THEOREMS AS CONSTRUCTIVE VISIONS[1]

Giuseppe Longo

CNRS, Ecole Normale Supérieure,

et CREA, Ecole Polytechnique

45, Rue D'Ulm

75005 Paris (France)

http://www.di.ens.fr/users/longo

Abstract *This paper briefly reviews some epistemological perspectives on the foundation of mathematical concepts and proofs. It provides examples of axioms and proofs, from Euclid to recent "concrete incompleteness" theorems. In reference to basic cognitive phenomena, the paper focuses on order and symmetries as core "construction principles" for mathematical knowledge. A distinction is then made between these principles and the "proof principles" of modern Mathemaical Logic. The role of the blend of these different forms of founding principles will be stressed, both for the purposes of proving and of understanding and communicating the proof.*

1. THE CONSTRUCTIVE CONTENT OF EUCLID'S AXIOMS.

From the time of Euclid to the age of super-computers, Western mathematicians have continually tried to develop and refine the foundations of proof and proving. Many of these attempts have been based on analyses logically and historically linked to the prevailing philosophical notions of the day. However, they have all exhibited, more or less explcitly, some basic cognitive principles – for example, the notions of symmetry and order. Here I trace some of the major steps in the evolution of notion of proof, linking them to these cognitive basics.

For this purpose, let's take as a starting point Euclid's *Aithemata* (Requests), the *minimal constructions required* to do geometry:

1 *Invited lecture,* **ICMI 19 conference on Proof and Proving**, Taipei, Taiwan, May 10 - 15, 2009, (Hanna, de Villiers eds.) Springer, 2010.

1. To draw a straight line from any point to any point.

2. To extend a finite straight line continuously in a straight line.

3. To draw a circle with any center and distance.

4. That all right angles are equal to one another.

5. That, if a straight line falling on two straight lines make the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which are the angles less than the two right angles. (Heath, 1908; pp. 190-200).

These “Requests” are *constructions performed by ruler and compass*: an abstract ruler and compass, of course, not the carpenter’s tools but tools for a dialogue with the Gods. They provide the minimal “constructions principles” the geometer should be able to apply.

Note that these requests follow a “maximal symmetry principle”. Drawing a straight line between two points, one obtains the most symmetric possible structure: any other line, different from this one, would introduce a-symmetries by breaking at least the axial symmetry of the straight line. The same can be said for the second axiom, where any other extension of a finite line would yield fewer symmetries. Similarly, the third, a complete *rotation symmetry*, generates the most symmetric figure for a line enclosing a point. In the fourth, equality is defined by congruence; that is, by a *translation symmetry*. Finally, the fifth construction again is a matter of drawing, intersecting and then extending. The most symmetric construction occurs when the two given lines do *not* intersect: then the two inner angles are right angles on both sides of the line intersecting the two given lines. The other two cases, as negations of this one (once the theorem in book I n. 29 in Euclid’s Elements has been shown), would reduce the number of symmetries. Their equivalent formulations (more than one parallel in one point to a line, no parallel at all) both yield fewer symmetries, *on a Euclidian plane*, than having exactly one parallel line.

Euclid’s requests found geometry by actions on figures, implicitly *governed by symmetries*. Now, “symmetries” are at the core of Greek culture, art and science. They refer to “balanced” situations or, more precisely, “measurable” entities or forms. But the meaning we give to symmetries today underlies Greek “aesthetics” (in the Greek sense of the word) and their sensitivity, knowledge and art, from sculpture to myth and tragedy. Moreover, loss of symmetries (symmetry-breakings) originate the world as well as human tragedy; as breakings of equilibria between the Gods, they underlie the very sense of human life. As tools for mathematical construction, they participate in the "original formation of sense", as Husserl would say (see below and Weyl, 1952).

Concerning the axioms of geometry, the formalist universal-existential version ("For *any* two points on a plane, there *exists one and only* one segment between these points" etc.) misses the constructive sense and misleads the foundational analysis into the anguishing quest for formal, thus finitistic, consistency proofs[2]. We know how this quest ended: by Gödel's theorem, there is no such proof for the paradigm of finitism in mathematics, formal arithmetic.

2. FROM AXIOMS TO THEOREMS

"Theorem" derives from "*theoria*" in Greek; it means "vision", as in "theater": a theorem *shows*, by constructing. So, the first theorem of Euclid's first book shows how to take a segment and trace the (semi-)circles centered on the extremes of the segment, with the segment as radius. These intersect in one point. Draw straight lines form the extremes of the segment to that point: this produces an equilateral triangle.

