Mathematical Intelligence, Infinity and Machines: beyond the Gödelitis*

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Abstract

We informally discuss some recent results on the incompleteness of formal systems. These theorems, which are of great importance to contemporary mathematical epistemology, are proved using a variety of conceptual tools provably stronger than those of finitary axiomatisations. Those tools require no mathematical ontology, but rather constitute particularly concrete human constructions and acts of comprehending infinity and space rooted in different forms of knowledge. We shall also discuss, albeit very briefly, the mathematical intelligence both of Our Good Almighty Lord and of Computers. We hope in this manner to help the reader overcome formalist reductionism, while avoiding naive Platonist ontologies, typical symptoms of the Gödelitis which affected many in the last 70 years.

0 Introduction

When one thinks of the foundations of the mathematical form of "intelligence", of the ways in which it comprehends and describes the world, it is "the rule" which comes to mind, or in fact the "regulae ad directionem ingenii", the fundamental constitutive norms of mathematics and of thought itself. It is in particular during this century that the foundational analysis of mathematics has focussed on the analysis of mathematical deduction. This in its turn has been based on the play of logical and formal rules as described perfectly well by one of

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the best and most rigorous scientific programmes of our times, namely the formalist programme of the foundations of mathematics (and of knowledge). The influence of this programme in the analysis of human cognition has been enormous: in the '30s formal computations and machines were rigorously defined, for the purposes of the investigation of formal deduction. And there began the modern adventure of the "computational mind" or the the various functionalist approaches to cognition. Reflections on the notions of mathematical proof and certainty, which started the project, may help in its revision.

Now, if there is no doubt that the notion of "proof" does indeed lie at the heart of mathematics and that this proof must follow (or must be able to be reconstructed by) "rules", yet mathematics is not just about proofs, and moreover it should be noted that these exhibit a wide variety of methods of construction, of reference and meaning... . Proofs and constructions are carried on with the greatest of rigour, however there is no reason to think that in mathematics rigour should be understood only as the application of rules *without meaning*. That, as we shall see, is the central hypothesis of the formalist programme: certainty is obtainable only in the absence of meaning in the course of a deduction which, for that very reason, must be "finitary" and "potentially mechanisable"; any reference to meaning may lead to semantic ambiguities, may involve "intuition" ... . Moreover, there is no doubt that mathematics is "normative", as far as it follows (or it may be reconstructed by) "rules", and that it is "abstract" and "symbolic". Yet, these complex notions do not need to be identified or reduced to "formal", in the restricted sense given in this century. A fortiori, then, this reduction does not need to apply to general human reasoning and intelligence.

It is not a question of denying the logical component of mathematical deduction, namely the "if...then" constructions which are always present, any more than the role of deduction conceived of as a pure calculation: many proofs in algebra or in logical systems based on "rewriting" techniques (or purely syntactic rules of "deductions-as-calculations") are simply non-self-evident sequences of applications of rules of formal manipulation. As such, these rules may appear to be "without meaning"- their application in the proof must at no stage make reference to possible meanings (for example, one can develop an equation and reach a surprising result without ever "interpreting" the equation in possible analytic spaces, but rather using only rules of algebraic computation)[1].

[1] This is welknown in Algebra. Relevant examples, related to the theories analysed here may

These two components of mathematical thought, namely logic, a system of rules of thought with meaning (Boole, Frege, Russell...), and formalism, a system of rules of calculation without meaning (Hilbert, Bernays, Post, Curry...), therefore make up an essential part of the proof. The problem is that they are not sufficient for the foundational analysis of mathematics. The search for foundations and "certainty" exclusively within these systems has been a kind of *unilateral diet* of most foundational reflections; for the logicists, who are fundamentally opposed to the formalists, often look for the foundation only in "signifying" logical rules, while the formalists want only to use calculation free of the ambiguities of signification, even logical signification. Both approaches then basically exclude each other as well as other forms of foundation (and rigor), such as reference to "meaning" broadly construed, geometric meaning, for example. This diet, a knid of philosophical obsession which has tried to reduce the mathematical project to a single conceptual level, has led in particular to the lack of analyses of the role of meaning in the construction and foundation of mathematical concepts (and proofs): the role played for example by the concept of mathematical infinity (and its "constructed meaning", see later), the phenomenal reference to space and time, or to those "actual experiences" which not only root mathematics in a plurality of forms of knowledge about the world (I will refer for this to Poincaré, Husserl, H. Weyl, Enriques ...), but which also enter into the proof itself, as I shall try to demonstrate. These constitutive elements of mathematical thought and practice must now become the object of genuine scientific analysis and no longer be consigned to the shadowy realm of "intuition", that black hole of the ontological mystery (a typical symptom of the gödelitis[2]), last recourse of the "working mathematician" who, fed up with the rules constraining his thinking, all too often resorts to a naive Platonism which allows him/her to dispense with these logico-formal systems which restrict the scope of his/her practices and are incapable of justifying them.

In short, I will try here to point out how (provably) non-formalizable arguments, yet based on robust, abstract, symbolic and ... human conceptual constructions, step in proofs even in Arithmetic, the core theory of the formalist

be found in the introductory textbooks: Hindley J.R., Seldin J.P. *Introduction to Combinators and Lambda-Calculus*, Cambridge University Press, 1986, and Baader F., Nipkow T. *Term Rewriting and all that.* Cambridge University Press. 1998.

[2]The virus of the gödelitis was first isolated, by naming it, by one of the leading authors mentioned below, whose friendship, I hope, will not be affected by the use of the word in this paper.

program. By this, one may avoid the dilemma between the Scilla of mechanicist formalism and the Cariddi of the naif platonist answer, largely prevailing in mathematical and philosophical circles of this century.

The presentation is meant for the general reader, in particular for the non-mathematician who is tired or puzzled of hearing that some incompleteness results "prove" either that all mathematics can be actually computerized, since man, in the act of proving, is not better than a Turing Machine, or, alternatively, that mathematics is "God-given". The purpose then is "reclaiming cognition" to our human being, as we developed one of our most fantastic conceptual constructions, mathematics, in the full rigor of a living process throughout history, yet rooted in deep cognitive process.

1 Incompleteness of formal systems: from Gödel to Girard

One of the major successes of mathematical formalism has been its ability to show its own limits—using its own methods of proof! Such a result is both remarkable and rare among the sciences, where 'crises' usually arise 'from the outside', by a change of paradigm and/or method.

Later on, we shall present various theorems of the last thirty years (often called of "concrete incompleteness" for Arithmetic or Formal Number Theory), proved by methods which do exit the given formal system, and for which it is *proven* that one must leave that system, in which the proposition is stated, in order to prove it. Note that this also applies to logical systems (Frege, Russell), since their principles are inscribed into the formal systems in question.

The truth is that a single level of expression, a fixed formal language, does not allow a complete representation of the mathematical structures in question, such as the natural numbers, for example. In other words, one shows that the mere manipulation of arithmetic symbols cannot be 'complete', cannot capture all the properties of a conceptual construction of great mathematical importance and product of innumerable experiences, even in the apparently simple case of (natural or integer) numbers. Or again, that the 'definitive certainty' of reasoning, sought by the original foundationalists, can not reside in a single conceptual level, that of linguistic formal systems, which should have allowed the codification of the 'multidimensional space' of mathematical construction, independently of meaning as reference to the underlying mathematical structure (e.g. the integer number line or the finite ordinals). Or, also, that some methods or 'conceptual tools', borrowed from other mahematical experiences or grounded on the intended meaning, cannot be 'removed'.

We take 'number theory', or Arithmetic, as an example in order to show the way in which even forms of intelligence as apparently 'isolated' as this one, actually make use of also non-formal tools of understanding the world. This, in my opinion, is the true meaning of the various 'incompleteness theorems' in mathematics. In particular, the interaction between the plurality of linguistic levels and the concept of infinity in mathematics will be at the centre of our analysis.

But what do we mean by logical/formal calculus, and the axiomatic method of Hilbert? First we must create a language of *well-formed formulae* using a precise syntactical structure, which is defined *independently of meaning*, or, more exactly, by juxtaposing letters and symbols in a specified manner ("A `and` B" is well-formed; "A `and` " is not). Then we must identify certain well-formed formulae as axioms, and fix some rules of deduction, where the passage effected by the rule from hypotheses to consequences, is based *exclusively* on their syntactic structure. If, for example, we posit the formulae A and $A \Rightarrow B$, then B follows mechanically—without reference to the possible meanings of A, $\Rightarrow$, and B. (Of course, if one interprets $\Rightarrow$ as *implies*, then we recover the classical *Modus Ponens*; though this 'meaning', which is human and historical, is unnecessary for the formal deduction.) This, then, is the level of formal language, or object of study, at which one may analyse proofs.

