Synthetic Philosophy of Mathematics and Natural Sciences
Conceptual analyses from a Grothendieckian Perspective

Reflections on "Synthetic Philosophy of Contemporary Mathematics" by
FERNANDO ZALAMEA,
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Introduction

Zalamea's book is as original as it is *belated*. It is indeed surprising, if we give it a moment's thought, just how greatly behind schedule philosophical reflection on contemporary mathematics lags, especially considering the momentous changes that took place in the second half of the twentieth century. Zalamea compares this situation with that of the philosophy of physics: he mentions D'Espagnat's work on quantum mechanics, but we could add several others who, in the last few decades, have elaborated an extremely timely philosophy of contemporary physics (see for example Bitbol 2000; Bitbol *et al.* 2009). As was the case in biology, philosophy – since Kant's crucial observations in the *Critique of Judgment*, at least – has often "run ahead" of life sciences, exploring and opening up a space for reflections that are not derived from or integrated with its contemporary scientific practice. Part of these reflections are still very much auspicious today. And indeed, some philosophers today are saying something truly new about biology.

Often Zalamea points the finger at the hegemony of analytic philosophy – and the associated "linguistic turn"

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– and the associated foundationalist projects in mathematics, highlighting the limits of a thought that, by and large, in spite of ongoing major advances, remains stuck to a debate on Hilbert's program (1900-1920) and Gödel's theorem (1931) – respectively an extremely important program and an equally important (negative) result, certainly. However, we should do well to consider that something important happened in the decades that followed, both in mathematics and in the correlations between the foundations of mathematics and physics, topics to which Zalamea dedicates several pages of his book. The conceptual and technical frames invented by Grothendieck are a fundamental part of these novelties.

At this juncture, I would like to introduce a first personal consideration: for far too long philosophical reflection on mathematics has, with a few rare exceptions,[3] remained within the limits of the debate going "From Frege to Gödel" (as per the title of a classic collection) a debate at best reaching the *statement* of Gödel's theorem, or indeed a simplified reduction of it which deprives it of its meaning. The meaning of a theorem is mostly (but not only) to be found in its proof, but in the case of Gödel's, it is found *only* by looking closely to its proof (see Longo 2010). Thus, with a limited range of references going from Euclid to, at best, the statement of Gödel's theorem, passing through Frege and Hilbert (often skimming over a great deal– Riemann and Poincaré being cases in point), for far too long we have debated ontologies and formalisms, thus moving, as Enriques had already foreseen in 1935, between the Scylla of ontologism and the Charybdis of formalism, a kind of new scholasticism.[4] I think, for example, that even within Logic, the beautiful results of Normalization in *Impredicative* Type Theory (see Girard, 1971, Girard et al. 1989), and of *concrete* Arithmetical incompleteness, as in the Kruskal-Friedman Theorem (see Harrington and Simpson 1985) – which allow for a breakout from this scholasticism (see Longo 2011) – or indeed the more recent progress in Set Theory, have not yet received a sufficient and *properly philosophical* attention. This attention should not be confused with, but should be a further insight w.r. to the remarkable ongoing work on "the philosophy of mathematical practice". First, because mathematics is a (historical) practice, no more no less, as any human cognitive and knowledge construction; yet, its specificity is in the proposal of maximally invariant principles, as in no other form of knowledge, largely grounded on cognitive activities that are common or accessible to mankind, often preceding language, while becoming linguistic. Ordering, tracing borders … in space and time are examples of them. Second, because the founding principles that, in particular, allow (and justify) the proofs in, say, the theory and results just mentioned (see Longo 2011), deserve a *foundational* analysis, grounding them in those cognitive praxes, as active "gestures" as proposed by (Châtelet 1993). The invention of the first and fundamental mathematical structure, the "line with no thickness" in Euclid's books (see below), is a paradigmatic blend of tracing a trajectory and describing in language an impossible property to draw, its absence of thickness. No Euclidean line is possible without acting a trace and defining it "with no thickness", then deriving its properties by both drawing and Logic. No gestures alone, no Logic alone found it.

3 Among these exceptions, an excellent collection is Mancosu 2008.

4"If we refuse to look for the object of logic in the operations of thought we open the door to this "ontology" which scientific philosophy must to fight as the greatest nonsense. … On the other hand, guarding oneself from the Scylla of ontologism, one falls into the Charybdis of nominalism: how could an empty and tautological system of signs satisfy our scientific reason?" … "On both sides I see emerging the spectre of a new scholastics". F. Enriques, 'Philosophie Scientifique', *Actes du Congrès International de Philosophie Scientifique*, Paris, 1935, vol. I-VII.

Zalamea's book is thematically vast. It is truly astounding to behold the rich range of mathematical themes that are touched upon, arguably including all of the most important objects of contemporary exploration. I can only single out a few of them, in an attempt to hint here to an "epistemology of new interfaces", and to emphasize, for my own account, the timeliness and epistemological relevance of the triadic relation mathematics-physics-biology which, obviously, is not the theme of this book.

Modes of Conceptualization, Categories, and Worldviews

> 6.5.1 Nowadays we may want to overturn Galileo's phrase: Is the book of mathematics written in a natural language?
> (Lochak 2015).

I would like to begin with what Zalamea considers, if I am not misreading his argument, the highest and most revolutionary point reached by post-World War II mathematics: Grothendieck's work. With a daring table (43) – as daring as it is arbitrary, like any such schematization – Zalamea sums up the principal "modes of conceptualization and construction pertaining to contemporary mathematics […]: arithmetical mixing, geometrization, schematization, structural fluxion and reflexitity". In his text, he gradually develops the meaning of each of these modes, attributing to Grothendieck alone the distinction of having contributed to every one of these forms of mathematical construction.