For a century we have been told that this is not a proof (in Hilbert's sense!): one must formally prove the existence of the point of intersection. These detractors could use more of the Greeks' dialogue with their Gods.[3] Lines are ideal objects, they are *a cohesive continuum with no thickness*. Both points and continuous lines are founding notions, but the conceptual path relating them is the inverse of the point-wise constructions that have dominated mathematics since Cantor. Points, in Euclid, are obtained as a *result* of an intersection of lines: two *thickless* (one-dimensional) lines, suitably intersecting, produce a point, *with no parts* (no dimension) The immense step towards abstraction in Greek geometry is the invention of continuous lines with no thickness, as abstract as a divine construction. As a matter of fact, how else can one propose a general *Measure Theory* of surfaces, the aim of "geo-metry"? If a plane figure has thick borders, which is the surface of the figure?

Thus came this amazing conceptual (and metaphysical) invention, done within the Greek dialogue with the Gods: the continuous line with no thickness. Points – with no

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[2] In my interpretation, existence, in the first axiom, is by *construction* and unicity by *symmetry*.

[3] Schrödinger stresses that a fundamental feature of Greek philosophy is the absence of "the unbearable division, which affected us for centuries… : the division between science and religion," (quoted in Fraisopi, 2009).

dimension, but nameable, as Euclid defines them[4] – are then produced by intersecting lines or sit at the extremes of a line or segment (definition γ). But lines are not composed of signs-points. A line, either continuous or discrete, is a *gestalt*, not a set of points.

Greek geometric figures and their theatrical properties derive by constructions from these fundamental gestalts, signs-points and lines, in a game of rotations and translations, of constructing and breaking symmetries. These gestalts inherently penetrate proofs even now.

3. ON INTUITION

Mathematical intuition is the result of an historical praxis; it is a constituted frame for active constructions, grounded on action in space, stabilized by language and writing in inter-subjectivity.

A pure intuition refers to *what can be done*, instead of to *what it is*. It is the seeing of a mental construction; it is the appreciation of an active experience, of an active (re-)construction of the world. We can intuit, because we *actively construct* (mathematical) knowledge on the phenomenal screen between us and the world.

As for that early and fundamental gestalt, the continuous line, our evolutionary and historical brain sets contours that are not in the world, beginning with the activity of the primary cortex. There is a big gap – actually, an abyss – between the biological-evolutionary path and the historical-conceptual construction; yet, I'll try to bridge it in a few lines.

The neurons of the primary cortex activate by *contiguity* and *connectivity* along non-existent lines and "project" by this non-existing continuous contours on objects (at most, contours are singularities). More precisely, recent analyses of the primary cortex (see Petitot, 2003) highlight the role of intra-cortical synaptic linkages in the perceptual construction of edges and of trajectories. In the primary cortex, neurons are sensitive to "directions": they activate when oriented along the tangent of a detected direction or contour. More precisely, the neurons which activate for *almost* parallel directions, possibly along a straight line, are more connected than the others. In other words, neurons whose receptive field, *approximately* and *locally*, is upon a straight line (or along parallel lines) have a larger amount of synaptic connections among themselves. Thus, the activation of a neuron stimulates or prepares for activation neurons that are *almost* aligned with it or that are *almost* parallel – like tangents

[4] Actually "signs" (σημεια, definition α): Boetius first used the word and the meaning of "point". Note that a sign-point (σημειον) in Euclid is *identified* with the letter that names it (see Toth, 2002).

along a continuous virtual line in the primary cortex. We detect the continuity of an edge by a global "gluing" of these tangents, in the precise geometrical (differential) sense of gluing. More exactly, our brain "imposes" by continuity the unity of an edge by relating neurons which are structured and linked together in a continuous way and locally *almost* in parallel. Their "integral" gives the line (Petitit, 2003).

The humans who first drew the *contours* of a bison on the walls of a cavern (as in Lascaux) instead of painting a brown or black body, communicated to other humans with the same brain cortex and life experience. A real bison is not made just of thick contours as in some drawings on those walls. Yet, those images evoke the animal by a re-construction of it on that phenomenal screen which is the constructed interface between us and the world. The structures of mathematics originate also from such drawings, through their abstract lines. The Greek "limit" definition and construction of the ideal line with no thickness is the last plank of our long bridge: a constructed but "critical" transition to the pure concept (see Bailly & Longo, 2006), far from the world of senses and action, well beyond all we can say by just looking at the brain, but grounded on and made possible by our brain and its action in this world.

