# Mathematical intuition and the cognitive roots of mathematical concepts[1]

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**Abstract**. The foundation of Mathematics is both a logico-formal issue and an epistemological one. By the first, we mean the explicitation and analysis of formal proof principles, which, largely a posteriori, ground proof on general deduction rules and schemata. By the second, we mean the investigation of the constitutive genesis of concepts and structures, the aim of this paper. This « genealogy of concepts », so dear to Riemann, Poincaré and Enriques among others, is necessary both in order to enrich the foundational analysis by this too often disregarded aspect (the cognitive and historical constitution of mathematical structures) and because of the provable incompleteness of proof principles also in the analysis of deduction. For the purposes of our investigation, we will hint here to the philosophical frame as well as to the some recent advances in Cognition that support our claim, the cognitive origin and the constitutive role of mathematical intuition.

## 1. From Logic to Cognition

Over the course of the XXth century, the relationships between Philosophy and Mathematics have been dominated by Mathematical Logic. A most interesting area of Mathematics which, from 1931 onwards, year of one of the major mathematical results of the century (Gödelian Incompleteness), enjoyed the double status of a discipline that is both technically profound and philosophically fundamental. From the foundational point of view, Proof Theory constituted its main aspect, also on account of other remarkable results (Ordinal Analysis, after Gentzen, Type Theory in the manner of Church-Gödel-Girard, various forms of incompleteness-independence in Set Theory and Arithmetics), and produced spin-offs which are in the course of changing the world: the functions for the computation of proofs (Herbrand, Gödel, Church), the Logical Computing Machine (Turing) and hence, our digital machines.

The questions having arisen at the end of the XIXth century, due to the foundational debacle of Euclidean certitudes, motivated the centrality of the analysis of proofs. In particular, the investigation of the formal consistency of Arithmetics (as Formal Number Theory, does it yield contradictions?), and of the (non-euclidean) geometries that can be encoded by analytic tools in Arithmetics (all of them - Hilbert, 1899 -: are they at least consistent?). For many, during the XXth century, all of foundational analysis could be reduced to the spillovers produced by these major technical questions (provable consistency and completeness), brought to the limelight, by an immense mathematician, Hilbert.

And here we forget that in Mathematics, if it is necessary to produce proofs, as key part of the mathematician's job, the mathematical activity is first of all *grounded* on the proposition or on the construction of *concepts* and *structures*. In fact, any slightly original proof requires the invention of new concepts and structures; the purely deductive component will follow.

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Now, as the keenest among the founding fathers would say, let's put aside this "heuristic" and rather focus our attention on the *a posteriori* reconstruction of the logical certitude of proof. An absolutely indispensable program, as we were saying, at the beginning of the XXth century, following the technical richness as well as confusion of the XIXth - a century having produced many results, among which several were false or unproven or badly stated -, a program, however, which excluded from foundational analysis any scientific examination of the *constitutive process* of mathematical concepts and structures. Such is the aim of the new project regarding the cognitive foundations of Mathematics, which is also an epistemological project. It is certainly not a question of discarding proof, with its logical and formal components, but simply of steering away from the formal (and computational) monomania, formerly justified and which dominated the previous century. It produced these marvelous logico-formal machines surrounding us, acting without meaning. The analysis of the constitution of sense and meaning in Mathematics, through cognition and history, is the purpose of the investigations in Mathematical Cognition, also in order to compensate the provable incompleteness of formalisms.

In order to grasp this issue, it is necessary to be precise regarding the term "formal", proper to the formal systems extensively taken to be the only locus for foundation. There exists a very widespread ambiguity, particularly in the field of physics: for a physicist, to "formalize" a physical process means to *mathematize* it. Since Hilbert's program, formal instead has meant: a system given by *finite* sequences of meaningless signs, governed by rules, themselves being *finite* sequences of signs, which only operate by purely mechanical "sequence-matching" and "sequence-replacement" – as do lambda-calculus and Turing machines, for instance, two paradigms for any effective formalism, following the equivalence results. To be fair, some recent revitalization of Hilbert's program try to extend this notion of "formal", sometimes in reference to writings by Hilbert as well. One may surely propose new notions, yet the *definition* of what a formal system is, is formally given by Gödel's incompleteness theorem and Turing's work. As Gödel says in his "added note" of 1963, we have, since Turing, a "certain, precise, adequate notion of the concept of formal system". And, not only Turing Machines, but, we stress, also the incompleteness theorem, by its use of the same notion of formal system makes it perfectly stable: definitions, in mathematics, are definitely stabilized by the (important) theorems where they apply.

Of course and "per se", there is circularity in any formal definition of a formal system, in the same way there is circularity when the notion of word is defined by means of other words: Turing Machines also need to be defined as a formal system (yet, a very basic one). Nothing serious, we are accustomed to this, just as when we do not stop speaking because of the loopholes of sentences which refer to themselves, like in the Liar's paradox, and... we continue to talk about sentences using sentences. But there is far more than that: the formalist foundation of Mathematics refers to the absolute certainty of notions such as "finite" and "discrete". Yet, as for finiteness, following the Overspill Lemma in Arithmetics and, more notably, since incompleteness, we know that we can not formally define the notion of "finite". In fact, an axiom of infinity is required in order to formally "isolate" the standard finite integers. A result which demolishes the core of any finitist certitude, including that which is internal to logical formalism. In short, the *notion* of finitude is very complex, requires infinity, if one wants to grasp it formally: it is far from being "obvious", in the Cartesian sense. And it is not an absolute: finiteness makes little absolute sense, for instance, in cosmology – consider the question: *Is the Relativistic Universe finite or infinite?* The Riemann sphere is finite but unlimited against the Greek identification between finite and limited. Likewise for the notions of continuity (in the manner of Cantor-Dedekind, typically) and of discreteness (the latter to be defined as any structure of which the discrete topology is "natural", we would say from the viewpoint of Mathematics): Do they make sense in Quantum Mechanics, where the discrete energy spectrum has as a counterpart, in space, non-locality, non-separability properties, the opposite of those of discrete topology whose points are well separated, and where a more adapted mathematical continuum should, perhaps, not be made of Cantorian points either?