Before delving deeper into the arguments, and maintaining a rather survey-like approach (an inevitability when trying to sum up a book this rich) I think that I can single out the core node of Zalamea's thought in this statement: 'contemporary mathematics systematically studies deformations of the representations of concepts' (172). In more classical fashion, I would rephrase this by saying that mathematics is, *in primis*, the analysis of invariants and of the transformations that preserve them (including the analysis of non-preservations, deformations and symmetry breakings). This does not aim to be an exhaustive framing of mathematical construction, but rather the proposal of a different point of view, in opposition to, for example, the set-theoretical analytical one.

I will also try to show how Grothendieck, in particular, went beyond this vision of mathematics inherited from Klein's Erlangen Program and developed by many others (that of symmetries, invariants, and transformations). Gorthendieck proposed notions and structures of an intrinsic mathematical "purity", free from any contingency requiring proof of invariance, presented in an highly abstract (yet not formal) mode, always rich of mathematical sense, particularly thanks to the analysis of relations with other structures.

Symmetries have clearly laid at the heart of mathematics since well before Klein's work or before 1931. Indeed we can trace its centrality to Euclid, whose geometry is entirely constructed out of rotations and translations (symmetry groups as invariants *and* as transformations), through Erlangen Program, Noether's Theorems (1918) and Weyl's work between the two World Wars. I would like to highlight, more than Zalamea's text does, the correlations with the foundations of physics which these last two mathematicians put at the very core of their work – and, in Weyl's case, of his philosophical thought (see Weyl 1932; 1949; 1952; 1987).

Weyl's work profoundly marked the period examined by Zalamea, moving within a framework which we could legitimately define as that of Category Theory, with frequent mention, for example, of Topos Theory. Mac Lane, one of the founders, along with Eilenberg, of this theory, had spent a year in Göttingen in the early 1930s, in close contact with Weyl, the great "geometer" (and mathematician, and physicist…). Category Theory, considering the role it plays in the analysis of invariants and their transformations, is indeed a profoundly geometrical theory, so much so that it led, in Grothendieck and Lawvere, to the geometrization of logic, a topic I shall consider later (see Johnstone 1982; Mac Lane and Moerdijk 1992). I should also mention (again echoing Zalamea but with an even stronger emphasis) the role of physical theory in mathematical invention, with particular reference to Connes. But we cannot do everything, and I – not being a geometer, and thus unable to adjudicate on many of Zalamea's conceptual and technical analyses – shall attempt to read the text though my contemporary lens, shaped by several years of cooperation with physicists and biologists on the interface between the foundations of these disciplines (see Bailly and Longo 2011; Longo and Montévil 2014).

I am no geometer and Zalamea's text, one could say, is dominated by geometrical work, if intended in an extremely broad and modern sense. It is partially this central role assigned to geometry that motivates Zalamea's vigorous polemic against analytic philosophy. The latter has done nothing but increase its focus on linguistic play and logico-formal axiomatics, without any programmatic relationship with space and the constructions of physics; without paying attention to the constitution of mathematics in the world, and to the interface between ourselves and the world described by physics. Frege and Hilbert, in different ways, both programmatically wanted to avoid founding mathematics in relation to the 'delirium' (Frege 1884) or to the challenges of meaning of non-euclidean geometry and physical (lived and intuited) space (Hilbert 1901). And they did so for very good reasons. In order to give certainty to mathematics, it was necessary to keep in check

1. The dramatic break between the common-sense intuition of space and a physics in which "all that happens are continuous changes in the curvature of space" (Clifford, referring to Reimann 1854);

2. The unpredictability of dynamical systems (Poincaré 1892): a result of *undecidability* of future state of affairs for non-linear deterministic systems – that is, for formalizable systems of equations – at the interface between mathematics and physics (see Longo 2010). It was considered necessary to make sure that, at least in pure mathematics, every well-formalized statement could be decided (Hilbert). This is by principle far, therefore, from the undecidability and chaos that systems of non-linear equations had already started to reveal in the context of physical dynamics.

3. The new and bewildering role played by measurement in physics, where (classical) approximation or (quantum) non-commutativity had introduced unpredictability (Poincaré) and indetermination (Plank) in the interface between physics and mathematics.

The exactitude of the whole number, a "logical and absolute" concept (Frege) and its theory – Arithmetic – were supposed to guarantee "unshakable certainties" (Hilbert), thanks to the demonstrable coherence and to the formal decidability of pure mathematics: a far cry from the protean, approximate, unpredictable, and indeterminate world of physics. And so it happened that a century of debates on foundations remained trapped (and for good reasons) between programmatically meaningless formalisms and Platonist ontologies attempting to deliver a meaning from outside the world; outside, that is, of the difficult analysis of conceptual construction, the latter being the real bearer of meaning. It is precisely this latter kind of project that lies at the heart of Zalamea's philosophical work.

From physics, Zalamea borrows a methodological question: "the great paradigm of Grothendieck's work, with its profound conception of a relative mathematics [140-141] interspersed with changes of base of every sort in very general topoi [141 -142], should be fully understood as an 'Einsteinian turn' in mathematics" (270). And so Einstein's *Invariantentheorie* (as he preferred to call it) thoroughly becomes part of the *method* of this analysis of mathematical construction, broadly based on invariants and the transformations that preserve them.