In particular, Hilbert's program relies on a distinction between the *mathematical* (or theoretical) level, called 'object level', expressed in a language of formulae, and a *meta*mathematical (or metatheoretical) level, i.e. the 'mathematics' thanks to which one is able to talk about the object level. On top of this, a 'third level' (or new conceptual dimension) was added by Tarski in the 1930's: the semantic structures which allow the interpretation of formulae and formal operations. (Although the mathematicians of the previous century had already laid the foundations for such a distinction, for example, Argand/Gauss re the complex numbers, and Beltrami/Klein re non-Euclidean geometries). Thus, to take an example which deals with the geometric significance of algebraic formulae:

- $(x = \sqrt{-1})$ is a *formula* in the formal language of algebra, i.e. at the object or mathematical level of study;
- '$x = 2$ and $x = 5$ are contradictory' is a *metalinguistic phrase* — it affirms a *property of formulae*, the fact they are contradictory; and lastly,
- the interpretation of the x in $(x = \sqrt{-1})$ by a point in the cartesian plane furnishes a *geometrical semantics* for that algebraic formula, and, more generally that of all complex numbers (the Argand-Gauss interpretation).

In the case of Arithmetic, according to Hilbert, it was necessary to prove, working at the metamathematical level, that the object-level language, entirely formalised by axioms and rules of deduction, allows all the formilazable properties of the integers to be proven (that is the hypothesis of the 'completeness' of the formal system). In other words, that axioms and finitary rules can wholly account for mathematical deduction; that, in a certain sense, they wholly account for mathematical intelligence and certainty, or more precisely, that they are able to reconstruct these *a posteriori*, and give them a rigorous logical/mechanical foundation by providing them with a formal framework.

But why is Arithmetic granted such importance? For several reasons: firstly, the natural numbers constitute an 'elementary' starting point; at the same time, their theory is at the very heart of mathematics. (Remember that Cantor and Dedekind gave definitions of the real numbers in terms of integers.) Moreover, Frege had remarkably set the logical foundations of mathematics in terms of the natural numbers in the 1880's, in the texts *Ideography*, and the *Foundations of Arithmetic*.

It is worth recalling the enormous clarification this program brought about, as much by its methodological rigour as by the fact that it subsequently made possible the reification of the logical/arithmetic intelligence thus defined, in the form of prodigious electronic machines. For, once the logic has been translated into a formal system and the meaning forgotten, the machine simply needs to be taught to compare sequences of letters; when A and $A \Rightarrow B$ are given, just check *mechanically* if the two occurences of A are "identical" and then write B. 'Pattern matching' (letter-by-letter comparison), or the search for a common syntactic structure of formulae which are not trivially correlated (a process called 'unification' or identification modulo certain syntactic transformations) is indeed at the very heart of mechanical reasoning and, in our own times, proof by computers. With the appearance of machines, the idea that the formal level (calculation using signs without meaning) could express human intelligence in its entirety took on its modern form, going far beyond the pretensions of most of the original foundationalists, who aimed more modestly at the rigorous but *a posteriori* reconstruction of mathematics, constituting its only formal foundation. Certain thinkers even went so far as to affirm that "intelligence is to be defined as that which can be manifested by means of the communication of discrete symbols"[3], by means of meaningless manipulations of these discrete symbols.

Nevertheless, this program which aimed at the codification of mathematics and its formal foundation, did not take long to fail, precisely because of Arithmetic itself, and Gödel's (justifiably) notorious Incompleteness Theorem. It should be said, however, that the relevance of this theorem is limited by the absence of 'explanation' it furnishes of the incompleteness phoenomena. The subsequent discoveries, to which I would like to introduce the reader, can help us to understand better this game of intelligence which interests us, at the level of mathematical proof. So before proceeding to other, newer, and more informative results, I will try to add a few words *à propos* this classic, which has become the object of innumerable presentations and reflections, some of which have become quite popular, like those of Hofstadter and Penrose. Who knows if by trying—insofar as possible—to be concise, and by working in a level of informal rigour, we will be able to avoid those transcendant and ontological elements which (mis)lead many to believe that 'in Mathematics, there are propositions which are true, but not provable', while at the same time leaving this notion of truth vague and mysterious. All we ask is that the reader pay careful attention to each word in the next few dozen lines for if we affirm that 'one proves that the proposition $\mathcal{G}$ is unprovable', for example, this phrase must be understood with particular care: what is said here is simply that, once a certain system of axioms and rules of deduction are fixed, one may prove that, inside this system, $\mathcal{G}$ can not be proven, i.e. deduced using its axioms and rules. Some close attention is required, since in these contexts, one often uses words and phrases which refer to themselves ('proving unprovability', or 'this phrase is unprovable'); yet, what I am saying is absolutely informal and literal: the mathematical proof itself being long and extremely technical.

[3] J. Hodges, in R. Herken (ed.) *The Universal Turing Machine*, Springer-Verlag, 1995

Recall in the first place that the First Incompleteness Theorem of Gödel (1931) states solely (I repeat, *solely*) that there exists a proposition or formal Arithmetic phrase which is *undecidable* in the framework of Arithmetic, *under the assumption* that the latter is consistent (i.e. not self-contradictory). It is an 'undecidability' theorem, in the sense that it gives us a proposition, call it $\mathcal{G}$, which can not be proven by the formal theory, and moreover, whose negation can also not be proven by the formal theory. In the statement of the theorem and in its proof, absolutely nothing is said about the truth of $\mathcal{G}$ or its negation (which of them is true?). The Second Incompleteness Theorem then proves that, *within Arithmetic*, one can demonstrate the logical equivalence of the proposition $\mathcal{G}$ and the formalised statement of consistency. As a consequence, since $\mathcal{G}$ is unprovable if Arithmetic is consistent, also the consistency of Arithmetic itself cannot be proven, by "arithmetic tools". More precisely, no finitary metamathematics, which is as such numerically 'codifiable' (see below), can prove the consistency of Arithmetic[4]. This unprovability property (of its own consistency) moreover, can be extended to *any* mathematical theory, which is sufficiently expressive to allow the codification of its own meta-theory (note the interaction between the theoretic and metatheoretic levels here.)

The above is one of the keys of Gödel's proof: the remarkable idea of *numerically codifying Arithmetic formulae*, by means of a laborious but conceptually simple technique (later called "gödel-numbering"). Once this has been achieved, formulae which describe properties of numbers, for example $(x+4=1+x+3)$, can 'speak' about formulae, in the sense that they can be applied to the numeric codes of formulae. If, for example, $(x+4=1+x+3)$ has numeric code 76, then it implies in particular that $(76+4=1+76+3)$, which is a fact about an instance of 'itself', where 'it' is understood as 'formula #76'. In fact the codification of a (well-formed) formula does not depend in any way on its 'meaning', but only on its syntactic structure: the finite sequence of symbols of which it is composed. But, recall, syntactic structure is sufficient for the formal analysis

[4]A formal theory is consistent if, in its language, one may write an unprovable sentence. Thus, by the Second Theorem for Arithmetic, consistency is unprovable exactly when it is assumed to be true. Or, consistency is unprovable (in Arithmetic) if and only if Arithmetic *is* consistent. In Gödel's theorems one has to make a subtle distinction between consistency and "omega-consistency": a technical nuance which is outside the scope of a discussion like the present one. A detailed presentation of Gödel's theorems may be found in Smorinsky C. "The incompleteness theorems", in Barwise J. (ed.) *Handbook of Mathematical Logic*, North-Holland, 1978.

of deduction.

By this, and in a very specific sense, Arithmetic formulae can "speak" about themselves, or even about their own properties. Consistency, for example, is a property of formulae, *and therefore* a property of numbers, once given that numbers, as codes, can be put in one-to-one correspondance with formulae. From then on *meta*mathematics, which studies the properties of formulae, becomes a sub-field of Arithmetic. Briefly, as a mathematical theory, Arithmetic describes its own metatheory. This is a mathematically difficult observation, which has opened the door to so much beautiful mathematics—and to so much extravagant speculation: Arithmetic speaks or refer to itself or, even, it is 'conscious' of itself, or about the infinite regressions of self-references, as if looking in facing mirrors, etc.. Some of the most severe patologies related to the gödelitis, for which, of course, the immense Gödel has no responsibility whatsoever.

Now, Arithmetic *is* consistent, in the sense that one cannot derive a contradiction from its axioms and rules (and thus the proposition $\mathcal{G}$ holds, as it is provably equivalent to consistency). The proof of consistency, however, must be made *outside* of Arithmetic, as a consequence of the two theorems of Gödel; in other words, one can not do it with 'purely formal reasoning', which is mechanisable and hence codifiable within Arithmetic. Again, one can not do it in a *formal theory*, i.e. a theory within which one avoids the *meaning* of axioms and rules, or, as we will try to explain later on, within which an ordinary axiom can not 'speak of infinity' nor of the standard order structure of numbers, or where the rules do not mix the theoretic, metatheoretic, and semantic levels. As already mentioned, if one assumes consistency or proves it, then it becomes banal to observe that $\mathcal{G}$ is true: the Second Incompleteness Theorem actually proves the equivalence between the two, inside Arithmetic. Indeed, there is no method to *affirm* the truth of $\mathcal{G}$, other than specifying a notion of truth for formulae and *proving* it to be true. Thus, in order to know that $\mathcal{G}$ is true one must assume or prove consistency, to which it is provably equivalent[5].

