comprehend the parts, the elements, without comprehending at the same time the whole, or a big part of the whole.

5. Also for Leibnitz and Kant the continuum cannot be decomposed into its elements, it is not formed from simpler unities: it presents itself simultaneously as a totality and its parts (see [Panza,1989]).

6. In the sequence of his fundamental reflections Weyl first joined the ranks of Intuitionism, then he embraced a more open view of mathematical knowledge. But the mathematics of Brouwer and the logical systems of Heyting are compatible with the impredicative notions: in fact even the definition of an intuitionist proof is impredicative (see [Longo, 1987] for more: the interplay is between theory and metatheory, so it is acceptable for many). After that, in his logical-philosophical writings (see the French version of [Weyl, 1918] for many references), Weyl will never go back to his Predicative Analysis. On the contrary, he will develop a very rich vision of the connections between Mathematics and Physics which will culminate in [Weyl, 1953] his last book, a Husserlian masterpiece, clearly anti-formalist:

Mathematics emerge from the effort to know the world (physically, chemically, artistically ...) as a 'transcendental objectivity'. See also [Weyl, 1985] for a very balanced and 'secular' view of the instruments of demonstration in mathematics.

7. In Intuitionistic Mathematics we distinguish between sequences given by a 'lawlike' or 'lawless' ones. Here the 'law' is an algorithmic (or effective) rule. For example, π is the limit of a lawlike sequence (the algorithm for constructing it), whereas a real whose decimals are given by successively playing a die is the limit of a lawless sequence. But even a convergent lawless sequence obeys a rule and follows a convergence criterium: for example play the dice and add its results as decimal numbers. The limit is unique and well-defined: the criterium of convergence is given by the fact that 'we add as decimals' the results of the throw. It is the existence (of the sequence) that is weak, non-effective.

8. "Among the usual spaces that better embody the ideal of the continuum, there are two that appear almost immediately: the euclidian line and the euclidian plan; the line for its mechanical and physical realisations (the extended thread, the luminous ray) ..." [Thom, 1992; p.142].

9. Leibniz infinitesimals became the new real numbers, smaller than any other standard real number. Then $x \approx y$ if $x - y$ is infinitesimal and hence a function is continuous if $f(x) \approx f(x+h)$ for all infinitesimal h.

10. Axiom of choice (AC): "For all non-empty collection of non-empty sets, we can contruct a set which constains exactly one element from each set of the family". The Axiom of Choice is essential in many demonstrations, including some that concern the continuum: without AC the definitions of limit based on neighbourhoods and the one based on sequences are not equivalent (it is necessary to *construct* a sequence, by *choosing* a point for each succesive environment).

11. The few technical notions in this section will not be used in the sequel: they are just examples of elementary connections between principles of proof and principles of construction. For more details see amongst others [Lambek&Scott,1986], [Asperti&Longo,1991].

12. We could say the same about Girard's Linear Logic as its nature makes even Classical Linear Logic ... "constructive" [Girard, 1991].

13. The research on the "unshakeable certainties" of Hilbert and Brouwer (see [Brouwer, 1927]) has given us this century a very solid notion of mathematical rigor: the finitist deduction, formal and effective. Over this basis, the incredible Thirties of Logic have seen the birth of one precise notion of calculation and of machine, the foundations of the modern programming languages (Turing: the imperative languages; Church: the functional ones; Herbrand: the logic ones).

14. See the note "D'Anaxagore à Dedekind", 1926, in the French version of [Weyl, 1918].

15. We can continue with ϵ_1, ϵ_2,..., ϵ_ω and having understood the mechanism, which after ϵ_0 is not that simple, we can continue with ϵ_{ϵ_0} ... .

16. But this method of proof was not considered very convincing by many, for the heart of the problem of consistency for Arithmetic is the *consistency of induction*, the key principle of Peano's Arithmetic: one shouldn't use an even more powerful induction to show it. There are other ways of proving it, tough, using formally equivalent, yet more conving, methods.

17. This means reiterate the constructions formalised by the axiom of power of a set, of replacement– image of a set by a function, etc ... so long as they are definable in the language of Set Theory.