Such are the certitudes, the alleged foundational absolutes, of any arithmetizing logicism, the finite, the discrete, which regain their modest status of biased constructions through a very specific constitutive history (a "contingent" one, see below). Now, the cognitive practice and construction of the finite and of the discrete, in our spaces of sensitive action, are undoubtedly good (they are what they are), but the concepts of which they are the origin and which they make significant, may be inadequate beyond these spaces, in Cosmology and in Microphysics, for instance, as they may depend on the choice of the reference system and the measure on it (the metrics). This remark hints also to the distance between elementary cognitive practices and the finesse of mathematical conceptual constructions that have been accomplished, along history and language.

As regards the formal/mathematical, physicists may very well preserve their ambiguity of language, where the formal is identified with the mathematical ("formal" structures instead of "mathematical" structures, as we said), once the issue has been clarified; because, only through this distinction may the recent results of the *mathematical* incompleteness of *formalisms* be understood. In short, in the difference between the formal and the mathematical, there is no less than some of the most important results in Proof Theory of the last 30 years (see (Paris, Harrington, 1977), Kruskal-Friedman in (Harrington et al., 1985) among others; see (Longo, 2002) and (Longo, 2005) for surveys and reflections). In this difference actually lies the *structural significance* of integers, these entities constituted within our cognitive and historical spaces, the issue under discussion in this text.

In particular, it is the cognitive practice of counting, of ordering, even pre-human, such as described below, which lies in the "backdrop" of the notion of number (integer, finite). It should be clear, though, that between primitive animal counting and the concept of number there is an abyss, that of language and of history. And we must also be very careful in not going towards another serious theoretical shortcoming which directly associates a concept to the cerebral episode, as well as to animal practice. In these shortcoming, it is the history of the communicating human community which is lacking.

There is no doubt that we understand the concept and that, in fact, we have *produced* it, because we share ancient, often pre-human, practices, just as it is certain that when we count (or think), something happens in the brain. But, we insist, no identification should be made between the explicit linguistic conceptualization and its cognitive (necessary) background. The former also journeys through intersubjectivity and history, it is grounded on numerous different practices, including the practice of writing, their plurality and difference being what leads us to the *conceptual invariant*.

Let us also observe that this error of some neuropsychologists is the same as that of the formalist, who confuses the formal sign without signification (in this case, the cerebral signal) with the signifying concept and who conjectures the completeness of formalism (or of the chain of cerebral signals). The fact remains that Mathematical Cognition, with its new neuropsychological technicality, is a grand novelty for the foundation of Mathematics: it is only necessary to integrate it within an analysis of conceptualization, by language, and of meaning, within history, as well as with an analysis of proof with its logic.

## 2. **Cognition and Invariance**

Beginning with Mathematics' Cognitive Analysis, the perspective assumed here, we stress the epistemological content of this investigation. In our view, any epistemology should also refer to the "genealogy of concepts", following Riemann (see (Bottazzini et al., 1995)) and Enriques' "progressive and historical conceptualizations" (see (Enriques, 1935), (Faracovi et al., 1998)) proper to Mathematics, beginning with "our human activity in our spaces of humanity" (Husserl, 1933) and continuing through history. What can be said, for instance, about the concept of infinity without making reference to its history? The iterated gesture and the metaphoric mapping of its conclusion (the limit) are informative ideas, yet they do not suffice. There is no constitutive history of this concept without an analysis of the historical debate that brought (actual) infinity to today's robust conceptual status, from Aristotle to Saint Thomas and Projective Geometry in Italian Renaissance Painting up to Cantor. Likewise for real numbers in Cantor's continuum: *their objectivity is in their construction*, which is the result of a historical practice, as is their *effectiveness*.

We now arrive at a crucial point; numerous authors refer to the "great stability and reliability" of Mathematics, which would need to be accounted for. And there is the customary and pre-scientific astonishment before the "unreasonable effectiveness of Mathematics", an exceedingly famous title for a quite modest article, (Wigner, 1960) (but why did such a great thinker as Wigner not find more profound examples to support his thesis? Wigner's article, which everybody quotes – due to its so very effective and memorable title – and which very few read, presents examples that are not astounding).

Do linguists (cognitivists, for example) consider the following problem: *What a miracle! How languages are unreasonably effective! When we talk, we understand one another!* Languages were born for purposes of communication and *while communicating*, to tell each other things – possibly non-existing things (this is why human language was invented), to understand one another. As regards stability and invariance, as said extensively in (Bailly, Longo, 2006) in connection to Physics' theoretical principles, we can even define Mathematics as the fragment of our forms of construction of knowledge which is *maximally invariant and stable*, from a conceptual viewpoint. That is, that as soon as we impose, in our practice of communication and understanding of the world, maximal conceptual stability and invariance (we hate absolutes), we are doing Mathematics. Otherwise, it is prose, whether we know it or not.

Again, regarding effectiveness: Mathematics is consistent with the world (actually, with the physical sciences, from physical and sensible space, where Euclid's figures evolve), because it is co-constituted with its knowledge (physical), while organizing it. It was constructed in resonance with our construction of objectivity and of (physical) objects. Of course, it is possible that complex numbers, derived from an algebraic practice of equations, enriched with a nice correspondence with the Cartesian plane (Argan-Gauss interpretation), will, two centuries later, make Quantum Mechanics intelligible. But natural languages provide us with tools which can be transferred from one field to another, and which organize completely new experiences and forms of life. Thus, they enrich themselves and change, as does Mathematics. No miracle there. Such miracle is the great question underlying any Platonic approach (Mathematics is up there in the sky, as it has been for ever. Why does it function in this world? Standard answer: Because the world is shaped by the pre-constituted molds of mathematical determination). But the question can also be asked by the formalist: Why do these signs, potentially handled by mechanizable computations, sequences of "sequence-matching and of sequence-replacement", tell us something about the significant structures of the world? And some of them explain: Since the world is a very large Turing machine (the genome being a program, Evolution, an algorithm and the brain, a digital switchboard).