[5]There is a simple 'classical' argument which *proves* the truth of $\mathcal{G}$, *under the assumption* that Arithmetic is consistent and once the First Incompleteness Theorem has been shown. It is just a naive paraphrasis of one implication in Gödel's Second theorem, which derives formally $\mathcal{G}$ from consistency, i.e. *within* Arithmetic. In summary, by the previous note, $\mathcal{G}$ is unprovable if and only if (consistency is unprovable if and only if Arithmetic *is* consistent if and only if) $\mathcal{G}$ is true. Note then that a biconditional, "$\mathcal{G}$ is unprovable if and only if $\mathcal{G}$ is true" (remarkable, isn't it?), is *not* equivalent to an "and"; it implies an "and" if one "assumes" or proves consistency (or $\mathcal{G}$). In truth, though, one should also say what "truth" means for a formula of Arithmetic, exactly.

Now whither the famous ontological mystery, as claimed by Platonists who reject the formalist program? '$\mathcal{G}$ is true, but not provable' means nothing but '$\mathcal{G}$ is not provable in the framework of Arithmetic, if one assumes consistency' and, under this *necessary* assumption, one can prove its truth; or, more precisely, '$\mathcal{G}$ cannot be proven except by techniques stronger than Arithmetic—those which allow one to prove consistency' (and we shall see what these may be). In mathematics, when one affirms that something is true, *in one way or another*, one must (explicitly define "truth" and) *prove that it is true*, and that is all. This is what one should analyse, i.e.*how* one may prove things outside a specific formalism, before stooping to theological arguments. One should research what non-mechanisable, non-arithmetical forms of reasoning (dare we say 'forms of intelligence'?) allow the necessary kind of proof.

As a matter of fact, the statements *and the proofs* of Gödel's theorem do not ever mention any notion whatsoever of "truth", either magical or mathematical. They are an absolutely remarkable game of codes for formulae, formal fixpoints equations, explicit computations (Gödel invented "programming in Arithmetic", an amazingly difficult and original challenge) ... . At most, one may understand the statements by observing that they prove a "gap" between formal proofs and various possible notions of truth (Tarski's, Kripke's ... there are many) over the (standard) model of Arithmetic (the natural numbers). That is, that formal provability differs from any reasonable *definition* of truth for arithmetic formulae, in particular if this definition assumes that any proposition is either true or false (the so called "tertium non datur"). The further shift, from this *gap between* a *notion of truth* and *formal provability*, to the God given "set of true, but unprovable sentences", is just some sort of medieval confusion between mathematical provability and ontological arguments. As a matter of fact, in mathematics, it is not the existence of entities or objects that matters, but the objectivity of mathematical/conceptual constructions.

The results discussed below will better explain that this gap is actually between formal (arithmetical) provability and other forms of conceptual constructions, proper to mathematics.

Consistency of Arithmetic was soon proven by Gentzen, in 1934. However, his proof presupposes an extremely strong form of induction (that of 'transfinite' order), which made it unconvincing to many: the very essence of Arithmetic being ordinary induction (that of finite order)[6]. This is the reason it went largely unnoticed, even though its technique of 'cut-elimination' went on to become a pillar of Proof Theory. This proof can be given in a set theory which includes an axiom affirming that 'there exists an infinite set'. This axiom does not have any sense unless one 'understands' the meaning of 'infinity': its codification by an Arithmetical predicate is impossible since Arithmetic cannot 'say anything' about infinite sets (nor even " single out" the finite sets or the standard integer numbers). Gödel also gave a proof of the consistency of Arithmetic in 1958, in an interesting logical calculus (an extension of Church's $\lambda-$calculus of the '30s, a "Theory of Types and Proofs" called system T), which also uses transfinite induction.

We owe a more enlightening proof of the consistency of Arithmetic to (Tait and) Girard, in 1970. It is given in a framework, called system F, similar to, but much more expressive than, Gödel's system T; system F is still an extension of $\lambda-$calculus, but with second-order impredicative types[7]. Girard proves, 'à la Tait', a 'normalisation' theorem, which implies consistency, using, among others, a principle called 'second-order comprehension' which combines, in an unavoidable way, theory, metatheory and semantics. I will try to explain this very informally, by abusing of easy and short ways to render difficult concepts and techniques. At a certain point in the proof, one takes an infinite set of terms (this is a *metatheoretical* operation: one collects terms of the object level, which one perceives 'from on high', i.e. metatheoretically, an "easy" operation for us human beings, along the proof, as we do it in our ordinary language). Then, one puts them in the place of a set-valued variable inside a *term*—which amounts to saying, one works at the *theoretical level* here. Thus in the course of the proof itself, one mixes metalanguage, and even semantics, with the language of terms, since the operation can be done only if one agrees to interpret formal, set-valued variables as actual sets (the so called "semantic convention" of the

[6] Arithmetical Induction is nothing other than the assertion that if one can show (or if one assumes) $A(0)$ and, writing "$\forall$" for "for all", one shows (or assumes) that $\forall y(A(y) \Rightarrow A(y+1))$, then one can conclude $\forall y A(y)$. In an equivalent fashion, if $\forall z((\forall x < zA(x)) \Rightarrow A(z))$, then one can again deduce $\forall z A(z)$. Transfinite induction allows an infinite number of hypotheses, i.e. it may be informally understood as interpreting z (and x) above as 'infinite numbers' (called transfinite ordinals). Gentzen used transfinite induction over a restricted set of formulae, not all formulae of Arithmetic - a key point.

[7] A mathematical notion is impredicatively given (is *impredicative*), when one uses a totality in order to define, by this notion, an element of that very totality.

axiom of second-order comprehension)[8].

This passage of proof (it is not the only one) is not codifiable within Arithmetic, and Girard proves it, by showing that his theorem implies the consistency of Arithmetic and hence cannot be proven inside Arithmetic. The reasoning is impeccable, comprehensible, and human: one needs to understand the blend of meta/theory/semantics to carry on the proof and no purely formal/mechanical/finitary account of it can be given. In a certain sense it corroborates what Wittgenstein already claimed in the 1930s: that the distinction between mathematics and metamathematics is fictitious. More accurately, it is a fine and technically convenient conceptual distinction, for the pruposes of the proof-theoretic analysis of mathematics, but as humans, we may move freely, by means of the interaction of language and meaning, between one level and another, just as we do every day in real mathematics (or in ordinary language: e.g., "I never say true sentences with more then 34 words" ... a typical mixture of the levels above). Treatment by a single linguistic level does not allow this type of interplay, no more than the reasonings which concern it: the Tait-Girard and Gödel Theorems (and an observation of Tarski) prove it. Now, digital machines do not function above the formal linguistic level, the level which can be codified by sequences of zeroes and ones. Our human language is, on the other hand, a dynamic construction, built in a permanent resonance with meaning, and so is mathematics and its proofs. By this, they are in a position to capture at once language, metalanguage, and meaning, which is *demonstratably* undoable by formal/mechanical means, i.e. by giving a finite coding technique by a discrete set of meaningless signs. If one wants to keep them purely formal, i.e. mechanically manipulable, this immediately gives the distinct levels of metalanguage and semantics. If you try to encode these levels, by further formal

[8]For a unifying approach to both Gödel's and Girard's normalisation theorems, see Girard J.-Y., Lafont Y., Taylor P. *Proofs and Types*, Cambridge U.P., 1989. Type Theory, the frame of Gödel's '58 work as well as Girard's, is elegantly and deeply related to Category Theory, *the* theory of mathematical structures (see Lambek J., Scott P.J. *Introduction to higher order Categorical Logic*, Cambridge U. Press, 1986, and Asperti A., Longo G. *Categories, Types and Structures*, M.I.T.- Press, 1991). A very interesting categorical understanding of normalisation may also be found in Cubric D., Dybjer P., Scott P. "Normalization and the Yoneda embedding" *Mathematical Structures in Computer Science*, vol. 8, 2, 1998. Among the uncountably many applications and developments of the normalisation theorems, a technically intriguing one may be found in Castagna G., Ghelli G. and Longo G. "A calculus for overloaded functions with subtyping", *Information and Computation*, 117(1):115–135, 1995, which has had some fall-out in the mathematics of programming, as the work of many in Type Theories (the author's papers are downloadable from `http://www.dmi.ens.fr/users/longo`).

signs ... the game starts over again, by further metalanguage and semantics.

In truth, this is the central point: by definition, a digital calculator must codify (everything) in symbols, roughly 0 and 1, and the encoding can depend neither on meaning nor implementation. Hodges' definition, mentioned above, applies very accurately to *mechanic* deduction: '(mechanical) intelligence ... is effectively defined as that which can be expressed by the communication of discrete symbols', and this codification following the requirements of the functionalist hypothesis *must not* depend on the specific *hardware* which realises it. The goal of high-level programming languages is precisely to be transferable from one computer to another, from one programming environment to another, without problems. That is only possible because their level is exclusively formal/theoretic, codifiable with finite sequences of symbols, and neither depends nor should depend on any (ordinary) meaning any more than on contexts. Human intelligence, on the other hand, depends on the structure of our brain, the fact that it is housed in *our* cranial cavity, and the complexities of its biological and cultural history: it is a rich blend of invariant, general laws and contextual meanings. It bears no rigid distinction between "language" and "metalanguage"; moreover, the meaning of the processing which occurs at any point in the brain depends also on *where* the elaboration takes place, on its geometry, on the *type* of the preceding neurons. And so on, up to ... the actual position of one's hands—as much because of the rôle the hands played in the evolving complexity of our cerebral cortices, as for the more historical fact that one understands body language (in particular, hand-waving), no matter who is the Italian speaking! For some, this contextual dependence, rooted in evolutionary, social and cultural history, can represent a limit, when in fact it is about a richness: the "gesture" of the mathematician, who tries to explain the "construction of a limit", refers to a deep and shared conceptual construction (e.g. the notion of actual infinity used) and it is inscribed irreducibely into the proof. This gesture does not make reference to an "ontology", but to a constitutive route through the history of mathematical knowledge, it is a essential part of the metaphors that yield the conceptual invariant. The actual challenge is to understand how do we get to a (relatively) stable invariant, yet grounded in our material, contextual lifes[9]. In some cases, the conceptual invariant results

[9] Some more references and discussion may be found in Longo G. "Géométrie, Mouvement, Espace: Cognition et Mathématiques" *Intellectica*, 2, n. 25, 1997. and in Longo G. "Mémoire

from a stability gained through intersubjective exchange, rich in meaning: its formal representation is a remarkable 'attempt' of capturing its expressiveness, but it is essentially incomplete.