18. Consider the contrapositive of König's Lemma (KL), which is called the FAN Theorem since Brouwer: "if in a finitely branching tree, each branch is finite, then the tree is finite". FAN says that if we 'get stopped' along each of the descendent branches, then a finitely branching tree is uniformly limited: it is thus a compactness property. Most intuitionists (e.g. [Troelstra, 1973]) accept FAN, which does not imply KL, for the equivalence between FAN and KL, its classical contrapositive, is not intuituionistically valid. (In general, in Intuitionistic Logic, $\neg\neg A$ is not the same as A and we cannot go from $\neg B \to \neg A$ to $A \to B$ (read KL as "A [infinite tree] implies B [there exists an infinite branch]").) In [Troelstra, 1973] a relevant variant of FAN is proposed, the Uniformity Principle (UP).

19. The theorems of cut-elimination and normalisation for the systems of higher order give extremely solid bases to the impredicative definitions. The consequence is that every proof in the system can be simplified to a 'minimal form' (a normal form or without cuts), or that there are no 'incontrollable propositions' that can introduce themselves into proofs. We must note that the second principle of proof mentioned here is sufficient to prove the theorem of normalisation, but the proof of Girard, which uses both principles displays very clearly, for its elegance, the issues of the construction. See also [Fruchart&Longo, 1995] for an application of a recent theorem to the justification of impredicative definitions.

20. In particular the formalised proposition, which says "this proposition is not provable".

21. Computer Science has given new motivations to the work of logicians, for without reference to the mathematical structures, they try to analyse the practice of programming and the conception of the architectures of computers (and to propose new designs). In the good practice of computing the unity of Mathematics imposes itself again, in the research for a mathematical and structural meaning for these theories (semantics of programming languages: denotational, algebraic,..).

22. We could mention that beyond products and coproducts, which correspond so well to the intuitionist conjunction and disjunction, the Effective Topos, which is the basis of the sketched constructions of second order, is constructed using principles that go beyond the other 'pure' intuitionist rules (the principles of Uniformity and of Markov, amongst others): hence the Effective Topos shows the truth of non-demonstrable propositions of the systems of intuitionistic logic, of which it is a model. The Genericity Theorem, see [Longo, 1995; Fruchart&Longo, 1997], gives another mismatch between the Topos and Intuitionistic Logic.

23. See also [Petitot,1992; par. II.1].

24. For a philosophical introduction to the "geometry of perception", and also for its numerous applications to which it referes, see [Petitot, 1995]. About the role of the continuum in linguistics, see [Fuchs&Victorri, 1994].

References

Asperti A., Longo G., 1991, **Categories, Types and Structures**, M.I.T. Press.

Augustine St, 401, **Confessiones**.

Barreau F., Hartong J. (eds.), 1989, **La Mathématique non standard**, Editions du CNRS.

Bouveresse J.,1992, "Weyl, Wittgenstein et le Problème du Continu", dans [Salanskis&Sinaceur,1992].

Brouwer L.E., 1927, "Intuitionistic reflections on Formalism" (dans Van Heijenoort, **From Frege to Gödel**, 1966.)

Chatelet G., 1993, **Les enjeux du mobile**, Seuil.

Chandarasekharan K., 1986, **Hemann Weyl**, Springer Verlag.

Cutland N. (ed.), 1988, **Non-standard Analysis and its applications**, Cambridge U.P..

Devlin K.J., 1973, **Aspects of Constructibility**, LNM, Springer-Verlag.

Feferman S., 1988, "Weyl vindicated: "Das Kontinuum" 70 years later", in Atti del congresso, Temi e prospettive della logica e della filosofia della scienza contemporanee, Volume I, **Logica**, edited by Carlo Cellucci and Giovanni Sambin, CLUEB, Bologna.

Feferman S., 1999, "The developement of programs for the foundations of mathematics: logicism, predicativism, set-theoretical platonism, intuitionism and formalism", **Enciclopedia Italiana della Scienza**, to appear.

Fruchart T., Longo G., 1997, "Carnap's remarks on Impredicative Definitions and the Genericity Theorem" in Logic, Methodology and Philosophy of Science: **Logic in Florence**, 1995, Cantini et al. (eds.), Kluwer, to appear.

Fuchs C., Victorri B., 1994, **Continuity in Linguistic Semantics**, Amsterdam.

Girard J.-Y., Lafont Y., Taulor P., 1989, **Proof and Types**, Cambridge Univ. Press.

Girard J.Y., 1994, "A new constructive Logic: Classical Logic" **Mathematical Structures in Computer Science**, vol. 1, n. 3.