Consider now the other main origin of our mathematical activities: the counting and ordering of small quantities, a talent that we share with many animals (see Dehaene, 1998). By language we learn to iterate further; we stabilize the resulting sequence with names; we propose numbers for these *actions*. These numbers were first associated, by common names, with parts of the human body, beginning with the fingers. With writing, their notation departed from just iterating fingers or strokes; yet, in all historical notations, we still write with strokes up to 3, which is given by three parallel segments interconnected by continuous drawing, like 2, which is given by two connected segments. However, conceptual iteration has no reason to stop: it may be "*apeiron*", "without limit", in Greek. Thus, since that early conceptual practice of potential infinity, we started seeing the endless number line, a discrete gestalt, because we iterate an action-schema in space (counting, ordering …) and we *well order* it by this very conceptual gesture. For example, we look at that discrete endless line, which goes from left to right (in our Western culture, the opposite for Arabs (Dehaene, 1998)), and observe "a generic non-empty subset has a least element" (the reader should pause here and observe this in his – enough mathematized – mind). This is the principle of *well-ordering* as used every day by mathematicians. It is a consequence of the discrete spatial construction, a geometric invariant resulting from different practices of discrete ordering and counting into mental spaces. It originates in small counting and ordering that we share with many animals (Dehaene, 1998; Longo, Viarouge, 2010) Further on, in a long path, via language, from those early active forms of ordering and counting objects, arithmetic (logico-formal) induction follows from them rather than founding them, contrary to Frege's and Hilbert's views (see below). The mathematical construction, induction, is the result of these ancient practices, by action and language, and, then, it organizes the world and allows proofs. Yet, it is grounded on a "gestalt", the discrete well-ordering where individual points make no sense without their ordered context.

4. LITTLE GAUSS' PROOF

At the age of 7 or 8, Gauss was asked by his school teacher to produce the result of the sum of the first n integers (or, perhaps, the question was slightly less general …).[5] He then proved a theorem, by the following method. He wrote on the first line the increasing sequence 1,… , n, then, below it and inverted, the same sequence; finally, he added the vertical lines:

$$\begin{array}{ccc} 1 & 2 \;\ldots. & n \\ n & (n\text{-}1)\ldots & 1 \\ \hline (n+1) & \ldots & (n+1) \end{array}$$

Then the result is obvious: $\Sigma_i^n\ i = n(n+1)/2$.

This proof is *not* by induction. Given n, it proposes a uniform argument which works for *any* integer n. Following Herbrand (Longo, 2002), we may call this kind of proof a *prototype*: it provides a (geometric) prototype or schema for any intended parameter of the proof. Of course, once the formula $\Sigma_i^n\ i = n(n+1)/2$ is given, we can very easily prove it by induction as well. But one must know the formula or, more generally, the 'induction load'. Little Gauss did not know the formula; he had to construct it as a result of the proof. On the contrary, we have the belief induced by the formalist myth: that proving a theorem is proving an *already given formula*! We learn, more or less implicitly, from the formal approach, that mathematics is "the use of the axioms to prove a given formula" – an incomplete foundation and a parody of mathematical theorem proving.

Except in a few easy cases, even when the formula to be proved is already given (best

5 This section is partly borrowed from the introduction to Longo, 2002.

known example: Fermat's last theorem), the proof requires the invention of an induction load and of a novel deductive path which may be very far from the formula. In Fermat's example, the detour requires the invention of an extraordinary amount of new mathematics. The same is true also in Automatic Theorem Proving, where human intervention is required even in inductive proofs because except in a few trivial cases, the assumption required in the inductive step (the induction load) may be much stronger than the thesis or have no trivial relation to it. Clearly, *a posteriori* the induction load may be generally described within the formalism, but its "choice", out of infinitely many possibilities, may require some external heuristics (typically: analogies, symmetries, symmetry-breaking, etc.).

More generally, *proving a theorem is answering a question*, like Gauss' teacher's question, about a property of a mathematical structure or about relating different structures; *it is not proving an already given formula*.