It is clear then why this approach assigns a central role to the notion of the Category. This is not a Newtonian universe anymore, a unique and absolute framework, the Universe of Sets, with an absolute origin of time and space (the empty set). It is rather the realm of a plurality of Categories and of an analysis of transformations, functors, and natural transformations that allow their correlation (preserving what is interesting to preserve). Among them, the Category of Sets is surely one of the most interesting, but just one of many. We are presented with an *open universe* of categories, then, to which new categories are constantly added; new invariants, and new transformations. Concepts are created by being correlating with existent ones, and by *deforming* one into the other, thus enriching them, paying attention to the meaning (the mathematical meaning, at least) of what is being done.

Thus Zalamea also retrieves an operational relation with the supposed delirium or desorder we refered to at the interface of geometry with physics : "Advanced mathematics are, by contrast [to the elementary mathematics analyzed in most philosophical reflections], essentially dynamic, open, unstable, 'chaotic' […] the 'geometry' of mathematical creativity is replete with unpredictable singularities and vortices" (39). Yet there is an order, a dynamical organization to all this since, as Lautman puts it, we continuously reconstruct "a hierarchization of mathematical geneses […] a structural explanation of mathematics' applicability to the sensible universe" (58). And this, in particular, is possible thanks to structural dualities at the heart of any attempt to organize the world, like those between "local/global, whole/part, extrinsic/intrinsic, continuous/discrete, etc.", as Zalamea, writes, again quoting Lautman (64). Indeed, "Lautman intuits a mathematics of structural relations beyond a mathematics of objects – which is to say, he prefigures the path of category theory" (68), which was indeed born just a few years after his death.

The conceptual node that must be added to the analysis of proof, which was the dominant preoccupation of foundational projects in twentieth-century mathematics, is that of the analysis of the constitution of concepts and structures (where these latter are seen as an additional organization of mathematical concepts).[5] This is what Zalamea aims at: for him, Lautman and Cavaillès are frequent points of reference, two philosophers utterly forgotten

5 Proof theory is an extremely important and elegant branch of mathematics (and by working with its varieties (with and without Types), its "categorial semantics" and its applications I have managed to earn a living for most of my life). However, in philosophy, to omit this or that pillar of foundational analysis is a typically analytic limit. Corfield (2003) and Mancosu (2008) have worked to overcome this limit and to avoid both the Scylla and the Charybdis I mentioned above, by referring to "Mathematical Practice" (or "Real Mathematics"), as if there were a mathematics which is not a *very real praxis*: a way to underline the delay of philosophical reflection on contemporary mathematics, something that Zalamea does more explicitly. Among the interesting analyses of the contemporary mathematical work that these volumes present, I want to single out the articles by McLarty on the notion of "scheme" (a topological space with a sheaf of rings or more), and of Urquhart on mathematical inventiveness in physics, often non-rigorous or presenting an original informal rigour, a co-constitution of sense and therefore, gradually, of new mathematical structures (see Mancosu 2008).

by logico-linguistic approaches to mathematics (yet enjoying a more flattering oblivion than Poincaré and Weyl, who have been subject to offensive caricature as, for example, half-hearted Brouwers or semi-intuitionists).

I omit several passages and citations from the opening chapters of the book, where I find myself somewhat perplexed by what seems to me the excessive space dedicated to those, like Badiou and Maddy, who place the category of Sets in the usual role of absolute, Newtonian universe – albeit (in Badiou's case) with some dynamical inflection. Badiou, for example, in a recent seminar at the École Normale Supérieure (Paris) has explained – referring uniquely to the (original) statement of the Yoneda Lemma – that every (locally small) category is reducible to (embeddable in) the Universe of Sets (**Set)**, modulo a Topos of prescheaves (on **Set).** This would definitely prove the absolute role of **Set** for mathematics. Now, the *proof* of the Lemma yields a more general result. The functional embedding just described is possible within every Topos considered as a Universe in which one sees the given (locally small) category as an object: the embedding is then possible towards the presheaves on *any* Topos.[6] Therefore, by this construction, every Topos (typically a pre-sheaves category, but I shall come back to this) can play an analogous 'relativizing' role, without for all that becoming an indispensable absolute.[7] Similarly, Maddy identifies mathematical practice with the work done upon a structureless set theory and identifies, in this non-structured assembling of points and elements, the cognitive foundations of mathematics. These approaches are in explicit contrast with the key ideas of Zalamea's book which, centered upon categorical universes of geometrical inspiration, attempts to make us appreciate the structural sense of mathematical construction.

Luckily, soon afterwards, a reference to Châtelet enlightens us with a much different insight. References (perhaps too cursory) to that masterpiece that is Châtelet 1993, bring our attention back to the "gesture" constitutive of mathematical objectivity, which lies "on the border of the virtual and the actual", in a tight interrelation between the construction of objects of study and objectivity in physics and the analysis of the organizational structures of the world, starting with symmetries. Châtelet's book, it should be emphasized, is also an history; rather, it is a historico-rational reconstruction of the rich entanglement between physics and mathematics running through the 1800s up to, and stopping short of, the advent of Set Theory. Regarding some related aspects of contemporary mathematics, Patras 2001 (a book that Zalamea cursorily mentions), has retrieved the point of view of "structural mathematics" with a philosophical competence rare to find in a mathematician. Patras exhibits the weaving together of structures and transformations that governs mathematical construction from the inside, from the point of view of mathematical practice and invention.