If we re-consider the problem, it will be then possible to perform the cognitive analysis of Mathematics, as a necessary component of a sound epistemological investigation of it. It may help to see it the way the linguist sees languages, which are quite effective as we said, although relatively *incomplete*: relatively, because each week, we find a word, an expression in Italian which says something that cannot be told exactly in English or French, or vice-versa (not to mention Chinese). So there are fragments (situations, feelings, mostly) that each specific language does not grasp, or not very well, or not exactly in the way another can. But languages and Mathematics (and even formalisms, in their mathematical incompleteness) remain quite effective. This is because Mathematics sticks to the world, as an interface which is constituted, in the way that bark sticks to a tree, with no miracle to account for. And it grows and evolves with it. It is necessary to account, by means of a *possible history*, as would any scientific report, for the phylogenetic and historical (cognitive) constitution of mathematical concepts and structures, with their maximal conceptual invariance (mathematical concepts are maximally stable, we insist, by definition). Its effectiveness and dynamic stability will follow.

Now if Mathematics is maximally stable and invariant by construction, among our forms of knowledge and of communication, that is also where its limitations are to be found. It is born around the invariants and transformations which preserve them, beginning with the rotations and translations in Euclid's geometry. We manage to mathematize the most "chaotic" of situations, precisely when we find the global determination (the dynamic law, the invariant), which makes intelligible the strangest of attractors. Even the shapes of clouds or of the coast line have marvelous mathematical *imitations* (but not models, see (Longo, 2007)), once an appropriate law, an invariant fractal, has been proposed. But the moment the invariance becomes difficult to grasp or if it is not at the center of the process (it is possibly not "that which counts"), Mathematics, as we conceive of it today, faces a stalemate. What are the great invariants in Biology? If we refer to Molecular Biology, we do find some invariants, but, despite their being present only within life phenomena, they pertain to Chemistry, not Biology. Life phenomena are very stable, globally, but proceed within local instability, which in turn can modify at once even the global structure (e.g., the morphogenesis of species; punctuated equilibrium/evolution according to (Gould, 1989)). We need to account for this instability/stability, variance/invariance, order/disorder, integration/separation…, which the Mathematics of Physics describes badly (see (Longo, 2009))).

One can surely not go into the details of this subsequent questioning, but it is part of the epistemological project: If Mathematics is constituted, in History, starting with animal practices, then the analysis of its constitutive journeys can help us explain new dynamics of knowledge, around new intentional views. And it can help us avoid applying everywhere the same tools of Mathematical Physics, as if they were Platonic absolutes or complete formalisms of the world, including to life phenomena, where new tools and observables (new invariants) are needed.

## 3. Intuition

The notion of intuition plays a large part in mathematics, yet the term "intuition" is highly polysemic, and may refers to different meanings depending on the context in which it is used. In particular, a distinction is commonly made between the intuition that takes place in the practice of mathematics, and the intuition conceived as grounding the constitution of a mathematical concept or the progress of mathematics itself. In both cases the way the notion of mathematical intuition is approached often involves inquiring into its legitimacy, i.e., the extent to which one should rely on it in the practice or, in a more foundational perspective, in the construction and progress of mathematics (as we said, the proposal of actual infinity and (non-)Euclidean geometries is particularly important in this regard). This is particularly true when dealing with an intuition of a sensible kind, or a geometrical intuition, in reference to the constitution of a sensible intuition, a posteriori, for example when it is grounded on empirical data (e.g. reasoning on graphs, etc….).

In fact, the progress of mathematics in history could easily be seen as going hand in hand with a gradual renunciation of the geometrical intuition, to the benefit of more and more abstraction. However, it seems more appropriate to view the progress of mathematics as a constant alternation and interaction between the consideration of sensible intuitions, and some more abstract developments. This is what we find especially in the phases of the history of number theory (described by Kline, 1980), with, for example, the analogy between integers and measures (lengths) by the Pythagoreans in the fifth century B.C., or the development of a science of the incommensurables relating irrationals to geometrical shapes.

Thus, it seems difficult to conceive of mathematics without making the notion of intuition stepping in. Many mathematicians are ready to attach a significant value to a phenomenon that they call « intuition » in their experience of mathematics. We insist here also on the epistemological, thus foundational interest of its cognitive analysis.

The value of intuition is stressed by Poincaré (1905, 1908), when he carries out an introspection upon his experience as a mathematician in his discussions about mathematical invention (1908), insisting on the creative aspect of intuition. In these reflections, Poincaré appeals to psychologists to investigate this phenomenon of intuition, which he describes similarly to a gestaltist « insight », wondering: « isn't the "subliminal self" superior to the "conscious self"? » (ibid.).

Indeed, Psychology at this time seemed particularly suitable to investigate on this topic, especially regarding its subjective nature. Yet nowadays the experimental methodology in cognitive psychology is no longer centred on introspection. On the contrary, the goal of this field of research (and of cognitive neurosciences in general) seems to impose, if one wants to describe processes that are common to an entire population, a reduction of the part taken by subjectivity, and so by introspection. And this investigation on mathematical intuition via cognitive science is presumably better able to answer to questions relative to the foundations of mathematics, as we will see below.

The Kantian conception of intuition, placing the latter in the field of the sensible, can be considered, in our view, if one sticks to a certain level of generality, as a first step towards this cognitive investigation. Indeed, according to Kant, the intuition doesn't provide us with knowledge of the object itself, but of its mode of presentation. Kant said in the Introduction to the Transcendantal Logic that intuition « only contains the way in which we are affected by objects » (Kant, 1944/2004, p.77). In his Transcendantal Aesthetic, Kant defines the intuition by « the mode by which knowledge immediately refers to objects and to which every thought tend as to the goal in view of which it is the means. » (ibid., p.53). Then he explains, « this is by mean of sensibility that objects are given to us, only this can provide us with the intuition » (ibid., p.53).