Consider a possible way to Gauss' proof. Little Gauss "saw" the discrete number line, as we all do, well ordered from left to right. But then he had a typical hit of mathematical genius: He dared to invert it, to force it to go backwards in his mind, an amazing step. This is a paradigmatic mathematical invention: constructing a new symmetry, in this case by an audacious space rotation or order inversion. That reverse-reflection (or mirror) symmetry gives the equality of the vertical sums. The rest is obvious.

In this case, order and symmetries both *produce* and *found* Gauss' proof. Even *a posteriori*, the proof cannot be founded on formal induction, as this would assume the knowledge of the formula.

4.1 Arithmetic induction and the foundation of Mathematical Proof.

Above, I hinted at an understanding of the ordering of numbers with reference to a mental construction in space (or time). Frege would have called this approach "psychologism", Herbart's style, according to (Frege, 1884). Poincaré instead could be a reference for this view on the certainty and meaning of induction as grounded on intuition in space. In Brouwer's (1948) foundational proposal, the mathematician's intuition of the sequence of natural numbers, which founds mathematics, relies on another phenomenal experience: It should be grounded on the "discrete falling apart of time", as "twoness" ("the falling apart of a life moment into two distinct things, one which gives way to the other, but is retained by memory"; Brouwer, 1948). Thus, "Brouwer's number line" originates from (a discrete form of) phenomenal time and induction derives meaning and certainty from it.

Intuition of the (discrete and increasing) ordering in space and time contributes to establishing the well-ordered number line as an *invariant* of these different active phenomenal experiences: Formal induction follows from and is *founded* on this intuition, in Poincaré's and Brouwer's philosophy. Recent scientific evidence (see Longo, Viarouge, 2010), suggests that we use extensively, in reasoning and computations, the "intuitive" number line as an order in space; those remarkable neuropsychological investigations take us beyond the "introspection" that the founding fathers used as the only way to ground mathematics on intuition. We are probably in the process of transforming the analysis of intuition from naive introspection to a scientific investigation of our cognitive performances, in which the "Origin of Geometry" and the intuition of numbers blend in an indissoluble whole.

I return now to the sum of the first *n* integers and induction. About 80 years later, Peano and Dedekind suggested that little Gauss' proof was certainly a remarkable achievement (in particular for a child), but that adults had to prove theorems in Number Theory by a "formal and uniform method", defined as a "potentially mechanisable" one, by Peano and Padoa. Then Peano definitely specified "formal induction" as *the* proof principle for arithmetic thus defining Peano Arithmetic, or PA (Kennedy, 2006).

Frege set induction at the basis of his logical approach to mathematics; he considered it a founding (and absolute) logical principle, and thus gave PA the foundational status that it still has. Of course, Frege thought that logical induction (or PA) was "categorical" (in modern terms); that is, that induction exactly captured the "theory of numbers" or that everything was said within PA: This logical theory simply coincided, in his view, with the structure and properties of numbers. (Frege didn't even make the distinction "theory vs. model" and never accepted it; the logic was exactly the mathematics, for him.)

The story continues. In The Foundation of Geometry (1899), Hilbert set geometry on formal grounds, as a solution for the incredible situation where many claimed that rigid bodies could be not so rigid and that light rays could go along curved geodetics. Riemann's (1854) work (under Gauss' supervision) had started this "delirium", as Frege called the intuitive–spatial meaning of the new geometry (1884, p.20). Later, Helmholtz, Poincaré and Enriques (see Boi, 1995 & Bottazzini, 1995) developed both the geometry and Riemann's epistemological approach to mathematics as a "genealogy of concepts", partly grounded on action in space.

For these mathematicians, *meaning*, as a reference to phenomenal space and its mathematical structuring, preceded rigor and provided "foundation". Thus, through mathematics, geometry in particular, Poincaré and Enriques wanted to make the physical world intelligible. For them, proving theorems by rigorous tools and conceptal constructions did not coincide with a formal/mechanical game of symbols. Hilbert (1899) had a very different foundational attitude: For the purposes of foundations (but only for these purposes), one has to forget the meaning in physical spaces of the axioms of non-Euclidean geometries and interpret their purely formal presentation in PA. In his 1899 book, he fully formalized a unified approach to geometry and "interpreted" it in PA. Formal rigor and proof principles as effective-finitistic reduction lie at the core of his analysis[6].