2 Infinity and Proofs

In proving the consistency theorems mentioned above, the use of the notion of infinity turns out to be inevitable, and this fact is made even more explicit in other, more recent "incompleteness theorems". If I continue to speak of the use of *actual infinity* in the theory of integers (Arithmetic) it is deliberately to "play into the enemy's hand": it would seem too easy to maintain that the proof of theorems about infinite-dimensional differentiable manifolds (which are very abstract spaces), require mathematics to be able to speak of infinity; but what proof theory has taught us is that even our good old positive integers sometimes require the concept of (actual) infinity, if one takes as a frame the usual set-theoretic approach[10].

There are other properties stemming from Arithmetic, which are not codifications of metatheoretic properties like consistency, but actual properties of numbers (like, for all x there exists a y such that $(6 + x = y + 2)$ —just a little more complicated), which can be shown to be *unprovable* by finitary techniques, i.e. by deductions that can be codified inside Arithmetic. But one *can* prove that these formulae are true for the natural numbers, by proofs which use "infinity" in an essential way, in a set-theoretic perspective. Our goal here is to reflect on the manner in which these theorems place the mathematical proof in a plurality of forms of intelligence, not just "formal", in particular the concept of infinity is applied.

The Paris-Harrington theorem (PH) and the "Friedman Finite Form (of Kruskal's theorem)" (FFF) are two arithmetic statements of the type "for all x there exists a y such that blah, blah, blah...". Here "blah, blah, blah..." is a property of numbers which may be complicated, but not overly so[11]. Both (PH)

et Objectivité en Mathématiques", Colloque *Le réel en Mathematiques*, Cérisy, Septembre 1999 (actes à paraître).

[10]Yet, other approaches may be followed, see the forthcoming footnotes: we hint, in the text, to the mainstream set-theoretic approach, but different proofs of the same unprovable statements, may be non-arithmetizable for different reasons.

[11]See Paris J., Harrington L., "A mathematical incompleteness in Peano Arithmetic", in Barwise J. (ed.) *Handbook of Mathematical Logic*, North-Holland, 1978. As for (FFF), see

and (FFF) *imply* the consistency of Arithmetic, by a proof *within* Arithmetic; but, given that consistency, when formalised as a proposition of Arithmetic, can not be proven in Arithmetic, neither can either of the two statements. Moreover, both statemetns describe more or less "concrete" properties of numbers (partitions or "coloring", as for (PH), inclusions of finite trees, as for (FFF)[12])

Even though unprovable in Arithmetic, (PH) and (FFF) are *true*. For a mathematician who is speaking of interesting propositions (FFF, in particular, is very interesting—it is a variant of a well-known theorem by Kruskal which is rich in applications), that means that (s)he can prove them *and can mean nothing else*; alternatively, that (s)he possesses convincing techniques for deducing the truth of these propositions, relative to the structure in question: the natural numbers. These techniques, and I am now thinking particularly of (FFF), base themselves essentially on the order-structure of the natural numbers, a geometric or an infinitary property, and on sequences of finite and infinite "trees": they involve making "instruments of proof" out of our mathematical experience of reasoning about the well-ordered sequence of natural numbers, and confronting simple planar (infinite) structures, trees, by inclusion, and describing the difference between finite and infinite[13].

Harrington, L. et al. (eds.) *H. Friedman's Research on the Foundation of Mathematics*, North-Holland, 1985.

[12]A partly informal and simplified statement of (FFF) may be given as follows. In mathematics, trees have a "root", "branches" and "nodes" and may be included one into the other, in a roughly ordinary sense. Then (FFF) says: "For any n, there exists an m such that for any sequence of finite trees $T_1, T_2, ..., T_m$, such that each T_i has at most n(i+1) nodes, there exist j and k such that $j < k < m+1$ and T_j is included in T_k." Finite trees may be coded by numbers, thus the statement is a formal statement of (first-order) Arithmetic, a Π^0_2 statement of the arithmetical hierarchy, to be precise.

[13]In a comparison, observe that the proposition $\mathcal{G}$ of Gödel, in section 1, affirms that there "does not exist a proof of $\mathcal{G}$" (nor of its negation) or that $\mathcal{G}$ (more accurately: its numeric code) is a solution in x of the following equation (x = *the code of "there does not exist a proof of the proposition coded by x"*). This is a very fine game between metatheory (the notion of proof), theory (the formal proposition $\mathcal{G}$) and semantics (the integer numbers, where x must be found). Yet, it is "artificial" or ad hoc (it is the arthmetic coding of a metatheoretic statement), it is not an "interesting" property of numbers per se or of finite codable structures, such as trees, say. Finally, the statement and its informal meaning suggest why $\mathcal{G}$ must be unprovable, if Arithmetic is consistent: this is exactly what $\mathcal{G}$ says! The difficulty of the theorem entirely lies in the construction of $\mathcal{G}$ and not in the proof of its unprovability nor in its truth, if consistency is assumed. Indeed, the "apparent evidence" of G has had a major misleading role in many philosophical reflections. This is not so for statements such as (FFF): its truth is far from "evident" and, yet, it is a "ordinary" theoretical expression about numbers as codes of finite trees (no self-reference, no metatheory involved). Its formal unprovability is extremely hard to be proved; it uses an ordering of trees by inclusion and transfers its properties, by isomorphic immersion, into the order of (very large) transfinite ordinals. This, following Gentzen, implies the consistency of

This is no miracle in the truth of (FFF): it is quite simply the "laborious conquest" of a proof handling (countable) infinities, organised in particular as the totally (well-)ordered set of the natural numbers, or as a partially ordered set of "nodes" and "branches" of trees[14]. As indicated in the footnote, that is a relatively easy proof, but formally underivable: but ... why must foundation be solely formal/mechanical? Do we not have other things to say about mathematical proof? What is the *constitutive process* of the "certitude" of mathematical proof, even outside formalisations? The formalist program is only a component, necessary and important, but provably incomplete, of the analysis of proof. Moreover, on the analysis of the formal consistency, it achieves nothing without appealing to larger and larger infinities (see section 3): a conceptual abyss which must in turn be "founded".

So I speak here of the "laborious conquest" of the notion of infinity, since clarity about the infinite is a conceptual conquest which has been needed in mathematics for centuries; the analysis of this constitutive process is, for me, an essential subsequent component of foundational analysis.

We know in effect how the Greeks hesitated when faced with infinite sequences of converging points ($\sqrt{2}$; Zeno's paradox) and how they laboriously, even anguishedly, distinguished between potential infinity, and a more indefinite infinity, negatively defined by Euclid and Aristotle (*apeiron*), though the latter provided some degree of clarity on the difference between the two. Not until Thomas Aquinas, Duns Scotus, and the late Middle Ages did a real change of scientific paradigm take shape. Infinity became a positive attribute, characteristic of God and possibly created by God (infinity *in actu*, as opposed to infinity *in fieri*). As for the mathematics of the sixteenth to eighteenth centuries, we recall that it developed in a milieu of uncertainty, but with a forever increasing audacity with regard to the use of the notions of actual infinity, and of limit, which no

Arithmetic.

[14]The truth of (FFF) is not so obvious, as it requires a "simple", yet smart proof, under classical, but strong assumptions. Briefly, the set-theoretic proof of (the truth of) (FFF), via Kruskal's theorem, uses an "oracle" on a Σ^1_1 set, an impredicative and extremely non-effective infinitary construction. This proof is relatively easy: one or two pages which any mathematician today could reconstruct without much pain (indeed, some fun). The analysis of this "easyness" has been one of our motivations. On the other hand, as we already said, the proof of its unprovability (and its essential impredicativity) is very difficult; it is a major technical breakthrough obtained by Friedman. (The complete statment, the proofs and other technical remarks about (FFF) may be found in the reference above, edited by Harrington - in particular, two papers by Smorinsky in that volume brilliantly explain the role of impredicatively given sets, the place where syntax and semantics are entangled.)

longer represented a negative concept, but on the contrary, a manner of calculation (cf. Pascal, Galileo, Cavalieri, Newton, or the metaphysics of Leibniz)[15]. This culminated in the great casualness yet conceptual clarity of Cantor's treatment of infinity. His "paradise of infinities" constitutes another turning-point, one which has truly marked our mathematical era. Cantor introduced infinity in the context of operations: addition, multiplication, and iteration of limits on infinities of infinities. Such juggling with infinities of excessive scale, even led to paradoxes. A century of mathematical work accompanied by an increasing refinement of techniques and also an increasing solidity of definitions was needed. Today a mathematician really knows what it means to "give a proper definition", above all in the difficult cases which imply infinitude, thanks mostly to the stubborn effort of logicists and formalists! This allows us in the present to promote infinite trees, the well-order of the natural numbers, and ordinals beyond them, to the title of daily instruments of proof; and this, without falling into the same errors and paradoxes, to which the audacious "founding fathers" stepped in. This mathematical praxis, finally, allows a rigorous *definition*, in different contexts, of infinity, i.e. exactly that which was undefined, according to Euclid and Aristotle. (Cantor really was a mathematician who dared to conjecture and prove the most surprising observations about infinity; truly, he dared to "think beyond infinity"[16].)