It is in that sense, by approaching the intuition, not via the object it reaches, but instead via the conditions of possibility of its presentation to our mind, that the Kantian conception of intuition constitutes a step towards the cognitive investigation we are interested in. Because, according to the Kantian conception, one has access to this mode of presentation of the objects, « this subjective contribution to phenomena is both immutable and accessible itself to knowledge: one is able to comprehend the fixed laws of his own cognitive constitution, each one being a necessary characteristic of possible experience » (Dubucs, 1999). Today the field of numerical cognition, by studying the cognitive and cerebral basis of our mathematical (and more specifically arithmetical) skills, indicates to what extent these skills - especially our representations of numbers -, are partially constrained by the structure of our cognitive system. It is in this sense that this kind of investigation can constitute a development of the transcendental conception of intuition in Kant.

As a matter of fact, we need to broaden this conception. The gestaltist conception of intuition may reveal the limits of such an analogy between the cognitive investigation and the Kantian conception. In a biography of Wolfgang Köhler, V. Rosenthal and Y.-M. Visetti explain to what sense the notion of intuition for Gestalt psychology differs from the Kantian conception. The gestaltist conception « broadens the domain of what is accessible to » this immediate perception (Rosenthal & Visetti, 2003, p. 172), notably by reconsidering the dichotomy between formal and material. The authors give the specific example of causality, which is for Kant a category of the faculty of understanding, that the gestaltist conception returns into the domain of intuition. Using the exemple of the transfers of movements between two billiard balls knocking together, they say:

> *"In the gestaltist conception, the causality is intrinsic to the seizing of these movements because it is intuitive; it's part of the mode of presentation of the phenomenon. (...) But this suggests to not consider the intuition as a frame only filled by the sensible inputs of the moment ; to not consider it either as a « background » which formality would in fact be identical to the one from an empty space and time, that the schematism would then turn into geometry. It's impossible to fit a gestaltist perception in a kantian architecture, even an extensive one: because a gestaltist perception contains no sensation that would be as a « filling content », or a « sensory material », for schemes turning categorial syntheses - empowered to qualify the experience – into geometry." (Rosenthal & Visetti, 2003, p. 175)*

This gestaltist conception is, in our view, adapted to defend the idea of a constituting intuition of a concept, carrying meanings by itself. The "geometric judgment" of well ordering, for example, is an intuition of the spatial structure of numbers, likely to carry the well order of the integers. By this we mean the "to see in mental spaces" the discrete increasing sequence of integer numbers and pronouncing the judgment: "a generic non-empty subset of it has a least element" (any sufficiently mathematized person should see this… we will go back to this issue). It constitutes an example of « reflection on meanings », in the sense Gödel wrote about, concerning the intuition of concepts and of mathematical structures (which organizes concepts). It is largely used in the mathematical practice and in proofs (see below).

The shape (the spatial dimension as we will see below) of our mathematical intuition acts as a constitutive constraint on our semantic relation to the concept, a constraint that is linked with our actions in the world. In itself this cognitive aspect of intuition already is unlikely to be compatible with the restriction of the notion of intuition to the domain of senses, and with the Kantian distinction between the (sensible) intuition and the categories of the faculty of understanding.

**4. Intuition and the "Number Sense".**

The idea of natural digits being accessible to intuition is ancient, and this term of « natural » is probably linked to it. In Poincaré « the intuition of pure number », on which relies the principle of induction, reflects his conception of integers as directly provided by intuition: « The only natural object of the mathematical thought is the integer » (1905, p. 33).

Of course when one addresses the issue of a psychological investigation on the basis of our mathematical knowledge, and especially on our intuition of numbers, the work of Jean Piaget cannot be ignored. An essential component of Piaget's work concerning mathematical knowledge lies in the idea that a real conceptual knowledge in mathematics appears only when some logical skills are acquired. The notion of intuition for Piaget (Piaget, 1967, 1967a) can be conceived as referring in a broad sense to everything that is not formal (this idea can be found notably in Fischbein, 1987), or to the knowledge acquired during the so called "sensori-motor" stage. Thus, this intuition does not really constitute a mathematical knowledge, in Piaget's view. Yet it shows the Piaget's work constitutes a step towards the idea that mathematical intuition emerges from actions and one's interaction with the world. Besides, some authors argue that Piaget was not necessarily defending a logicist theory of the foundations of mathematics, and considered the logical skills as insufficient (see Smith, 2002). Finally, we shall mention the fact that Piaget, having noticed that some skills (such as the logic of class inclusion) could be found earlier in children when the experiments were restricted to small sets, may have called these small numbers « intuitive » (cf. Hunting, 2003, and Gelman & Gallistel, 1978, Chp. 6, where the notion of "intuitive number" in Piaget is also mentioned).

Recent studies of numerical cognition show that we possess an intuition of natural integers before the actual learning of their formal definition. Indeed, studies have shown the existence of what is notably called by Stanislas Dehaene a « number sense ». By using this expression, S. Dehaene emphasizes that the numerical skills shown by studies of numerical cognition allow us to « perceive numbers the same way we perceive color, shapes, or the position of objects » (Dehaene, 1997). Thus researchers in numerical cognition often refer to a « perception of numbers », or even to a « numerical acuity » (ibid.). The expression « number sense » indicates that the existence of mechanisms, in our cognitive system, that are specifically dedicated to the processing of numbers, leads us to consider the perception of number as an actual sensation. Our intuitions of numbers should then be compared to a perceptive intuition in that sense, without defending a platonistic conception of numbers.

An important characteristic of these studies is that they show a phylogenetic aspect of our mathematical intuition. Indeed, studies in monkeys show that we share this "number sense" with non-human primates, and that our mathematical intuitions are partially constrained by the structure of our cognitive system, inherited from a phylogenetic history. This aspect is missing in the Piagetian conception of the development of mathematical abilities since, according to this conception, everything happens as if mathematical abilities were constructed ex nihilo.

**5. Representations of Numbers**

Several systems for the representation of numbers have been described, some of them being found already in babies. The first one is the "subitizing" process, which corresponds to a spontaneous perception of the cardinal of small sets (until around 3 items). Indeed, when a participant is asked to give the cardinal number of a set of dots presented briefly on a computer screen, the error rate is low and the response times are low and almost constant for the sets with a cardinal smaller or equal to 3. And this intuition of small integers can also be found in the early child. For example, some experiments have shown that 6 months old babies presented with two slides on which common objects are drawn spend a greater time staring at the slide that has a number of items equal to the number of sounds played during the presentation (Starkey, Spelke & Gelman, 1983, 1990). This study is even more interesting since it shows, from the age of 6 months old, a capacity of abstraction of the number between several modalities, tending to indicate that this kind of intuition already contains the invariance peculiar to the concept of number. Thus, it seems that small integers have a particularly important status from a cognitive point of view.