Hauser M.D. et al., 1996,"Numerical representations in primates", **Proc. Natl. Acad. Sci.**, USA, vol 93, Feb..

Hodges A., 1995, "Alain Turing and the Turing Machine" in (Herken, ed.) **The Universal Turing Machine** (revised edition), Springer.

Hyland M., 1988, "A small complete category", **Annals of Pure and Applied Logic**, vol. 40.

Jech T., 1973, **The Axiom of Choice**, North-Holland.

Lambek J., Scott P.J., 1986, **Introduction to higher order Categorical Logic**, Cambridge University Press.

Lassègue J., 1994, **L'Intelligence Artifcielle et la question du Continu: remarques sur le modèle de Turing**, Thèse de Doctorat, Univ. Paris X.

Longo G., 1987, "Some aspects of impredicativity: notes on Weyl's philosophy of Mathematics and on todays Type Theory" **Logic Colloquium 87**, Studies in Logic (Ebbinghaus et al. eds), North-Holland.

Longo G., 1995, "Parametric and type-dependent polymorphism", **Fundamenta Informaticae**, 22(1-2):69–92.

Longo G., 1997, "The difference between Clocks and Turing Machines", Conference on **Models of Cognition and Complexity Theory**, Roma, 1994, **La Nuova Critica**, Roma, n. 29.

Longo G., Moggi E., 1991, "Constructive Natural Deduction and its "Omega-Set" Interpretation", **Mathematical Structures in Computer Science**, vol. 1, n. 2.

Moschovakis Y., 1980, **Descriptive set theory**, North - Holland.

Panza M., 1989, **La Statua di Fidia**, Unicopoli.

Panza M., 1992, "De la Continuité comme concept au Continu comme Objet", dans [Salanskis, Sinaceur,1992].

Parker S., Gibson K., 1990, **Language and intelligence in monkeys and apes**, Cambridge Univ. Press.

Petitot J., 1992, "Continu et Objectivité: la Bimodalité Objective du Continu et le Platonisme Transcendantal", dans [Salanskis, Sinaceur,1992].

Petitot J., 1995, "La Réorientation naturaliste de la Phénoménologie" **Archives de Philosophie**, Toma 58, cahier 4, Oct-Déc..

Pitts A., 1987, "Polymorphism is Set Theoretic, constructively" Symposium on **Category Theory and Comp. Sci.**, SLNCS 283 (Pitt et al. eds), Edinburgh.

Reynolds J., 1984, "Polymorphism is not set-theoretic," Symposium on **Semantics of Data Types**, (Kahn, MacQueen, Plotkin, eds.) LNCS 173, Springer-Verlag

Rosolini G., 1986, **Continuity and efffectiveness in Topoi**, D. Phil. Thesis, Oxford Univ..

Salanskis J.-M., 1991, **L'Herméneutique Formelle**, Editions du CNRS.

Salanskis J.-M., Sinaceur H., 1992, **Le Labyrinthe du Continu**, Springer-Verlag.

Shanker S., 1988, "Wittgenstein's remarks on the meaning of Gödel's Theorem" dans **Gödel's Theorem on focus** (Shanker ed.), Croom Helm, London.

Takeuti J., 1987, **Proof Theory**, 2nd edition, North-Holland.

Thom R., 1990, **L'Apologie du Logos**, Hachette.

Thom R., 1992, "L'Antériorité Ontologique du Continu sur le Discret", dans [Salanskis, Sinaceur,1992].

Troelstra A.S., 1973, **Metamathematical investigation of Intuistionistic Logic and Mathematics**, LNM 344, Springer-Verlag.

Weyl, H., 1918, **Das Kontinuum**, (italian version, par B. Veit, Bibliopolis, Napoli, 1977; french version, par J. Largeault, Vrin, 1994).

Weyl, H., 1953, **Symmetry**, Princeton University Press.

Weyl, H., 1985, "Axiomatic Versus Constructive Procedures in Mathematics." (Edited by T. Tonietti) **The Mathematical Intelligence**, Vol. 7, No. 4, Springer-Verlag, New York.

Wittgenstein L., 1939, **Cours sur les Fondéments des Mathématiques** (version française par E. Rigal, Mauzevin, 1995).

Wittgenstein L., 1956, **Remarks on the foundation of Mathematics**.

Wittgenstein L., 1964, **Philosophische Bermerkungen**, Blackwell, Oxford.