[15]Zellini P. *Breve storia dell'Infinito*, Adelphi, 1980 (trad. francaise, Seuil, 1986); Gardies G. *Pascal entre Eudoxe et Cantor*, Vrin, 1884.

[16]For us today, Cantor's transfinite Arithmetic is nothing particularly difficult. After counting $0, 1, 2, 3, \ldots$, we use ω to denote the "limit" of this process, its "closure" on the horizon. Then we continue counting $\omega + 1, \omega + 2, \omega + 3, \ldots \omega + \omega = \omega \times 2$. Similarly, $\omega \times 2, \omega \times 3, \ldots \omega \times \omega = \omega^2$. By now, the rule of the game should be clear, and one continues to apply exponents: $\omega^2, \omega^3, \ldots, \omega^\omega$. The limit of $\omega^\omega, \omega^{\omega^\omega}, \ldots$ should simply be "ω to the ω", ω times. This "ordinal" number is called ε_0. It is the least solution of the equation $x = \omega^x$. If one succeeds in proving (or if one assumes) that these ordinals are "well-ordered", then one can prove that Arithmetic is consistent (this is Gentzen's 1934 proof). The statements (PH) and (FFF) which we have mentioned imply the well-ordering of ε_0, and much more, for evidently one can also create $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_\omega$ and so on so forth, up to "huge" ordinals (whose *individual* definition requires a reference to the entire *collection* of ordinals: the "impredicatively defined" ordinals). In fact, this is a game children and mathematicians often play: give me a number, and I give you a bigger one. But the game is not arbitrary as is "takes meaning" in an interesting geometric structure, the well-ordering of the natural numbers. The game only *extends* this order, by extending the operations of sum, product, exponentiation, and "limiting" beyond ω. The deed of giving a name to infinity, in a coherent manner, has been inscribed in the conceptual area we call Mathematics. The ordinal ω is not of this world, but neither is it a convention nor a mere symbol; rather, it synthesizes of a *construction principle*, a "disciplined gesture" grounded in a historically rich mathematical practice (cf. Longo, G., "The mathematical continuum, from intuition to logic" in *Naturalizing Phenomenology: issues in contemporary Phenomenology and Cognitive Sciences* (J.

In other words, the order of the numbers, in space or time, their succession and extension into a discrete and well-ordered structure, i.e. their ordered extention beyond the infinity of the numbers themselves, the planar structure of possibly infinite trees, are all infinitary or geometric properties which allow the proof of these finitary arithmetic propositions, which *require* treatment beyond formal systems codifiable within Arithmetic. Evidently, one must *speak* of these structures, and the proofs which rest on them, with human words of finite length, but one can not do so in a "complete" manner with *formal calculi* which can be manipulated without reference to meaning: this, no more no less, is what all these results tell us; the *historical praxis* of infinity, the order according to which we organise, mentally or in the plane, numbers and trees (possibly infinite), are part of the foundations, are the extremely solid roots of unmechanisable methods of proof. This amounts to saying that in these proofs, in the feat of making hypotheses and passing from one line to the next, these lines certainly consisting of finite words of our language, "meaning" steps in, in a *provably* essential way. That is, along the proof, one must *understand* the concept of actual infinity or the geometric structure of well-ordering; or that one should be speaking of infinite sets of numbers or infinite trees in the plane, by situating oneself in a mental geometry, where one may "pick up" the least element of an arbitrary non-empty subset of the natural numbers (an absolutely infinitary operation, in set-theoretic terms). Although at the end of the day, these proofs should themselves be described by finite words, *for passing from one phrase to another*, in at least one step along the proof, it is therefore necessary to grasp that *behind* these words lie the significance of infinity or of orders, *provably uncodifiable* in Arithmetic. Or, in other words, neither codifiable nor manipulable by sequences of finite symbols without meaning, by a Turing Machine, say, as its computation could be equivalently expressed within Arithmetic . Thus, these recent incompleteness results show that even "simple" properties of the integers are provably true using tehniques which can not be represented at the solely theoretic level, codifiable as such with zeroes and ones, or with other mechanisable techniques[17]. The difference w.r. to Gödel's theorems should be

Petitot et al., eds) Standford U.P., 1999, for subsequent considerations.)

[17]The firm, but naif, formalist may still say: yes, but then, this proof, still given by finite sets of words, may be "formalized" in a "suitable" Set-Theory or in second-order Arithmetic! Of course, one may add the independent statement as a new axiom, but this is cheating and just moves forward the problem (by Gödel's technique one may give further independent statements). The point is that sufficiently expressive, but not "ad hoc" theories, such as

clear. As already mentioned, that classic is only an undecidability theorem: it gives no hint on how to prove $\mathcal{G}$ or the consistency of Arithmetic, to which it is provably equivalent. This helped to fall into the mysticism of an unproved truths, that "the mathematician could see by looking over the shoulder of God" (Barrow) - a very convenient position, indeed. In the latter, concrete, cases, the authors had to *prove* the truth of the arithmetic statement in question, actually on standard numbers; of course, this, jointly to unprovability, implies undecidability, but it is stronger.

Thus, such proofs have no need of reference to ontological miracles evoking "inaccessible mathematical truths". We humans are absolutely not constrained to reason with no reference to meaning and only with finite formal sequences codifiable in zeroes and ones, which is exactly what computers do, or, equivalently, to deduce only from "pattern matching" (if A is syntactically identical to the A in $A \Rightarrow B$, or it may be made so, then write B). Our rigor is not simply formal/linguistic: for example, we construct a praxis of "infinity" in different conceptual frameworks, and we make it into a rigorous mathematical (or geometrical) concept, admittedly a difficult acquisition which required centuries of work[18].

proper II order ones, are not "formal", in the rigourous sense of the hilbertian tradition: they are infinitary and use tools from infinitary logics or impredicative definitions. The set-theoretic treatment mentioned above, by the use of a Σ^1_1 formula, or the derivation in second-order Arithmetic (provably) require impredicative notions. By this, syntax and semantics get mixed (first, by the so-called second order convention in the comprehension axiom) and, as for Arithmetic, validity, as truth in all models, is no longer "effective", i.e. the set of valid propositions is not recursively enumerable (thus, they cannot be derived by any machine whatsoever, by Church Thesis). For this, and for the related use of totalities in order to define elements, the firm, but coherent and competent formalists reject proofs in these impredicative frames (see, for example, the life-long work of S. Feferman - for a collection of papers see Feferman S. *In the light of logic*, Oxford University Press, 1998 - or Simpson S. *Subsystems of Second Order Arithmetic*, Springer-Verlag, 1998).

[18]There is another way to understand how meaning steps in the proof of these recent incompleteness results. The three statements mentioned above (normalisation, (PH), (FFF)) may be all formalized in Arithmetic in the form $\forall x \exists y.P(x,y)$, that is "for all x, there exists y.P(x,y).", where P is a decidable predicate. In each case, $\forall x \exists y.P(x,y)$ is unprovable in Arithmetic. Yet, from (the proof of) its truth over the natural numbers, one may easily derive that "for all n, Arithmetic proves $\exists y.P(n,y)$ ". There is subtle but crucial distinction here, which is a "semantical" one: in order to prove, in Arithmetic, the statement $\forall x \exists y.P(x,y)$, the variable x has to range *only* on the natural numbers. That is, the proof of the universal statement "for all x ..." may be given just if interpreting x as a *generic* natural number, with no use of formal induction (similarly as one would prove a statement such as "for all r, real number, ... such and such a function is continuous ...", where induction on r is not possible and the proof should be given for a "generic" real). In other words, the fact that n in $\exists y.P(n,y)$ must be a generic natural number and *not a formal variable* (which could be interpreted also in non-standard models) is crucial to the proof and forbids first order

It is difficult to speak of the above, for one of the goals of this century's mathematical logic, has been exactly to "avoid infinity" in foundational analysis, even if one must consider it pertinent to the practice of mathematics: it is too dangerous to be foundational. Yet, infinity is a central element which today we can rest on, thanks to, in part, results obtained in mathematical logic. And infinity constitutes precisely one of these mathematical concepts, which, in order to have meaning, needs a plurality of references to other forms of "intelligence", considered even in their historical evolution, by the very fact that it has been given in different forms of knowledge.