The studies investigating this perception of small quantities in babies, and their manipulation in small calculus (e.g. Wynn, 1992), provide an explanation in cognitive terms for the intuitive aspect of some mathematical propositions. The obviousness of a proposition such as « 2+2=4 » (which Descartes considered as directly accessible via an intellectual intuition) is to be found before the already constructed proposition, in the perception of small quantities. Thus we may have an explanation in cognitive terms of cultural observations concerning the peculiar status of the first integers, notably regarding the numerical vocabulary. In fact, the evidence of this intuition can even be seen in symbolical notations, such as roman or Chinese ones, where the symbols representing the first three numbers seem to refer directly to the perception of three objects (I, II, III, and -, =, ≡, respectively, and this is still true for arabic notation). Besides, some Amazonian tribes don't have a lexicon for numbers greater than 4 (see the studies of numerical cognition performed in these populations, e.g. Pica et al., 2004), and, in several Amerindian languages, the words for numbers that are smaller than the limit of subitizing are the only ones that are not composed from other words of the language (Cauty, 1988). In this regard, note that the words for greater numbers are often constructed from words referring to body parts (e.g., the word meaning « hand » is frequently used to refer to the number « 5 »), reflecting again the heritage of practices, such as counting on the body, which constitutes our intuitions of numbers.

The second system of representation of numbers allows us to estimate when we are asked, for example, to give as quickly as possible the number corresponding to a set of dots with a cardinal greater than the limit of subitizing. This is an analogical intuition of numerical magnitudes. This system has several characteristics: a trend towards underestimation of the evaluated magnitude, and a growing imprecision with the latter. In line with the idea that our intuition of numbers constitutes a « sense », the perception of numerical magnitudes follows the well-known Weber's law, that is to say our ability to discriminate between two quantities changes proportionately with their ratio (see Izard & Dehaene, 2008). These observations have led the researchers in numerical cognition to assume the existence of an internal representation of numbers in the shape of a mental number line. To account for the law of Weber, underlying our intuition of magnitudes, it has notably been suggested that this mental number line was compressed (possibly logarithmic). Thus, our intuition of numerical distances would rather follow a logarithmic scale, the numbers being closer from each other when moving towards greater quantities on this line. Several studies confirm this hypothesis, notably by showing that the logatithmic intuition of numerical distances tends to become more linear with age, particularly at the beginning of mathematical education (Opfer & Siegler, 2007), or that one tends to perceive as random a series oversampling the smallest numbers of an interval (see Banks & Coleman, 1981, Banks & Hill, 1974, and also Viarouge et al., under revision). Although the mental number line is only a model to account for behavioural data, one can think that this spatial structure of our intuition of numbers has a real cognitive relevance. The influence of culture is undoubtedly crucial in our propensity to represent numbers as organized along a line. The trajectory of the counting on parts of the body that we already mentioned has probably also something to do with that. In 1880, Sir Francis Galton reported the introspections of people who perceived a spatial organization of numbers, with very precise trajectories (and sometimes colours), in a very pregnant way. An essential study of the internal representation of numbers, conducted by Shepard and his colleagues in 1974, allowed, via the method of multidimensional scaling, to obtain what could correspond to a graphical representation of our intuition of numbers from the results of similarity judgments, revealing the alignment and the compression towards big numbers that we mentioned above.

## 6. More on Numbers in mental spaces

The spatiality of our intuition of numbers is confirmed by a set of studies showing strong interactions between the representation of numbers and spatial cognition. A large number of these studies focus on an effect of automatic association between numerical size and the side of space. The SNARC effect (for « Spatio-Numerical Association of Response Code, Dehaene et al., 93) shows that when one is asked to perform a numerical task (such as indicating as quickly as possible on a laptop keyboard if a digit is odd or even), the largest numbers included in the experiment are responded faster with the right button, whereas the smallest numbers are responded faster with the left button. This automatic association of numbers with a side of space has often been considered as giving an orientation to our intuition of numbers organized on the spatial structure of the number line. Again, a frequently found hypothesis is that cultural practices are at the origin of this kind of association, such as the direction of writing (Zebian, 2005), confirming the idea that the constitution of our mathematical intuitions is grounded on a plurality of cognitive practices. It is the constructed *invariant* of the *transformation* from one practice to another.

The spatial dimension of our intuition of numbers is also confirmed by neuroimaging studies indicating an overlapping between cerebral regions dedicated to spatial cognition, and those known to be implicated in the representation of numerical magnitude (see Hubbard et al., 2005, for a review). This importance of the spatial component of our intuitions of numbers echoes back the role granted to the two types of pure intuition in Kant, namely space and time, as necessary shape of representation of the mathematical objects to our mind. The heritage of this conception in mathematicians has mostly been expressed via the role of the intuition of time, especially in intuitionists, such as Brouwer. A growing number of studies focus now on the interactions between numbers and time. For example, a recent study has revealed interactions between numerosity (of sets of dots or Arabic numbers) and temporal judgments (Xuan et al., 2007). The participants of this study judged as longer the presentation times (but also the temporal intervals between stimuli) of the stimuli corresponding to the largest numerosities. The cognitive analysis of our numerical intuitions can thus be considered as being in agreement with the kantian conception of intuition by showing the constitutive role of both intuitions of space and time. The question still remains open of the interdependence of the spatial, temporal and numerical abilities. The studies of numerical cognition tend to promote the idea of a system which would be specifically dedicated to the processing of numbers, in interaction with other systems of representation of space and time. But one could also consider that these three dimensions (number, time and space) all come from a common system. This is the thesis upheld by V. Walsh (2003), who postulates the existence of a global system for the processing of magnitude, covering the numerical, spatial and temporal dimensions. In our view, whether or not a specific brain-function may be associated to the notion of numeric magnitude, the concept of number is the constituted invariant, in language, of different cognitive praxes. That is, it is general and abstract exactly because it is invariant w.r. to the transformation from one cognitive frame to another: space, time or sound-rhythms. Of course, this does not exclude a formation of a brain area, stabilized in evolution and history, partly dedicated to the "number sense" (as pointed out by Dehaene, existing cerebral resources may be *used* – "*recycled*" - *for new activities).*