On one hand, that geometrization of physics, from Riemann (1854) to Einstein and Weyl, 1949 (via Helmholtz, Clifford and Poincaré, see Boi, 1995), brought a revolution in that discipline, originating by breathtaking physico-mathematical theories (and theorems). On the other, the attention to formal, potentially mechanisable rigor, independent of meaning and intuition, provided the strength of the modern axiomatic approach and fantastic logico-formal *machines*, from Peano and Hilbert to Turing and our digital computers (Longo, 2009).

At the 1900 Paris conference, Hilbert contributed to giving PA (and formal induction) its central status in foundation by suggesting one could prove (formally) the consistency of PA. In his analytic interpretation, the consistency of the geometric axiomatizations would have followed from that of formal Number Theory, with no need of reference to meaning.

Moreover, a few years later, Hilbert proposed a further conjecture, the "final solution", to all foundational problems, a jump into perfect rigor: Once shown the formal consistency of PA by finitistic tools, prove the completeness of the formal axioms for arithmetic. Independent of its heuristics, a proof's certainty had to ultimately be given by formal induction.

However, the thought of many mathematicians at the time (and even now) proposed more than that. That is, in addition to acting as a foundation for "*a posteriori* formalization", they dreamed the "potential mechanization" of mathematics was not only a locus for certainty, but also a "complete" method for proving theorems. The Italian logical school firmly insisted on this with their "pasigraphy": a universal formal language that was a mechanisable algebra for all aspects of human reasoning. Now the "sausage machine" for mathematics (and thought), as Poincaré ironically called it (Bottazzini, 2000), could be put to work: Provide pigs (or axioms) as input and produce theorems (or sausages) as output (traces of this mechanization may still be found in applications of AI or in teaching). The story of complete *a posteriori* formalization and, *a fortiori*, of potential mechanization of deduction ended badly. Hilbert's

[6] For more on the connections between "proof principles" *vs.* "construction principles" in mathematics and in physics, see Bailly & Longo, 2006.

conjectures on the formally provable consistency, decidability and completeness of PA turned out to be all wrong, as Gödel (1931) proved. Gödel's proof gave rise to (incomplete but) fantastic formal machines by the rigorous definition of "computable function". More precisely, Gödel's negative result initiated a major expansion of logic: Recursion Theory (in order to prove undecidability, Gödel had to define precisely what decidable/computable means), Model Theory (the fact that not all models of PA are elementarily equivalent strongly motivates further investigations) and Proof Theory (Gentzen) all got a new start. (Negative results matter immensely in science, see Longo, 2006.) The latter led to the results, among others, of Girard and Friedman (see Longo, 2002).

For Number Theory, the main consequence is that formal induction is incomplete and that one cannot avoid infinitary machinery in proofs (e.g., in the rigorous sense of Friedman, 1997). In some cases, this fact can be described in terms of the structure of "prototype proofs" or of "geometric judgments" (see below), with no explicit reference to infinity.

4.2 More on prototype proofs

> ". . . when we say that a theorem is true for all x, we mean that for each x individually it is possible to iterate its proof, which may just be considered a *prototype* of each individual proof." Herbrand,1930; see Goldfarb, 1987.

Little Gauss' theorem is an example of such a prototype proof. But any proof of a universally quantified statement over a structure that does not realize induction constitutes a "prototype". For example, consider Pythagoras' theorem: one *needs* to draw, possibly on the sand of a Greek beach, a right triangle, with a specific ratio among the sides. Yet, at the end of the proof, one makes a fundamental remark, the true beginning of mathematics: Look at the proof; it does not depend on the specific drawing, but only on the existence of a right angle. The right triangle is *generic* (it is an invariant of the proof) and the proof is a *prototype*. There is no need to scan all right triangles. By a similar structure of the proof one has to prove a property to hold for any element of (a sub-set of) real or complex numbers; that is, for elements of non-well ordered sets. However, in number theory, one has an extra and very strong proof principle: induction.

In a prototype proof, one must provide a reasoning which uniformly holds for all arguments; this uniformity allows (and is guaranteed by) the use of a generic argument. Induction provides an extra tool: the intended property doesn't need to hold for the same reasons for all arguments. Actually, it may hold for different reasons for each argument. One only has to give a proof for 0, and then provide a uniform proof from x to $x + 1$. That is, uniformity of the reasoning is required only in the inductive step. This is where the prototype proof steps in again: the argument from x to $x +1$. Yet, the situation may be more complicated: In the case of nested induction, the universally quantified formula of this inductive step may be given by induction on x. However, after a finite number of nestings, one has to get to a prototype proof going from x to $x + 1$ (induction is logically well-founded).