In conclusion, with reference in particular to the incompleteness phenomena, we would argue that *in mathematics a concept is proposed, a method is chosen, structures are built, theorems are proven, specifying where and by what means, in a manner which is certainly not arbitrary, as "context of proofs and meanings" are provided. In mathematics there exist no propositions which are "true and unprovable" and, at the same time, mathematics is not simply mechanisable calculations, since each time it is a question of proving, if necessary using infinitary methods or meaningful derivations, if and within what framework such and such a proposition is unprovable, and if and in what framework it is provably true.* Those who claim, in a mystic tone, that there are propositions which are "true but unprovable" must give an example of one, by singling it out: but, in doing so, they will also have to prove it, that is to say, prove its truth within a well-defined construction. That is just what had to be done, by Gentzen or normalization, for Gödel's $\mathcal{G}$ (or consistency of Arithmetic, to which it is formally equivalent), for (PH) and (FFF) with regard to the natural numbers, as well as for the "Continuum Hypothesis" and the "Axiom of Choice", each in a different set theoretic construction, as hinted next[19].

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induction over arithmetic formulae. As a matter of fact, this "meaningfull" property (n is a natural number), easy for us, provably cannot be formalized in Arithmetic (a consequence of the so called "overspill lemma" in the model theory of Arithmetic); in Set Theory, its proof requires infinitary assumptions.

[19]An alternative proof of (FFF) can be derived form the work in Rathjen M., Weierman A. "Proof theoretic investigations on Kruskal's theorem" *Annals of Pure and Applied Logic*, 60:49-88, 1993. The authors give an infinitary, non arithmetizable proof, yet much more "constructive" than the set-theoretic one, mentioned here (their proof is "intuitionistically" acceptable, at least by "open minded" intuitionists ... .) The proof uses the constructive theory of Inductive Definitions and avoids a (classically) crucial passage "per absurdum" of the set-theoretic approach. Infinity also is used in a slightly different or finer way: the infinitary ordinal structure mentioned above shows up, in the inductive proof, in a minimal way, with no use of Σ^1_1 sets, but "just" of least impredicative ordinals.

3 Infinity and Metaphors

The foundation of infinity has been largely advocated by (formal) Set Theory. However, I believe that today there can be no *a priori* foundation or *a posteriori* formal justification for mathematical infinity other than 'metaphors". Actual infinity does not belong to our sensory experiences, be it direct or indirect; nor even to the practice of counting or of classical geometry based on shapes: for that, potential infinity will do, both with regard to adding "+1" and endless motion. The historical praxis of actual infinity, which I have mentioned several times, constitutes a further progress: the limit of this unending "+1" must be conceived, the horizon closed, while positing a point at the "limit". This practice is, as I have stated, established on the basis of innumerable reflections situated between the mystical and the emotional: starting with the Greeks, continuing with the controversies surrounding the infinite grace of Mary (Duns Scotus's actual infinity), the discussions of perspective in painting (the vanishing point at infinity of Renaissance painting, which is an entirely artificial construct, one among many other possible kinds), the anchorage in monads and the metaphysical infinity of Leibniz which aimed to give sense to the infinitesimal calculus. And it is precisely this "constitution" of the concept of actual infinity, so rich in emotivity sedimented over the centuries and *rendered "objective' by mathematical practice*, which has become an essential part of its mathematical specification, by the very fact of its entry into proofs. Nevertheless, mathematical infinity is not the same as the different mental experiences of infinity mentioned above, since it is *the invariant concept* which we posit after all of these various experiences. At the same time, however, its foundation is

This paper focuses on the notion of infinity, as a crucial tool of the existing analysis of proofs. However, a further approach should be more closely explored, if possible. Instead of forcing (formal) induction, by stretching it along the ordinals, we should just rely on the order-structure of natural numbers and use generic elements (see the proof by generic n in footnote 18). A different philosophical attitude, giving up the absolutely central role of formal induction (since Peano and Frege), can lead to a more "finitary" approach, yet non formalisable, as referring to an essentially geometric argument (the order of natural numbers, which is, formally, fully expressed only by second order impredicative principles: every non-empty subset as a least element). But spelling this out may be a difficult, if ever possible, project, which would allow to propose an alternative to the deep analyses given in the set-theoretic frame and, perhaps, even to the more constructive approach by Inductive Definitions. These approaches "explain" natural numbers by a (very interesting) detour via infinity. Infinity is fine and good, a beautiful and very human conceptual construction, but perhaps, in Arithmetic, it may be replaced by a geometric insight into numbers and an analysis of proofs also by generic elements (which involve meaning along the deduction).

to be found precisely in the (vectorial) sum of these experiences, each one being a metaphor, an opening onto other forms of knowledge and other forms of intelligence[20].

In reality, it has been a vain effort to try to give a foundation to infinity using methods within mathematics, let alone within some formal system or another. As we have already recalled, it was Cantor who first defined and treated infinity with a wholly mathematical rigour, objectifying it in notation and in computation, and, furthermore, unifying it in a theory of (possibly infinite) sets. Frege was to give this theory a rigorous logical form, which was corrected later on and formulated into a formal Set Theory, under the influence of the Hilbert school. But, ever since the start of this adventure, which was to change the face of mathematics, the infinite had posed serious problems.

Cantor had developed his theory with the aim of analysing the continuum of the real numbers: he proved that these are strictly more numerous than the natural numbers; and his proof suggests how one can carry on building ever larger infinities, indefinitely. He then spent many years trying to prove the *Continuum Hypothesis*: that the infinite number of real numbers, their *cardinality* as he termed it, is the "immediate successor" to that of the natural numbers. He failed in his attempt, as did his successors, Zermelo, Bernays, von Neuman and many others. All of these formalisers of Cantorian Set Theory were also unable to prove the derivability of another key property of infinity: the *Axiom of Choice*, which affirms that one can choose an element for each set belonging to a (possibly infinite) collection of sets.

These properties of infinity, now well-defined as a mathematical concept, seem to escape formal treatment by set theory: thanks to a result of Gödel in 1938 and to a theorem of Cohen's (1963), it would be proven that formal Set Theory had not managed to say anything about the Continuum Hypothesis or the Axiom of Choice. Or, in other words, these propositions are undecidable

[20]The notion of metaphor used here is close to Lakoff and Nunez's (see Lakoff, G. and Nunez, R. *Where Mathematics Comes From: How the Embodied Mathematics Creates Mathematics*. New York: Basic Books, 2000.): it is in fact the meaning of a certain concept, in our case the concept of mathematical infinity, which, as I will try to explain, is constructed with reference to a plurality of metaphors. Their notion of "conceptual blend", at the core of the unity of our forms of knowledge, as a the permanent "transfer of meaning and conceptual practices" nicely underlies or allows the "vectorial sum" of different constructions I refer to here. Before getting acquainted with the approach proposed by Lakoff and Nunez, I was influenced by the use of metaphors in the conceptual constructions of mathematics, by a remarkable book on the philosophy of mathematical physics: Chatelet G. *Les enjeux du mobile*, Seuil, 1993.

in the Formal Set Theory, as is Gödel's proposition $\mathcal{G}$ in Arithmetic. One might then ask whether they are true. One must then *prove* if and within what framework they are true, just as for Gödel's proposition: if one then proceeds to a certain construction of a universe of sets, this construction being due to Gödel, one proves that the Continuum Hypothesis and the Axiom of Choice are true in this universe; if one goes on to another construction, namely Cohen's, both of them are shown to be false in this other set-theoretic universe. Those who believe in mathematical propositions which are true in God's spirit, who believe in absolute and undemonstratable mathematical truths, would then have to say whether they believe that, for God, the Continuum Hypothesis and the Axiom of Choice are true or false. All that we humans can say is that, for example, Gödel's construction is "simpler", in the sense which is suggested to us by certain regularities in the world, such as the principles of "minimal" structures: but to do this, it is necessary to specify what one understands by "minimal" in this precise context, to propose an infinitary mathematical construction, and to prove, within it, the truth of the Continuum Hypothesis and the Axiom of Choice. Now, although it is not "minimal", even Cohen's construction is not arbitrary: it uses a notion of "generic element" which is very relevant in mathematics (and in Arithmetic, as we noticed in a footnote).

To summarize briefly the question of infinity in formal set theory, the independence of the Continuum Hypothesis and the Axiom of Choice proves that, if one remains at the level of the formal system, without it taking on or before it even takes on a meaning in a mathematical construction, then this formal theory, though created in order to be able to speak of these crucial properties of mathematical infinity, remains once again completely silent. But the formal theory would have to be at least formally consistent (in particular, if one adds to it the Continuum Hypothesis and the Axiom of Choice as axioms). A consequence of (certain extensions of) Gödel's incompleteness theorems is that this consistency cannot be proven—without assuming the ability to construct further infinities, extremely large cardinal numbers, which go beyond the formal theory whose consistency one aims to prove. That is, if one wants to prove the consistency of a formal Theory of Sets which may express a given infinite (cardinal) alpha, say, then one *needs* "to assume the existence" of a cardinal number beta, strictly larger than alpha.

There is nothing metaphysical about this, nor is any ontology of mathematical infinity involved: what one is stating is simply that, if we are capable of making, or we assume that we are capable of making certain conceptual constructions of "very large" infinities (by iterating power-set operations, limits and much more), then, using these same constructions, we can prove the consistency (*construct models*) of such and such a set theory; and this is what certain difficult results of Set Theory of recent decades have led to.