This investigation of the cognitive origins of mathematical intuition constitutes, in our opinion, a contribution to epistemological questions. S. Dehaene, for example, suggests that « our number sense » is an argument « against the hypothesis that our brain does mathematics as a logical machine » (Dehaene, 1997, p. 322). These intuitions are not reducible « to the definition that [the axioms of Arithmetic] pose » (ibid., P. 325). Further, the author specifies that « a formal definition is not necessary, we intuitively know what the natural integers are » (ibid., p. 326). We can observe that some studies now try to investigate our perception of more complexe mathematical objects, such as fractions, sometimes guides by the idea of an accessibility of these objects via the same system of analogical representation that we previously described (Jacob & Nieder, 2009). Such a cognitive analysis provides the very first steps of the constitution of our mathematical knowledge by describing the shape of our first intuitions of numbers. Thus we saw that we possess, from the early age on, different systems of representation of numbers, which provide us our first intuitions of the concept of number. If the phylogenetic heritage is an essential component of the constitution of our mathematical intuition, this notion of heritage is to be extended to the idea, more general, of a heritage of practices. The cultural practices, the language and the learning of mathematical formalism, all are the driving force behind the dynamic of our mathematical intuitions. Moreover, the strong spatial component of our intuitions of numbers, notably revealed by the SNARC effect, is also in agreement with a role of action in the constitution of our intuitions. The recourse to intuitions of a geometrical type in our practice of mathematics (included the one of Arithmetic) is the sign of these cognitive roots of our mathematical knowledge, progressively constituted from a plurality of cognitive *active* experiences. This role is confirmed by the studies conducted on the SNARC effect and which indicate a great flexibility of these number/space associations.

This entire constitutive route makes the intervention of very complex intuitions, needing a high level of expertise in mathematics before cognitive and historical praxes, like the « geometrical judgment of well-ordering », become possible. By this judgement we mean that, once integer numbers are organized as a gestalt, the trained mathematician sees them as a well-ordering. That is, he/she accepts as an immediately and intuitively sound and perfectly robust statement the sentence « a non-empty subset of numbers has a least element » (don't you see this property on your mental number line?). This statement, according to the logical approach, is highly infinitary (a *for all* statement over uncountably many subsets), yet it has a finitistic flavour in our gestaltist understanding. If the *supposed* set is assumed to have an element, then the way the rest of the set "goes to infinity" doesn't really matter, in order to understand that it must have a least element: the element supposed to exist (by the non-emptiness of the set) must be somewhere, in the finite, and the least one will be among the finitely many which precede it, even if there is no way to present it effectively. This is well-ordering as a property of the space organization of integer numbers, in the increasing discrete. Every mathematician « sees » this property and uses it in proofs every day. The formalist myth wanted to avoid it by claiming the completeness of formal induction, that is, that all arithmetic proofs could be reduced to first-order arithmetic induction, or that the "intuition" of well-ordering could be replaced by formal induction in the proofs of all finitistic statements. Yet, this intuition, which derives from a very complex and manyfolded constitutive route and of which our proto-mathematical abilities are the very first step are, unavoidably shows up in proofs. In particular, the recent « concrete incompleteness » results (Paris-Harrington, Friedman-Kruskal theorems for example) provide formally unprovable arithmetic statements. Yet, they are proved to be true by using the geometric judgment of well-ordering (see (Paris, Harrington, 1978), (Harrington et al., 1985) for a proof and (Longo, 2002) for an analysis of the proof).

We are forced here to refer to further writings, (Longo, 2002; 2005) in particular, for a closer analysis and references to the "ordinal analysis" that tries to transfer the foundational problem from the use of well-ordering, as a gestalt, to the construction of large ordinals, a further extension of our practices of order. Yet and again, also the endless extension of induction along the ordinals is provably insufficient to complete the incompletable: the normalization theorems in Type Theory (system F) require higher order arithmetic systems and escape also the ordinal analysis (see (Longo, 2002)). Only proper second order induction, based on the above "geometric judgment" yields completeness, exactly because it lies outside formalisms and is grounded on cognitive performances. Besides its (inter-)disciplinary interest, the cognitive approach to the foundation of Mathematics seems a needed tool to fill the gap between formalisms, which are provably incomplete, and actual (meaningful) mathematical deduction and conceptual constructions.

## 7. **Structures and historical constitution in Mathematics**

Reference should be made also to the work of Lakoff and Nunez (2000). The limits of this very interesting attempt which has significant merits in terms of originality, is due to the fact that it is centered only around two notions: that of the metaphor and that of conceptual and metaphoric mapping. The idea is very stimulating, but it is impossible to consider the entire foundation of Mathematics while adopting just this point of view, in particular because it overlooks pre-linguistic activities in space as well as history.

Beyond and before the linguistic investigation, the work by Piaget, Dehaene, Berthoz among others (see references and (Berthoz, 1997)), gave us important elements to this foundational novelty: the Cognitive Analysis of Mathematics (only slightly brushed upon by Poincaré and Enriques, see references). Also, there are the texts of P. Maddy (see Tieszen, 2005) for reference and a discussion). Contrarily to the richness of empirical data, although indubitably questionable, but methodical, among the aforementioned linguists, cognitivists, neuropsychologists, P. Maddy proceeds by "common sense". With a bias, rooted in the purported centrality of Set Theory in the foundations of Mathematics, she sees the objects of the world come together in *sets*, before our eyes, collected by predicates. Now, the processes of set-theoretical common sense are opposite to the *constitution of meaning* we are seeking to detect in the constitutive process of Mathematics. There is an abyss between common sense and the constitution of sense and this abyss is called the construction of scientific objectivity to which Mathematics participates. But Set Theory is a (still) very common trend. No mathematician encodes his or her work in ZF or NBG, but almost all prefaces to mathematical books, in order to justify themselves in the face of the century's orthodoxy, guarantee that the work they introduce could be; then it actually uses, at most, little and common set-theoretical notations, a very convenient language which has no doubt unified Mathematics, but which does not say much from the foundational standpoint.