In general, the origin of meaning in mathematics is to be found in the ways in which it allows us to organize, to structure, the world. Only then does it detach itself from the world in the autonomy of constitutive gestures,

6 One of the few required properties is the "locally small" hypothesis: every collection of morphisms Hom(A,B), must be a set (see Mac Lane and Moerdijk 1992). Once more, a close look at the assumptions and the proof (its right level of generality, in this case) is essential for the understanding of a theorem.

7 Many (all?) categorial objects can be codified as sets, even **Set**, the paradoxical "set of all sets". In every such occasion an ad hoc construction or codification is necessary, and in such a case, we pay the price of "stretching" the sets, up to cardinals as "inaccessible" (Kanamori 2003) as they are far from the construction one wants to interpret. These are codifications that push the meaning of categorial structures out of sight. The point, indeed, is not the possibility of a coding, perhaps a meaningless one: it is rather the relativizing -- and geometrical – diagrammatical knowing proper of categories, which is "sensitive to coding", as we might put it, that makes all the difference.

between the virtual and the actual where, at a farther remove from the original constitution of meaning, one obtains relevant results at the intersection between constructions of diverse origin. From classic algebraic geometry and differential geometry, two very productive blends, to sheaf-cohomology and cohomology-sheaves, between complex analysis and algebra (179), where, as Serre puts it, "such problems are not group theory, nor topology, nor number theory: they are just mathematics". Structural continuity becomes conceptual continuity, a navigation between concepts as a "sophisticated technical transits over a continuous conceptual ground".

In brief, the study of structures, of their continuous *enchaînements* and deformations, is an essential component of foundational analysis; without it one can at best hope to do Set Theory.[8] The latter is an extremely interesting theory and category: the error is to make an absolute out of it and to posit sets of meaningless points at the root of every mathematical construction, in what amounts to a ruinous disintegration of sense. The origin of mathematics and its principle of construction are located in that which is meaningful, in thought operations that structure and organize the world, but which then go to intersect on planes far removed from the world and acquire by these conceptual interactions a proper mathematical sense.

Thus Zalamea cites the "Langlands Program". Langlands dared to write to the more famous André Weil proposing an "extensive web of conjectures by which number theory, algebra, and analysis are interrelated in a precise manner, eliminating the official divisions between the subdisciplines", and suggesting that one "approach the world of the complex variable and the world of algebraic extensions functorially, by way of group actions". This will indicate an "unexpected equivalence between certain differentiable structures associated with an extended modularity (the automorphic forms associated with the linear group) and certain arithmetical structures associated with analytic continuations (the L-representations of the Galois group)" (180-182). Here we see groups again, and thus transformations and symmetries, both technical and conceptual, which allow for this splendid structural unity which lies at the heart of mathematics: in a certain sense, Langlands program extends Erlangen's program to Number Theory. So technical and conceptual invariants get transformed, like the generalized analysis of continuity that underlies the notion of fibration, and the subtle interplay between continuous and discrete, "the founding aporia of mathematics […] that drives the discipline", as Thom puts it (138).

Zalamea recognizes that "nothing could therefore be further from an understanding of mathematical invention than a philosophical posture that tries to mimic the set-theoretical analytic, and presumes to indulge in such 'antiseptic' procedures as the elimination of the inevitable contradictions of doing mathematics or the reduction of the continuous/discrete dialectic" (183-184). This, I would add, extends all the way to the discrete-computational approaches, flat (or better: unidimensional) visions of the world, according to which the Universe (Wolfram and others), the brain (too many to mention), or DNA (Monod, Jacob, Crick…) would be a (large, medium or small) Turing Machine (see Longo 2009, 2012). The great invention of Gödel, Turing and others in the 1930s, the theory of *logical-formal* - computability, instantiated in machines that today are changing our world, is projected by these stances to the world and *identified* with it, even while it was originally developed, within (Frege and) Hilbert's logical systems, thus to explicitly distinguish itself from the world. Nowadays these approaches are not so

8 Consider that the axioms of Set Theory, essentially created in order to adjudicate the validity of principles of "well-ordering" and "choice", are silent on them: a failure for a whole program. A refined analysis has been conducted, in structured environments wherein these constructions can be relativized, by Blass (1983).

counterproductive in physics, where they are mostly ignored: in biology, instead, such frameworks and methods exclusively grounded on discrete sets of strings of code have profoundly impaired the comprehension of biological phenomena. It is here that I will introduce a correlation of outlooks, the necessity of which I hope to convince the reader of.

Let us begin with an example. The discrete-computational outlook has not helped us (or has not permitted us) to detect the role of endocrine perturbators of the 80.000 (sic) artificial molecules that we produced in the twentieth century. These were mostly presumed to be innocuous, below arbitrarily imposed individual thresholds, since not stereo-specific (not in exact physico-chemical-geometric correspondence) and *thus* unable to interfere with *molecular-computational cascades, necessarily stereo-specific*, going "from DNA to RNA to proteins" (the Central Dogma of molecular biology), and with hormonal pathways. It should be noted, indeed, that exact molecular stereo-specificity was *deduced*, against experimental evidence that were already available (since 1957, see Elowitz and Levine 2002; Raj and Oudernaaden 2008): it is 'necessary', as Monod (1972) puts it, for the transmission of computational information and for the genetic programme to function. Thus, negating the role of context in genetic expression and hormonal control, the consequences (direct and indirect) of the finite combinations of said 80.000 molecules on the organism and on the chemical ecosystem of the living have receded from view. Cancer incidence has grown in the last half century, across all age groups, jointly to the *halving* (sic) of the average density of human spermatozoa in Western countries (Diamanti-Kandarakis et al. 2009; Soto and Sonnenschein 1999, 2010). As for cancer, the failure of the fifty years old, DNA centered, molecular approach has been recently aknowledged even by one if its founding fathers, Weinberg (2014).