Thus, induction provides a rigorous proof principle, which, over well-orderings, holds in addition to uniform (prototype) proofs, though sooner or later a prototype proof steps in. However, the prototype/uniform argument in an inductive proof allows one to derive, from the assumption of the thesis for x, its validity for $x + 1$, in any possible model. On the other hand, by induction one may inherit properties from x to x+1 (e.g., totality of a function of x; see Longo, 2002).

As we already observed, in an inductive proof, one must know in advance the formula (the statement) to be proved: little Gauss did not know it. Indeed, (straight) induction (i.e., induction with no problem in the choice of the inductive statement or load) is closer to proof-checking than to "mathematical theorem proving"; if one already has the formal proof, a computer can check it.

5. Induction *vs.* well-ordering in Concrete Incompleteness Theorems

Since the 1970s several examples of "concrete incompleteness results" have been proved.[7] That is, some interesting properties of number theory can be shown to be true, but their proofs cannot be given within number theory's formal counterpart, PA. A particularly relevant case is Friedman Finite Form (FFF) of Kruskal Theorem (KT), a well-known theorem on sequences of "finite trees" in infinite combinatorics (and with many applications).[8] The difficult part is the proof of unprovability of FFF in PA. Here, I am interested only in the proof that FFF holds over the structure of natural numbers (the standard model of PA). FFF is easily derived from KT, so the problems of its formal unprovability lies somewhere in the proof of KT. Without entering into the details even of the statements of FFF or KT (see Harrington, 1985; Gallier, 1991; Longo, 2002), I skip to the place where

[7] Concerning "concrete" incompleteness, an analysis of the nonprovability of normalization for nonpredicative Type Theory, Girard's system F, in terms of prototype proofs is proposed in Longo, 2002.

[8] For a close proof-theoretic investigation of KT, see Harrington, 1985; Gallier, 1991. I borrow here a few remarks from Longo, 2002, which proposes a further analysis.

"meaning," or the geometric structure of integer numbers in space or time (the gestalt of well-ordering) steps into the proof.

The set-theoretic proof of KT (Harrington, 1985; Gallier, 1991) goes by a strong non-effective argument. It is non-effective for several reasons. First, one argues "*ad absurdum*"; that is, one shows that a certain set of possibly infinite sequences of trees is empty by deriving an absurd if it were not so ("not empty implies a contradiction; thus it is empty"). More precisely, one assumes that a certain set of "bad sequences" (sequences without ordered pairs of trees, as required in the statement of KT) is not empty and defines a minimal bad sequence from this assumption. Then one shows that *that* minimal sequence cannot exist, as a smaller one can be easily defined from it. This minimal sequence is obtained by using a quantification on a set that is going to be proved to be empty, a rather non-effective procedure. Moreover, the to-be-empty set is defined by a $\sum_1{}^1$ predicate, well outside PA (a proper, impredicative second order quantification over sets). For a non-intuitionist who accepts a definition ad absurdum of a mathematical object (a sequence in this case), as well as an impredicatively defined set, the proof poses no problem. It is abstract, but very convincing (and relatively easy). The key non-arithmetizable steps are in the $\sum_1{}^1$ definition of a set and in the definition of a new sequence by taking, iteratively, the least element of this set.

Yet, the readers (and the graduate students to whom I lecture) have no problem in applying their shared mental experience of the "number line" to accept this formally non-constructive proof: From the assumption that the intended set is non-empty, one understands ("sees") that it has a least element, without caring about its formal (infinitary, $\sum_1{}^1$) definition. If the set is assumed to contain an element, then the way the rest of the set "goes to infinity" doesn't really matter: the element supposed to exist (by the non-emptiness of the set) must be somewhere in the finite, and the least element will be among the finitely many preceding elements, even if there is no way to present it explicitly. This is well-ordering. Finally, the sequence defined *ad absurdum*, in this highly non-constructive way, will never be used: it would be absurd for it to exist. So its actual "construction" is irrelevant. Of course, this is far from PA, but it is convincing to anyone accepting the "geometric judgment" of well-ordering: "A *generic* non-empty subset of the number line has a least element". This vision of a property, a fundamental judgment, is grounded in the gestalt discussed above.