As a matter of fact, Mathematics is a science of structures, it makes intelligible fragments of the world because it organizes them, it correlates them, *structures* them: for this reason the cognitive analysis of Mathematics is the exact opposite of the set-theoretical approach and of cognitivist common sense, a gaze upon scattered points or objects. In Set Theory, elements or points precede structures: the latter are conceptually secondary. In our views, gestalts, as structures, precedes points, they are our primary, proto-mathematical, relations to the world. Even counting and sequencing, at the center of oft-quoted practices, organize, establish correlations, order, as gestalts: this is what mathematically matters first.

For example, the act of monitoring a trajectory, the constitution of its memory, the latter being abstract as it is the memory of a prediction, is a paradigm for this (see (Bailly, Longo, 2004)). The memory of a prediction is abstract in the double sense of being independent of the context, as our forgetful memory remembers just the invariant that matters for future protension, away from the specific context of the original action (memory is intentional, both when retaining and when recalling, it choses). And it is abstract as an invariant of action (gestalts originate, for us, as invariants of a variety of actions). Another fundamental structuring of the world is the creation of a boundary which is not there (see the gestalts from Kanitza's triangles in (Petitot, 2008): active saccades draw them and complete non existing lines). This is where Mathematics originates and this is why we believe that there is no Mathematics without structure: its constitutive analysis must be the opposite of the unstructured assembly which is the primary foundation and the conceptual origin of Set Theory.

## 8. **More on lines and orderings as Gestalts.**

Let us finally discuss the example of a gestalt which is constitutive of Mathematics: the widthless line. From the cognitive standpoint, it is possible to refer first to the conceptually

simultaneous role of:
- The saccadic eye movements preceding the prey, tracing its continuous path,
- The "vestibular line" (the inertial stability of the body, gided by the vestibular system, which contributes to the memorization and continuation of inertial movement),
- The visual line (which includes the direction detected and anticipated by the primary cortex, see below).

The isomorphism proposed by Bernard Teissier (from Poincaré-Berthoz, according to his definition, (Teissier, 2005)) is between the two latter cognitive experiences: action and movement impose (make us practice) an identification (an isomorphism) between the experience of inertial movement, in a straight line, and the glance ahead, which precedes subjective movement. This isomorphism is to be extended by the ocular pursuit (the saccadic eye movements), also an action, and has for result, as a pre-conceptual practice, a pure direction, with no thickness. The invariant of these three cognitive practices is a barely traced line, a form of (pre-conceptual) abstraction: it is that which matters, that which is in common, distilled in the memory of action, in the view of new action.

This practice gives a direction and is at the origin of (enables) the conceptual, linguistic and historic baggage by which we come to propose a continuous line, parametered over the reals in the manner of Cantor-Dedekind. In other words, this widthless continuous line is the pre-conceptual invariant of the mathematical concept, an invariant as regards several active experiences; it is irreducible to a single one. The invariant of praxes is not the concept, but is the foundational element and the locus of meaning: we do not understand what a line is, we do not manage to conceive of it, to propose it, even in its formal explanation, without the seen gesture, or even without it being drawn on a blackboard, or felt, appreciated by the body, in the evocation made by the gesture of teacher.

We believe it is necessary to stress, in all of this, the key role of memory, in one of its most important characteristics: forgetfulness. Intentional forgetfulness, as the result of an aim (even pre-conscious, if we accept to enrich Husserlian intentionality), is constitutive of invariance: from the selective (intentional) role of vision, an active sight, a palpation through vision (Merleau-Ponty), up to the reconstruction in memory which intentionally selects (even subconsciously) that which is important regarding the aim of the action. Up to the conceptual construction, it is the forgetting of that which is not important, relatively to the goal in question, which precedes the proposal and the understanding of the invariant, of that which is stable with respect to a plurality of actions-perceptions.

So there is, starting with primitive animal counting, with the SNARC effect as seen by Dehaene, this cognitive mountain which we see, for instance, even behind an elementary judgment (irreducible to a finitary formalism), like the well-ordering of integers. Its construction and intelligibility lies also in the structured order of integers given by the "sense of movement", in the action which arranges them on a line, a gestalt which remains in the backdrop, but which contributes in organizing them, in setting them, in well-ordering them up to infinity, in a highly mathematized conceptual space. Moreover, it is not excluded that in order to grasp the statement of well-order, so complex despite its elementarity (it cannot be further decomposed, this is the incompleteness of formalisms), that we need a mastery of all the line, hence of projective geometry (or of perspective in painting). So there is this constitutive network of Mathematics, as a structured discipline, which participates in proof by forcing complex gestalts even in Number Theory.

But how to prove this? There is so little "gestaltist" work, along the lines we hinted, regarding the foundations of Mathematics or even Mathematical Cognition! The books by P. Maddy accentuate this lack: following the orthodoxy, Mathematics according to her is founded on Set Theory, hence the choice of looking out only for point-collecting cognition. Now, as we have said, it is as a *science of structures* that Mathematics organizes the world:

points, isolated, non-structured, are derived (as extreme of segments or intersection of two lines, as Wittgenstein said and as practiced by Euclid, see (Longo, 2010) for more on this). In fact, a line is not a set of points! It is a gestalt. It can be reconstructed by means of points (Cantor-Dedekind), but also without points (in certain toposes, Lawvere-Bell).

So there lies the immense cognitive (and historical) complexity of elementary (irreducible) judgments, such as the well-order of integers (of which the ad hoc reconstructions by induction along large ordinals are also very complex). These judgments make proof not a chain of formulas, but an organization of meaningful and complex geometries, of referrals, of threads which link mathematical reasoning to a plurality of acts of experience, both conceptual and pre-conceptual, as well as to other already constituted structures of Mathematics. The strength of reasoning, even of its certitude, lies therefore in its stability and invariance, as a dynamic uniformity (and one which changes throughout history) of deductive methods, but also in the richness of the lattice of connections which fastens it to a whole universe of practices and of meanings, even pre-mathematical or non-mathematical.