Recall now that, conscious of the difficulties involved with infinity, a number of mathematicians/logicians from the beginning of the century, Hilbert and Brouwer to name only two, had sought to exclude actual infinity from foundational theories. Hilbert had recognised the centrality of infinity in mathematics; more importantly, he had affirmed the indispensable character of this notion for mathematical thought, and claimed that the mathematician should work in Cantor's "paradise of infinities". However, Hilbert also claimed that, in order to guarantee the certainty of reasoning, a finitistic foundational analysis must be its basis, since "operations on the infinite can only be guaranteed using the finite as a basis". That is, the mathematics of infinite and ideal objects had to be saved by a finitistic metamathematics, the frame for consistency proofs. (Two writings of 1925, published as appendices to the French edition of *The Foundations of Geometry*, as well as in van Heijnoort[21] are fine elaborations of such a programme.)

By contrast, as we have seen, infinity reappears, not only in consistency proofs, but, in view of the recent incompleteness theorems, even in proofs of "proper" statements of theories of the finite par excellence such as Arithmetic. And this is so in an essential manner, at least within the well established frames mentioned here, Set Theory and intuitionistically acceptable theory of Inductive Definitions, see footnote 19. In other words, not only are we unable to guarantee the consistency of theories which speak of infinity using the finite, as in the case of different set theories (a key aim of the formalist program), but we can even need infinity to prove, by induction and/or within Set Theory, consistency of Arithmetic as well as certain finitary statements of Arithmetic, such as (PH) or (FFF) mentioned above.

Nothing too bad so far, precisely because the certainty of our use of infinity is, in my opinion, extremely "robust", or as much as mathematics itself: it stems from the interaction and mutual support of numerous mental and historical

[21] *From Frege to Gödel*, Harvard University Press, Cambridge, 1967.

experiences which can even come from outside mathematics. Its conceptual solidity stems from its rootedness in a plurality of conceptual constructions which have allowed us to conceive of, propose and define the mathematical invariant gradually across history. I return to this plurality one last time, since it is the nub of the whole issue: mathematical infinity is not a metaphor, but our very proposal of an invariant (stable) concept constructed on the basis of (and as an invariant w.r.t.) a plurality of mental experiences including religious metaphors, the vanishing point in perspectival painting, distant points of convergence—to name but a few conceptual practices which work with (and ground by a praxis) infinite mathematical sets and structures.

Gödel's splendid 1931 theorem left us with a dramatic metaphysics of mathematics: because it concerns "only" a theorem of undecidability (it says nothing about how one proves the undecidable proposition in Arithmetic, or the consistency which is formally equivalent to it), for decades people have not stopped talking about absolute "truths" of mathematics, of "looking over God's shoulder", instead of going and studying the many beautiful proofs of consistency elaborated since 1934. Since undecidability does not help us to understand what methods of proof might exist outside Arithmetic (by simply stating that certain propositions cannot be proven in Arithmetic), the debate has become trapped in a Manichean conflict: on the one side those who say that the limits of man (the "human computor", in the act of proof) are the same as those of the machine; on the other, those who have hymned the praises of the manifold mysteries of mathematics, precisely because, in sticking uniquely to Gödel's theorems, which says nothing on how to prove the unprovable statement (consistency), one passively accepts the notion of demonstrability (and even mathematical rigour!) as being exclusive to formal systems, indeed the mechanisable ones: the rest belongs to the obscure realm of intuition. A closer analysis of more recent undemonstrable statements of Arithmetic, but demonstrably true, to which I have just alluded, may allow the "foundation" of a crucial practice, namely the rigorous use of the mathematical concept of infinity. A use to which, for now, I can give no other foundation than a practical, historical, and yet extremely solid one, which makes reference to metaphorical meanings, in a blend of conceptual practices.

4 Metaphors and analogies, between intelligence, emotions and affection

Before introducing a daring discourse on forms of mathematical intelligence and their relation to "affectivity", I would like to open a parenthesis. In the act of disseminating the foundations of mathematics into a variety of forms of knowledge and intelligence, in this attempt to analyse their evolutionary and historical genesis, I, as logician/mathematician, am explicitly violating one of the most established dogma of the philosophy of science of this century, the dogma which forbids any confusion between "genesis" and "foundation", "creativity" and "deduction", "logic of discovery" and "internal rationality" of a discipline. But it is precisely "this dogma of the principial fracture between epistemological elucidation and historical explanation [...] between epistemological origin and genetic origin [...] [which must] be overturned completely", as Husserl stated in the so rarely read *The Origin of Geometry* (1933-36). This is a crucial question concerning the whole of scientific knowledge, because in each of these forms of knowledge there exists the difficult interplay between epistemological autonomy—on the level of logic and internal justification, and the genesis of the knowledge—on the level, amongst other things, of its relationship to other forms of knowledge and the course of its evolution and history.

There is no such a think as the set of "universal Laws of Thought", with no genesis and at top of which lie the perfectly formal rules of Mathematics. The hierarchy which has developed in our cultures represents a distorting mirror of human intelligence, whether this be in the idea of the logical formalism as the only (or ultimate) form of rationality, of the only form of scientific method (which are, in the last analysis, those of formalised mathematics) or in the idea of unbreachable compartmental boundaries. Instead, we have to discern the elements of continuity and the links between the different forms of intelligence, the different human forms of relationship to the world, and some of the different forms of scientific knowledge. For example, Damasio[22] gives the point of view of the neurophysiologists on the question. Indeed he sets out to explain the neurophysiological discoveries on the basis of which affectivity and intentionality are shown to be integral parts of rationality. For him, that is where Descartes' error lies, namely in his separation of the rational soul from the emotional soul

[22] *Descartes' Error*, 1994.

which, for the neurophysiologist, is completely impossible.

I think, thus, that we have to enrich this analysis of meaning and of human intelligence as grounded in our active existence of living beings, from our biological reality up to the dialogue between humans in history. Furthermore, I would like to add that intentionality and affectivity are not only essential "stimuli" for intelligence, a point on which everybody would agree, but that they affect the very content of intelligence. Or rather, that intentionality, affectivity and the emotions are not simply the possible bait for or the possible break of the "rational" machine, but that they help to determine its direction, and therefore its content. Personally, I see things as follows.

The understanding of a fact, up to and including the conjecture of the mathematician, are based on analogies, metaphors, and consequently on choices of direction in the representation and the contents of the conceptual construction, whose meaning is rich with affectivity: one proposes, chooses and understands an analogy, a metaphor in view of or because they have an emotional or affective content; one is therefore led by intentionality. In other words, one *chooses* to "build a certain bridge" between different kinds of knowledge, between different kinds of "intelligence" (and that is what constitutes analogy and metaphor), on grounds which are both affective and emotional or intentional (they have aims). That is why the very content of a "rational" practice, which is based on metaphors and analogies, is rich in intentionality and emotions, because through the direction given to the metaphor or the analogy-"bridge", they contribute to its determination. Now, this infinity, a key concept and unavoidable notion of a mathematics which remains the stronghold of rationality, can only be understood as the invariant concept of a plurality of conceptual experiences, both practical and emotional, running from religious metaphors to the metaphor of depth in painting, via the convergence of parallel lines at the horizon, and the limit of iterated movements. It is a kind of vector resulting from a set of vectors, a construction which is therefore different from all the given vectors, but nevertheless always dependant on them, be this in terms of direction or contents.

To conclude, Descartes provided us with an important intellectual clarification by helping us to found the modern scientific method and by purging "reasoning", amongst other things, of the residues of magic, of the empty logic and syllogisms of medieval times, and of pervasive religious mysticism. Now that we have understood and, on the whole, know fairly well how to put all of that into practice in scientific frameworks, we are in a position to balance the schism with which he endowed us with his bequest of a "method" and "rules *ad directionem ingenii*". This will allow us to go further, to understand better with the aid of scientific analysis, and to put our finger on those particularly difficult points where rationality and affectivity become confused and yet form the basis of one another reciprocally, in other words where a plurality of forms of human intelligence mix together, even in mathematics. This is particularly needed as its formal foundation, beautifully developped in Mathematical Logic, is essentially incomplete; in particular, it lacks the analysis of meaning as embedded in human intentionality.

5 Induction, machines, and the Lord Almighty

In this research programme, mathematics can lend valuable assistance, because if we manage to break its permanent siege-status, in other words the absolute and separate role ascribed to it, if we manage to destroy the ivory tower in which Platonists and formalists want to shut it up, it will be able to provide us with a good example of, among other things, a relevant cognitive practice. A relatively simple example, since even when profound and difficult, mathematics nevertheless remains conceptually simple: elegance and conceptual necessity are its watchwords, and among its *raisons d'être*. Elegance and necessity are combined with constructions of a profoundly human nature. To such a degree that (but don't say this too loudly) the Lord Almighty and computers are largely incapable of doing mathematics. The former is unable to keep the planets, which are nevertheless His greatest creation, in orbits which are sections of cones, as Kepler recommended. More precisely, the orbits of our own planets do not integrate a system of differential equations. Occasional omnipotent flicks are needed in order to sustain our orbits about the sun, as intuited by Newton[23]. This, if I dare say it, is more a pragmatic than a mathematical solution (surely not a limitation of His omnipotence: He just decided to organise matters in a different way from Kepler's expectation). And let us hope that He continues to make such large gestures, because there is no stability theorem for the solar

[23] Newton discovered that the orbits of the planets affect one another reciprocally by mutual gravitational attraction, and in particular that effects of "gravitational resonance" could even endanger the stability of the solar system—his profound religiosity provided him with that solution, the only one which can garanty stability for good (see the next footnote).

system[24]. Yet, we soundly try in every possible manner to describe its movement approximatively using the difficult instruments of the best language we possess for speaking about the movement of the bodies, curves and geometries of space, namely the mathematics of dynamical systems; but we only manage to obtain qualitative descriptions of the behaviour of systems of chaotic determinism.