In contrast with the claims of the informational analyses, macromolecular interactions – even within the cell, where the macromolecules in Brownian motion have quasi-chaotic entalpic oscillation – *are stochastic, and are given as probabilities*, *and these probabilities depend upon the context*; a strongly influential context, made of interactions, deformations, morphogenetic fields, biological networks and structures, and so on (see Elowits and Levine 2002; Noble 2006, among others. See also Longo and Montévil 2014). A context, then, made of *ecosystemic structures* and their *transformations*, very different from the fragmentation of the analysis of organisms as sets of molecules promoted by the still-dominant Laplacean reconstruction (a linear one, molecule after molecule, a "cartesian mechanisms" says Monod).

The discourse on the foundations of mathematics has played an enormous scientific, suggestive and metaphorical role in these events: the absolute certainty of the arithmetical discrete/finite, decidable (and thus programmable) has produced, on the one hand, original and powerful machines, perfectly artificial instruments for formal calculus allowing the "networking" of the world, while on the other it has contaminated our worldview – even though, originally, it had been lucidly and courageously originally proposed, by Frege and Hilbert, in order to detach those foundations from the world.

Logics, Topos, and Symmetries. In Brief.

Returning to less dramatic topics, another author Zalamea often refers to is Lawvere. The latter transferred Grothendieck's notions into an original analysis of Logic, grasping how Topos Theory and, more generally, Category Theory presents “a permanent back-and-forth between the three basic dimensions of the semiotic, emphasizing translations and pragmatic correlations (functorial comparisons, adjunctions) over both semantic aspects (canonical classes of models) and syntactic ones (orderings of types)” (191). Going back to my first scientific life, I remember the interest around the categorical interpretation of Type Theory, which owes much to many brilliant mathematicians who Zalamea has no space to mention (but who are cited in Longo 1988; Asperti and Longo 1991). A wonderful community, where a logical sensibility – and I am thinking of the challenge offered by Girard’s Impredicative Theories of Types – found in categorical semantics a strong link to the mathematics of structures that concerns Zalamea. The crucial point is the “geometrization” of logic and its “relativization” to Topoi that can have different internal logics, properly correlated by functors and natural transformations.

In these circles, Fregean quantifiers, for example, are interpreted in terms of adjunctions. More precisely, existential and universal quantifiers become right and left adjuncts to a sort of diagonal functor: the pullback along a projection. Then the existential quantifier is interpreted as the projection in a product of objects in well-defined Topoi, and the universal quantifier is its dual, modulo an adjunction. So the level of “effectivity” of the existential quantifier (the possibility of “effectively constructing” the mathematical object whose existence is predicated), a delicate issue that has been the object of a century-long debate, is relativized to the effective nature of morphisms in the intended Topos as a (relative) Universe – that is, to its “internal logic”. The meaning of logico-formal construction, then, is given by a reflexive interplay of invariances and symmetries (the duality present in an adjunction) without the need for an understanding of “for every” as meaning *for every,* or that “exists” really means *exists* – just as, for far too long, we have been told that “snow is white” is true just when *snow is white*, a truly remarkable mathematical discovery. When the “geometric” meaning of an adjunction is known, *qua* profound and omnipervasive construct of Category Theory, the meaning and the relation between the quantifiers is enriched with a new structural significance through the construction described above. That is, they become immersed in a geometric context, a universe of dynamic and modifiable structures. In particular, it becomes possible to go from one logic to another, from one Topos to another, studying their invariants and transformations, that is, the functorial immersions and the adjunctions correlating them. For this reason I often say, in provocative manner, that I am happy to leave the question of truth to priests and analytic philosophers: we operate constructions of sense, we organize the world by proposing and correlating structures that have a meaning because of our being world-bound active humans in different conceptual worlds which we strive to put into dialogue. Let us not confuse this with the fact that the judge seeks, in witnesses for example, the “truth”: science is not a testimony of, but an *action upon* the world, aimed at organizing it and giving meaning to it.

I will return shortly to this extremely timely geometrization of Logic, a “royal way out” of the narrow singlemindedness of the logico-linguistic turn. In this regard, Zalamea quotes Girard who, within Proof Theory, has subsumed the same structural sensibility, the same distance from Tarskian truth and its ontological flavours. I remember when I first attended, in the 1980s, a talk by Girard on Linear Logic; I asked him why, after having radically modified the “structural” rules of logics, changing their symmetries in formal notation, he had introduced a certain inference rule. He replied: for reasons of symmetry.[9]. Symmetries are at the core of the close relationship between physics and mathematics, ever since Archimedes asked himself: why doesn't a scale with equal weights on both sides move? And answered: For reasons of symmetry. Guided by the same symmetry reasons, Sacharov and Feynman proposed anti-matter, thus giving a meaning – faced with experimental phenomena in need of explanation – to the negative solution of Dirac's electron equation. Alas, unfortunately (or fortunately?) cellular reproduction is at the heart of ontogenesis and phylogenesis, also because it is asymmetrical.