An intuitionistically acceptable proof of KT was later given by Rathjen&Weierman, 1993. This proof of KT is still not formalizable in PA, of course, but it is "constructive", at least in the broad sense of infinitary inductive definitions as widely used in the contemporary intuitionist community. It is highly infinitary because it uses induction beyond the first impredicative ordinal Γ_0. Though another remarkable contribution to the ordinal classification of theorems and theories, this proof is in no way "more evident" that the one using well-ordering given above. In no way does it "found" arithmetic more than that geometric judgment, as the issue of consistency is postponed to the next ordinal, on which induction would allow one to derive the consistency of induction up to Γ_0.

6. The Origin of Logic

Just as for geometry or arithmetic, mathematicians have to pose the epistemological problem of logic itself. That is, we have to stop viewing formal properties and logical laws as meaningless games of signs or absolute laws preceding human activities. They are not a linguistic description of an independent reality; we have to move towards understanding them as a result of a *praxis* in analogy to our praxes in and of space and time, which create their geometric intelligibility, by their own *construction*.

The logical rules or proof principles have constituted the invariants of our practice of discourse and reasoning since the days of the Greek Agora; they are organized also, but not only, by language. Besides the geometry of figures with their borders with no thickness, which forced symmetries and order in space (our bodily symmetries, our need for order), the Greeks extracted the regularities of discourse. In the novelty of democracy, political power in the Agora was achieved by arguing and convincing. Some patterns of that common discourse were then stabilized and later theoretized, by Aristotle in particular, as rules of reasoning (Toth, 2002). These became established as invariants, transferable from one discourse to another (even in different areas: politics and philosophy, say). The Sophistic tradition dared to argue "per absurdum", by insisting on contradictions, and, later, this tool for reasoning became, in Euclid, a method of proof. All these codified rules made existing arguments justifiable and provided a standard of acceptability for any new argument, while nevertheless being themselves the *a posteriori* result of a shared activity in history.

Much later, the same type of social evolution of argument produced the practice of actual infinity, a difficult achievement which had required centuries of religious disputes in Europe over metaphysics (see Zellini, 2005). Actual infinity became rigorous mathematics (geometry) after developing first as perspective in Italian Renaissance painting. Masaccio first used the convergence point at the horizon in several (lost) Annunciations, mentioned one century later by Vasari, 1998; Piero della Francesca followed his master and theoretized this practice in a book on painting, the first text on projective geometry[9]. The conception of actual infinity enabled mathematics to *better organize the finite*. The advance of discourse helped to conceive infinity, initially as a metaphysical commitment, to be restored in space as a projective limit, a very effective tool to represent three dimensional finite spaces in (two dimensional) painting. Mathematicians later dared to manipulate the linguistic-algebraic representations of such inventions, abstracted from the world that originated them but simultaneously making that same world more intelligible. For example, infinity became an analytic tool which Newton and Leibniz used for understanding finite speed and acceleration through an asymptotic construction. In the XIX century, the extremely audacious step by Cantor (see Cantor, 1955) followed and turned infinity into an algebraic and logically sound notion: he objectivized infinity in a sign and dared to compute on it. A new praxis, the arithmetic of infinity (both on ordinal and cardinal numbers) started a branch of mathematics. Of course, this enrichment of discourse would have been difficult without the rigorous handling of quantification proposed in Frege's foundation of logic and arithmetic (Frege, 1884).

That fruitful resonance between linguistic constructions and the intelligibility of space contributed to the geometrization of physics. Klein's and Clifford's algebraic treatment of non-Euclidean geometries (see Boi, 1995) was crucial for the birth of Relativity Theory[10]. Hilbert's axiomatic approach, since his 1899 book, was also fundamental in this, despite his erroneous belief in the completeness and (auto-)consistency of the formal approach. In addition, physicists, like Boltzmann, conceived limit constructions, such as the thermodynamic integral, which asymptotically unified Newton's trajectories of gas particles and thermodynamics (Cercignani, 1998). Statistical physics, or re-normalization methods, play an important role in today's physics of criticality, where infinity is crucial (Binney et al., 1992). Logicians continued to propose purely linguistic infinitary proofs of finitary statements. The development of infinity is but one part of the never-ending dialogue between geometric construction principles and logical proof principles. It started with projective geometry, as a mathematization of the italian invention of perspecive in painting, first a

[9] Masaccio and Piero invented the modern perspective, in Annunciations first (1400-1450), by the explicit use of points of converging parallel lines. As a matter of fact, the Annunciation is the locus of the encounter of the Infinity of God with the Madonna, a (finite) woman (see Panovsky, 1991). Later, "infinity in painting", by the work of Piero himself, became a general technique to describe finite spaces better.