We can then understand why Mathematics has always been an essential component of the theories of knowledge, from Plato to Descartes, Kant, Husserl and Wittgenstein. One of the reasons for this attention, recurring throughout history, is due to its anchoring in fundamental processes of the shared forms of our deepest interactions with the world. The movement which traces a trajectory, a boundary, the following of a prey by saccadic eye movements, their memory, counting in view of gathering, dividing, comparing... are among the most ancient actions of our living beings and participate in mathematical construction. It even seems that writing began with the quantitative inscription of debts, among Sumerians (see (Herrenschmidt, 2007)). In short, it is possible that the first fundamental conceptual invariants were proposed with the maximality which is specifically mathematical, these invariants being complex developments, within human communication and language, of the original and most fundamental organization of human space and action. The Cognitive Sciences and Mathematics have all to gain in heightened, two-way interaction via the main issues in the foundations of Mathematics, the latter being one among several fields that were co-constituted with the intelligibility of the world, but one which distinguishes itself from all others through its maximal invariance and stability.

**Concluding remarks**

We would like to conclude with a strong remark, the goal being of completely toppling any absolute, any necessity, be it logicist, Platonic or otherwise, including those of Set Theory, a Newtonian Universe where all Mathematics would already be inscribed. This remark is inspired by the Husserl of "The Origin of Geometry" (1933) and helps us to understand the role of cognitive analyses: *any constitution is contingent*, in the sense of "*historical*", of an history that did *not need* to be there. It is the constitutive process, by its friction in the world, which ensures the objectivity and the (very reasonable) effectiveness of Mathematics. And this is so, precisely because it is contingent, since it is constituted along evolutionary and human history, in this world. Yet, we insist, Mathematics is *universal*, but it is so relatively to our phylogenetic and historical praxes, because it is grounded on the deepest components of our active experiences in the world (movement, action, retention, protension … see also below). By this rooting on action, well before language and, then, on language, Mathematics is maximally invariant and stable among our cognitive relations to the world. It is maximally effective and it is universal, for mankind and for the many animals that share our world, as long as they share it, yet it is not an absolute (we should always depart from absolutes, in analyzing science).

It is the friction with the actual world, that canalizes our action and linguistic proposals, and this is mostly interpreted as a necessity. But the world forces our conceptualizations along *possible* paths, as the concepts are not there, are not necessary, they *do not need* to be proposed. A falling stone does not move along a continuous trajectory, made of Cantorian points (how do the quanta it is composed of move, exactly?). Indeed, a frog, which does not have continuous saccades, sees it by well separated snapshots. It has been an evolutionary history, a possible one, that endowed us with protensive saccades and a cortical memory, whose "gluing" produces the apparent "continuum" of the trajectory: we *glue* the instantaneous retention and protension of a trajectory, in the sense formalized by differential geometry (Bailly, Longo, 2009). Continuity is in *our* active perception, in our exploring with eyes ("vision is palpation by the eyes"), an exploration, though, of the actual world, yet using our memory and expectation. On top of this cognitive mountain, a complex historical constitution, based on language as well, lead us from the cohesive pointless continuum of Euclid, with its conceptually derivable notion of "point with no dimension", to the incredibly audacious construction by Cantor, a possible complete mathematization of the continuum. The objectivity and effectiveness of Cantor's continuum is entirely in its contingent historical constitution[2] (we mentioned different ones, over Toposes with no "points", or with "not enough points").

Yet, Quantum Physics suggests that there is no such a thing as a "continuous trajectory", preceding our human activity. But, of course, trajectories are not discrete either: the "quantum jumps" traced by an electron in a bubble chamber are the macroscopic symptoms (they are revealed or measured by ionization) of what we better describe as the *continuous* evolution of a state vector in a (possibly infinite dimensional) Hilbert space. This evolution is described by Schrödinger differential equation, (Paul, 2007). The reader should give up any believe in the absoluteness and necessity as for his/her notion of "continuum" and "discrete", the latter naturally endowed with the discrete topology. They are "potentially universal", in the sense that our common humanity, with a shared phylogenetic history, may lead any human to understand the mathematical concepts, in their maximal stability and invariance; and this often after hard work as for the cultures so remote from ours, see the ones studied in (Pica et al., 2004). These concepts and structures are *possible*, but *not arbitrary* ways of making the world intelligible, at our historical best. And we can do better: non-commutative Geometry, (Connes, 1994), is entirely restructuring the physical space, with the aim of unifying Relativity and Quantum Mechanics, and we probably need of a new continuum for a better understanding of microphysics (some try with the Topos theoretic approach), or a renewed discrete, not naturally endowed with the discrete topology, which is rather unsuitable to grasp the quantum non-locality and non-separability phenomena (entanglement). But this may be the hardest part, since the discrete gestalts, well separated and "local" (the "discrete structures" in their discrete topology), are among our deepest cognitive experiences, as we stressed several times along this paper, far away form the largely counter-intuitive intelligibility of microphysics, given, most typically, by the non-separability of entangled quanta. As a matter of fact, microphysics is very remote from our life world, as it is always filtered by the strong theoretical commitment of the quantum measure instruments.

It is thus necessary to give to the words "history" and "contingency" the broad sense of an objectivity constructed by the cognitive subject, as the result of a phylogenetic and intersubjective history in its friction with the world, and not that of little psychological or sociological histories: the scientific challenge always lies in the difficult balance between possible subjective history and constructed objectivity of knowledge. And this challenge finds expression when we renounce absolutes, which are independent of our humanity and its

[2] For more on this epistemological perspective, see (Bailly, Longo, 2006) and related papers by Longo, since (Longo, 1998).

historical practices. We then try to see the mountain on which we are standing, that of language and of history, all the while looking towards the past, as far as that may be done, in order to perform an analysis of the epistemes that can help us look forward, to explain the philosophy of a new construction, beyond the concepts and structures which we have already provided ourselves with up till now.

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