More on Invariance and Symmetries, in Mathematics and the Natural Sciences

1-Between mathematics and physics: Symmetries, Gestures, and Measures.

I have been too critical, much more than Zalamea is, of Set Theory as a foundational discipline, since there is one concept about which it has been the field of a rigorous and useful foundational analysis: the question of the infinite.[10] This is a crucial concept in mathematics. All mathematics is construction to the limit, starting with the line with no thickness of Greek geometry, a limit construction, all the way to the higher constructs I have discussed above. It has come into relation with physics since Galileo's *asymptotic* principle of inertia. Great merit goes to Shelah, whose work Zalamea discusses at great length, for he demonstrated that "the theory of singular cardinals corresponds to the idea of seeking natural algebraic invariants (homotopies, homologies) for topology" (202). From there, we are referred to Serre's work on homotopy, which makes possible an algebraic-topological relativization of the notions of finite and infinite. Once again, it is a relativizing operation, breaking with the absolutes of logicist formalisms, according to which the "finite" is locus of certainty and absoluteness. Likewise, in physics, the "Riemann Sphere", a bidimensional model of the relativistic universe, is infinite for its surface-bound inhabitant moving towards the poles, whose meter stick progressively contracts; it is finite as observed from an external reference frame.

At the level of groups, however, a discrete combinatorics can be fundamental; indeed, Zalamea refers to the Grothendieck-Teichmüller groups, which "may come to govern certain correlations between the universal constants of physics (the speed of light, the Planck constant, the gravitational constant), while, conversely, certain mathematical theories originating in quantum mechanics (non-commutative geometry) may help to resolve difficult problems in arithmetic (the Riemann hypothesis)" (205). As Zalamea tells us, here we witness "absolutely unanticipated results, which bring together the most abstract mathematical inventions and the most concrete physical

9 Symmetry principles – or more precisely principles of "inversion" – were already present in Grentzen's sequent calculus, to which Girard explicitly refers to. They permit the "generation" of a calculus starting with logical connectives, and to finely analyze the properties of proof-theoretic normalization (see Negri and von Plato 2001).

10 This analysis extends all the way to the recent and daring "anti-Cantorian" explorations of Benci, Di Nasso and Forti (in Blass et al. 2012). According to them, as for Euclid, "the whole is larger than its parts", even for infinite sets (at least when denumerable: this approach, for the time being, is not extended beyond the denumerable. For this latter domain, we will probably have to look beyond the category of sets, towards other structural invariants).

universe" (206).

Through a back-and-forth between mathematics and physics, various intersections far from the world are drawn out, between domains with roots in diverse conceptual constructions, each originating in different organizational actions upon the physical world. It is neither unreasonable nor surprising that the locus of conceptual invariance and of the analysis of its transformations – mathematics – should influence theoretical physics. Beyond the strict relation mentioned above between mathematical symmetries and conservation principles in physics (Noether, Weyl), the physicist's theoretical work begins from the invention of appropriate, and very abstract, mathematical phase-spaces (observables and pertinent parameters) like the spaces of state-function in quantum mechanics or Hilbert spaces; all phase-spaces the physicist uses or builds to analyze (generic) objects and (specific) trajectories, result, in turn, from symmetries and invariances. I will try to sum up here analyses and notions which are central to attempts to differentiate and establish a dialogue between mathematics, physics, and biology (as exposed in Bailly and Longo 2011 and Longo and Montévil 2014).

Mathematics and physics share a common construction insofar as they isolate and draw pertinent objects, perfectly abstract and with pure contours – like Euclid's lines with no thickness, edges of figures drawn on the veil of phenomenality, at the interface between us and the world. Euclid, indeed, invents the difficult notion of border: his figures are nothing but borders, and thus without thickness – one thinks of Thom's cobordism (Rudyak 2008). These objects, in mathematics as in physics, are *generic*, that is interchangeable, symmetrical according to permutations within their definitional domains. A right-angled triangle in Euclid, a Banach space, or a sheaf, are all generic, as are Galileo's weight, an electron, a photon, and so on. These are generic insofar as they are invariants of theory and of physical experience, symmetrically permutable with any other. So that the same theory can deal with falling apples and planets as generic gravitational objects, just as the even more marked theoretical invariance of the theory of relativistic bodies allows us to unify gravitation and inertia. The genericity of objects and of structures, therefore, is the result of a fundamental symmetry/invariance, shared by both mathematics and physics.

Beginning with the genericity of its objects, physics analyzes "trajectories" in a suitable phase-space. The classical one based on momentum and position (or energy and time) is only one among many (thermodynamics, for example, operates within a space defined by pressure × volume × temperature, and has added a revolutionary observable: entropy). These trajectories are *specific*, unique, and are imposed by the geodetic principle in its various instances. Even in quantum mechanics, where the quanta certainly do not follow "trajectories" in space-time, the Hamiltonian allows the derivation of the Schrödinger equation, defining the trajectory of a probability amplitude in Hilbert space. But the Hamiltonian, or the extremization of a Lagrangian functional, follow from a conservation principle – a principle of symmetry – as Noether's theorems have explained (see Kosmann-Schwarzbach 2004; Bailly and Longo 2011). Here is the extraordinary unity, completely construed or better co-construed, of the physical-mathematical edifice. Here is the power of its intelligibility, utterly human, for we animals characterized by a fundamental bilateral symmetry who, in language and intersubjective practices, organize the world, our arts, and our knowledge in terms of symmetries (see Weyl 1949, 1952, followed by Van Fraassen 1993) and, subsequently, their breaking.