[10] Klein and Clifford, also stressed the role of symmetries in Euclidean Geometry: it is the only geometry which is closed under homotheties. That is, its group of automorphisms, and only its group, contains this form of symmetry.

praxis, a technique, in art. It is a subset of the ongoing historical interaction between invariants of action in space and time and their linguistic expressions, extended also by metaphisical discussions (on infnity), originating in human inter-subjectivity, including the invariants of historical, dialogical reasoning (logic). These interactions produce the constitutive history and the evolving, cognitive and historical, foundations of mathematics.

Conclusion

In my approach, I ground mathematics and its proofs, as conceptual constructions, in humans' "phenomenal lives" (Weyl, 1949): Concepts and structures are the result of a cognitive/historical knowledge process. They originate from our actions in space (and time) and are further extended by language and logic. Mathematics, for example, moved from Euclid's implicit use of connectivity to homotopy theory or to the topological analysis of dimensions. Symmetries lead from plane geometry to dualities and adjunctions in categories, some very abstract concepts. Likewise, the ordering of numbers is formally extended into transfinite ordinals and cardinals.

In this short essay, I have tried to spell-out the role of prototype proofs and of well-ordering vs. induction. I insisted on the role of symmetries both in our understanding of Euclid's axioms and in proofs; I stressed the creativity of the proof, which often requires the invention of new concepts and structures. These may be, in most cases, formalized, but *a posteriori* and each in some *ad hoc* way. However, there is no Newtonian absolute Unverse nor Zermelo-Fraenkel unique, absolute and complete set theory, nor ultimate foundations: this is a consequance of incompletenness (see Longo, 2011). More deeply, evidence and foundation are not completely captured by formalization, beginning with the axioms: "The primary evidence should not be interchanged with the evidence of the 'axioms'; as the axioms are mostly the result already of an original formation of meaning and they already have this formation itself always behind them", (Husserl, 1933). This is the perspective applied in my initial sketchy analysis of the symmetries "lying behind" Euclid's axioms.

Moreover, recent concrete incompleteness results show that the reference to this underlying and constitutive meaning cannot be avoided in proofs or in foundational analyses. The consistency issue is crucial in any formal derivation and cannot be solved within formalisms.

After the early references to geometry, I focused on arithmetic as foundational analyses have mostly done since Frege. Arithmetic has produced fantastic logico-arithmetical machines – and major incompleteness results. I have shown how geometric judgments penetrate proofs even in number theory; I argue, *a fortiori*, their relevance for general mathematical proofs. We need to ground mathematical proofs also on geometric judgments which are no less solid than logical ones: "symmetry", for example, is at least as fundamental as the logical "*modus ponens*"; it features heavily in mathematical constructions and proofs. Physicists have long argued "by symmetry". More generally, modern physics extended its analysis from the Newtonian "causal laws" – the analogue to the logico-formal and absolute "laws of thought" since Boole, 1854 and Frege, 1884 – to understanding the phenomenal world through an active geometric structuring. Take as examples the conservation laws as symmetries (Noether's theorem) and the geodetics of Relativity Theory.[11] The normative nature of geometric structures is currently providing a further understanding even of recent advances in microphysics (Connes, 1994). Similarly, mathematicians' foundational analyses and their applications should also be enriched by this broadening of the paradigm in scientific explanation: from *laws* to *geometric intelligibility* (we discussed symmetries, in particular, but also the geometric judgement of "well-ordering"). Mathematics is the result of an open-ended "game" between humans and the world in space and time; that is, it results from the inter-subjective construction of knowledge made in language and logic, along a friction over the world, which canalizes our praxes as well as our endeavor towards knowledge. It is effective and objective exactly because it is constituted by human action in the world, while by its own actions transforming that same world.

Acknowledgements. Many discussions with Imre Toth, and his comments, helped to set my (apparently new) understanding of Euclid on more sound historical underpinnings. Rossella Fabbrichesi helped me to understand Greek thought and the philosophical sense of my perspective. My daughter Sara taught me about "infinity in the painting" in the Italian Quattrocento. The referees proposed a very close revision, as for English and style.

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[11] See Weyl (1927) for an early mathematical and philosophical insight into this. For recent reflections, see van Fraassen (1993); and Bailly & Longo (2006).

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