As for digital machines, as hinted above, they cannot demonstrate the consistency of Arithmetic, via normalization say, or (FFF), because meaning (or an "impredicative" blend of syntax and semantics) is *provably* essential to the proof and, thus, formal-artithmetizable reasoning does not suffice; but much more should be said, as they are incapable of even giving proofs by arithmetical induction which scarcely reach beyond the banal. I shall explain what this means. By arithmetic induction, as we have already recalled, we mean the following finitary rule: if one proves $A(0)$ and if one proves that, "for all x, $A(x) \Rightarrow A(x+1)$", then one can deduce from this that "for all x, $A(x)$". What could be more mechanisable? Well, there are proofs which, though hardly complex, use this rule in an "a priori" non-mechanisable fashion, as it is relatively rare for one to manage to make the inductive step or to proof that "for all x, $A(x) \Rightarrow A(x+1)$", where A is exactly the proposition you want to prove. In a number of significant cases, a proposition B has to be found, stronger than A or such that $B \Rightarrow A$, and for which, by contrast, one manages to prove that "for all x, $B(x) \Rightarrow B(x+1)$". Proposition B is called the "inductive load". B can be much more complex than A. There exists no *a priori* criterion for choosing B, excepting a few vague heuristic indications according to which the inductive load must "contain everything which is needed", on the basis of the hypotheses, the structure of the proof one is constructing, and the thesis aimed for. In actual fact, the choice of B among an infinite number of possibilities is based on analogies or on setting bridges or embedding in broader mathematical frames. An analogy, for example, with a proof already found, which can be algebraic even when one is working in geometry, or, indeed, an analogy with an induction on the number of dimensions inspired by another, very different proof based on the length of the formulae etc.,etc... Analogies and bridges between different

[24]See the results in Laskar J., *The chaotic behaviour of the solar system*, Icarus, 88: 266-291, 1990: the orbit of the Earth is provably impredictable beyond 100 milion years. Similarly for the solar system, as a whole, beyond 1 milion years; just nothing when considering the expected life of the Sun (5 or more bilion years?) - the point is that Pluto's orbit is "very chaotic".

forms of knowledge, normally within mathematics, I would say, but not always, for the analogy, just like the metaphor, can easily take us outside this[25].

Of course, none of this affects the foundational programme of the formalist, who in such cases can always reconstruct *a posteriori* the logical/formal framework of the proof in which he will quite simply replace A with B where this proves necessary. But, this nevertheless constitutes an insurmountable obstacle for the logico-computational hypothesis, since there exists no machine which is capable of choosing B in the context of a new proof, out of infinitely many possble inductions loads, while an analogy or an intentional, meaningful choice may suggest it.

The question of the "inductive load" now represents a crucial point of interactivity for interesting programmes of proof by machines. There exist in fact many fine systems of automated calculation and deduction which are eminently interactive: the mathematician isolates enormous calculations and huge database searches which he then has the computer execute quickly and perfectly; he distils from these a number of terrifying lemmas, whose proof needs numerous mechanisable passages, and then he transfers these into the system, intervening finally in those crucial choices in the proof of a theorem, such as the choice of the inductive load, of the hypothesis rich in meaning, etc. Other automated proof assistants check, a posteriori, proofs or properties of programs, a major help for some work in Algebra and in programming. Finally, freed from the myths surrounding it, the computer, with its special powers of deductive/formal calculation, in certain cases quite literally gives wings to human calculations and proofs, thanks to an interaction between man and well-constructed machine. It is an interaction which leaves to man the use of analogy, metaphor and meaning, in other words to that ability to make connections within the network of integrated knowledge and forms of intelligence which makes up the specific unity

[25] Induction is particularly relevant also in view of "gödel-numbering" techniques. By these, one may encode in Arithmetic many mathematical structures, which apparently do not need to lie within numbers. For example, in Longo G., Moggi E. "The hereditary partial recursive functionals and recursion theory in higher types" *Journal of Symbolic Logic*, 49(4):1319 – 1332, 1984, some sort of computable functionals (functions of functions of functions ...) are hereditarely encoded into Arithmetic and can be easily defined by arithmetizable formulae (with some further work). Yet, even the proof that they are well defined in higher types requires a huge inductive load, calling for topological spaces and continuous functions. But this is very common in Mathematics. Another amazingly heavy inductive load (by the "candidates of reducibility") may be found in Tait-Girard's normalisation proofs, even in the arithmetizable fragment, i.e. the proof of "there exists y.P(n,y)." of footnote 18.

and force of human thought.

6 Conclusions

To conclude, the formalist hypothesis argues that only calculations using signs without meaning can allow the *a posteriori* reconstruction of any mathematic reasoning and the elaboration of its logical/formal foundation; its proponents nevertheless recognise the plurality of the forms of deduction (the famous "mathematical creativity") which must be able to be reinscribed *a posteriori* exclusively onto the formal level. The logico-computational hypothesis, which is much stronger, assumes that logical/formal intelligence, based on the manipulation of formulae conceived of as sequences of discrete symbols lacking meaning, themselves codified with, for example, zeroes and ones, allows the representation of all forms of intelligence, thus not only the *a posteriori* reconstruction of the formal skeleton of mathematics, but also the perfect simulation of the advances of those reasoning in each of these fields. Now, if the various Incompleteness Theorems mentioned here lead to the failure of the former of these two programmes, they imply *a fortiori* the same result for the latter; what is more, this second programme fails even when confronted with a question as "banal" as that of the inductive load in Arithmetic.

How then do the many defenders of these two programmes face up to this fact? The formalists, who are quite conscious of the metamathematical relevance of the Incompleteness Theorems, argue that it is more or less a question of metamathematical "tricks" (implying *ad hoc* metatheory), and that the proofs of all "interesting" propositions can be reconstructed formally. Now, deciding what is interesting is a matter of opinion, and I believe that Girard's Normalisation theorem is both interesting and rich in applications, particularly for the mathematics of Computer Science, although the Tait-Girard style proof requires metatheory. The same can be said of FFF (Kruskal's theorem has lots of applications), in which, and even more explicitly, the variety of our forms of knowledge enters into its set-theoretic proof, in particular through the concept of infinity.

As for the defenders of the logico-computational hypothesis, if I have understood properly, they either just ignore these findings, or give to them interpretations borrowed from more learned formalist arguments—which are nevertheless restricted exclusively to mathematics; in other words, they continue to believe that the machine's limits are man's limits, or that the "interesting" things that man knows how to do, the machine also knows how to do, and they consign everything else to the category of uninteresting or nonexistent things (some, such as Searle[26] call these theses "eliminationist" - an expressive, if somewhat sombre term). Finally, others bravely argue that the digital computer will, one day, go beyond working only on a formal/theoretical level. This is possible: I do not bet on future, when living clones of jupitarians will be our next generation computers .... Yet, so far, this possibility contradicts the two key hypotheses of functionalism (which deal with both the design of digital computers and their languages), namely the codifiability (on the theoretical level only) of any form of intelligence into discrete symbols, themselves in turn codifiable into formal Arithmetic or into similar theories, and the independence of this codifiability (but not of the code itself, obviously) with regard to any specific implementation. As I have already noted, the formal codifiability (the uniqueness of the conceptual level on which the zeroes and ones are found) and the independence "of what one knows how to do" with regard to specific contexts and implementations (the fundamental idea of the programmability of computers, namely "software portability") exclude precisely the network of connections characteristics of human thought which is based on the unity of its specific hardware and its software: our "modularised" brain with its history. For any monist, this network and this unity cannot be fragmented into metalanguage, language and semantics, and then, in a machine, into "software" and "hardware" (the soul and the body?), in order to carry on "meaning independent" computations, represented at the linguistic/theoretical level only. Some theorems I mentioned proved this for us, by showing the incompleteness of this artificial split of human reasoning.

By contrast, it is just this unity, this indivisible ego, or divisible purely for reasons of temporary mathematical commodity, or for the construction of machines, this contextual dependence, this specific hardware, living in the world and in history, which allow us to make these "bridges", metaphors or analogies between different forms of intelligence. These analogies and metaphors are essential elements of human reasoning, including mathematical reasoning; what is more, they are governed by intentionalities and emotions. It is on pre-

[26] *The Rediscovery of Mind*, M.I.T.-Press, Cambridge, 1982

cisely these constitutive elements of "meaning" that cognitive and foundational analysis should also be concentrating today, by focusing on the remarkable conceptual invariance and symbolic abstraction, so typical of Mathematics, but which (provably) cannot be defined as purely formal.