Such unity will be discovered in the symmetry breaking constituted by the non-Euclidean modifications of Euclid's fifth postulate – which yields the closure of the Euclidean plane under the group of homotheties – a breaking that will allow Einstein to give a mathematical foundation to relativist physics, beginning with the astonishing measurement of the invariance of the speed of light. Likewise, in Connes' non-commutative geometry, which includes physical measure in the foundations of his approach: Heisenberg's matrix algebras, from which it derives in analogy with Gefland's construction, are built starting with the non-commutable nature of quantum measurement. In a striking difference from arithmetical foundations, geometry, the privileged locus of invariance and transformations, has always had an origin in a constructive relationship of "access" to space and its processes: from the Greek compass and straightedge to Riemann's rigid body, to the algebras derived from Connes's quantum measurement, yet another bridge between mathematics and the universe of physics.

To sum up, a fundamental component of the unity we have delineated between mathematics and the theorization of the inert is this central role assigned to the *genericity* of objects and the *specificity* of their trajectories, both being definable in terms of symmetries. To this we should add an active relation to the world, grounded on both the constitutive gesture of the continuous line, of the trajectory – a movement at the origin of the phenomenic continuum – and on the access to the world as mediated by measurement: classic, relativistic, and quantum. Following Zalamea, I will return, in what follows, to some contemporary consequences of these considerations (which sum up ideas extensively developped in Bailly and Longo 2011 and in Longo and Montévil 2014, and are directed towards a discussion of biology).

2- What About Biology?

What can we say about the theorization of the living? The only great biological theory, Darwin's, was born by positing some principles: of which the first in particular, "descent with modification" (indispensable for the second, "selection"), stands in stark contrast to those conservation principles (symmetries) which, starting with Galileo's inertia and the geodetic principle (think of Hamilton's variational method, contemporary to Darwin), were taking center stage in physics. "Descent with modification" is a principle of *non-conservation* of the phenotype, of organisms, of species and of all the observables of Evolutionary Theory. The morphogenetic iterationin the living, in particular reproduction as conservation by *inheritance,* is never identical to itself, and this must be take its place as a fundamental principle, together with Darwin's, of the intelligibility of ontogenesis (see Longo *et al*. 2014).

We are working towards an understanding of onto-phylogenetic trajectories as "cascades of symmetry change", a kind of "extended critical transitions" (see below), borrowing a *method* from physics: a mathematical construction of objectivity, yet with dual principles. Critical transitions capture the continuity of change that is proper to reproduction. The challenge is to unify ontogenesis and phylogenesis, on the basis of the same, or similar, principles (see Longo et al. 2014), thus towards a "theory of organism" and therefore of ontogenesis, avoiding the prescientific metaphors of an Aristotelian homunculus codified in the DNA (even when the defenders of such "theories" dress their ideas in modern garments: the homunculus is in a machine code and the DNA contains both the program and the operating system (Danchin 2009)).

The problem is that biological trajectories, cascades of changes of symmetry in constant interaction with the ecosystem, must be considered as generic: they are "*possible*" trajectories among the many which are compatible with the ecosystem – the limbs of an elephant, of a kangaroo, of a whale (its vestigial forms) are so many possible evolutions originating form a same tetrapod vertebrate. What's particularly hard to grasp is that they are *possibilia* in phase-spaces (to use a physics jargon), not pre-given but rather co-constituted with trajectories: so an organism, in phylogenesis as well as in ontogenesis, co-constructs its ecosystem: consider how, two to three billion years ago, bacteria created oxygen, beginning with a primitive atmosphere which contained none or in negligible amounts. And so the pertinent observables – that is, the phenotypes – are modified up to speciation. The result of this evolutionary trajectory is an historical and individuated object, a specific organism, the result of a contingent cascade of change of symmetry (*qua* changes of the coherence between organism and ecosystem) channeled by massive historical "constraints". One of the most important of which is the DNA: the imposing chemical trace of an history, continuously employed by the organism throughout the course of ontogenesis.

To sum up: biological trajectories are generic, while their objects are specific – a radical duality, as opposed to the physical-mathematical realm, where we pointed out the genericity of the objects and the specificity of the trajectories. Such duality profoundly modifies the role – so rich in physics – of symmetries, invariances and transformation. To the impenitent reductionist, hellbent on an abstract physics (and not the physics of the historically-situated theories) to which everything must be reduced, we respond (see the introduction of Longo and Montévil 2014) with a recommendation, for example, to try to "reduce" the classical domain to the quantum one, or the hydrodynamics of incompressible fluids in a continuum to quantum mechanical principles, if she can – after all, there are both classical and quantum dynamics (and plenty of water) at play within a cell. The unity of knowledge and of its scientific instruments, starting with unity in physics, is a hard-won conquest – as in the case of quantum and relativistic physics – and not a theoretical *a priori*.

I mention these problems both because they are my current interests and because the construction of objects and structures in mathematics has proceeded in lockstep with a prodigious construction of objectivity in physics, simoultaneously locating in the richness of language and of historically located human gestures an autonomy that pushed it steadily away from physical experience (where is Euclid's thickless line to be found? Where is a Grothendieck pre-sheaf located?). And yet, considering the analogous approach in physics and mathematics to "objects" and "trajectories", this was a process of constitution capable of falling back again upon physics, through unexpected avenues: think of the marvelous story of Cardano's imaginary numbers, having an highly abstract algebraic origin and yet being today essential to talk about microphysics (yet Argand's and Gauss's interpretation allows us to discern a possible role for them in the description of wave amplitudes and their trajectories: before falling back upon the world, they became a rich geometric structure).

This parallel construction of objects and concepts does not merely concern the interaction of physics and mathematics. Indeed, even in the ambit of proof, mathematics does not proceed by way of demonstrations of already-given formulae – as the formalist caricature would have it – and physics does not construct theories as summations of experiences and facts. Neither proofs nor theories are "already there", not even in the most dynamical and weakly-Platonic sense. The construction of sense plays a powerful role in proof, even arithmetical proof (see Longo 2010, 2011); likewise, physical theory tells us which observables are to be isolated and analyzed, which experiences to have, which phenomena to observe. Mathematics and physics are the result of a laborious effort of knowledge construction, as Weyl has it, through a non-arbitrary friction with the world. Non-arbitrary and effective precisely because rich in history and contingency: mathematics and physics are thus a human praxis in and towards the world, as Peirce – a thinker Zalamea often likes to refer to – would say.

Contemporary biology poses enormous challenges: to face them we would need to combine the imagination of Newton (a Newton of the blade of grass, as Kant has it, without denying the possibility of such a science), with his differential calculus as infinitary construction to understand the movement of the finite; of Hamilton, with the variational method for the geodetic principle; of Dirac, with his delta, for a long time without any mathematical sense; and of Feynman, with his integral, the solution of a still-non-defined equation. The principal invariant in biology (fortunately not the only one) is *variability*: it allows diversity adaptability, at the heart of the structural stability of the living. What to do with our *invariantentheorien*?

Groups Everywhere, Metrics Everywhere

Among the omnipresent references to Grothendieck, Zalamea underlines time and again how his work incorporates "a transit between objects (variations, perturbations) so as to then proceed to determine certain partial stabilities (invariants) beneath the transit" (212). As for the invariants, I have often referred, as Zalamea does, to those correlated with symmetries, i.e. group structures. But together with groups (to be interpreted as instruments of action upon spaces, all the way to the most abstract ones due to Grothendieck), a crucial epistemic role should be assigned to semigroup structures. As it is observed in Bailly and Longo 2011, on the one hand we should consider the gnoseological and mathematical complex of {space, group, equivalence relation}, on the other that of {time, semigroup, ordering relation}. In the passage between the two we see a useful instrument to analyze the interplay between space and time in the natural sciences, as well as the difference between physics and biology: oriented/ordered time plays a crucial operatorial role in biology, as we say also in Longo and Montévil 2014, well beyond its role as parameter in physics. In this regard, Zalamea insists on the role of semigroups in the hyperbolic varieties of Lax and Phillips (218). These are collections of operators Z(*t*), with a parameter that can be interpreted as time, which permit the construction of "the deep connection that lets us unfold the 'intrinsic meaning' hidden in differential equations like the non-euclidean wave equation, a meaning that can be glimpsed precisely in virtue of the semigroup Z(t)" (220).

In this inexhaustible search for unity, not forced towards impossible reductions, but constructed with bridges, correlations, and structural passages, we can "naturally mediate" between "the Poincaré plane, seen as a non-Euclidean model, with its differential Riemannian geometry and analytic invariants, on the one hand; and the same plane, seen as a complex model, with its theory of automorphic functions and arithmetical invariants, on the other" (220). Here we arrive at Connes's programme for non-commutative geometry, a programme for the reconstruction of vast sections of mathematics, grounded on the non-commutativity of quantum measure (and its algebras). The objective of this geometrization of quantum mechanics is to contribute to its intelligibility and, ultimately, to deliver a unification with the relativistic universe, radically changing the theory of space – not a mere "background", as string theorists claim. Zalamea adroitly sums up several bridging aspects, correlating them with the work of other geometers, starting with the recent developments of Riemaniann differential geometry, with particular focus on

> the passage from infinitesimal manifolds (Riemann) to C*-algebras of compact operators (Hilbert, von Neumann), the passage from dual K-homology (Atiyah, Brown, Douglas, Filmore) to non-

commutative C*-algebras (Connes), the passage from the index theorem (Atiyah, Singer) to the handling of non-commutative convolutions in groupoids (Connes), the passage from the groups and algebras of modern differential geometry (Lie) to quantum groups and Hopf algebras, the passage from set-theoretic punctuality to the actions of non-commutative monoids in Grothendieck topoi, etc. (224).

There is no doubt in my mind that this allows for a correspondence *in fieri* between mathematics (as a study of quantities and organized in structures) and the cosmos (as order), as Zalamea argues, legitimately philosophizing from a conjecture of Cartier. But this shouldn't be considered a new Pythagoreanism, in my view: it is we who single out elements of order in the cosmos (those we can and want to see – symmetries for example). As Kontsevich, quoted by Zalamea, has it, in physics we begin with very little: "where one doesn't see structures so much as the symmetry, locality and linearity of observable quantities" (229). We then enlarge these almost Gestaltic elements (symmetries and locality), we generalize them, and we transform them into the language of a metaphysics-rich communicating community. Finally we project them back again upon the cosmos, recognizing it as orderly because intelligible, and intelligible because orderly. This process is legitimate because, in this theoretical back-and-forth, our friction and action upon the world are real: the world resists, it says "no", and channels our epistemic praxis, which is of an eminently *organizational* character, and it is always *active*.

Such knwoledge construction works because of this cognitive entanglement, beginning with the common genericity (of objects) and specificity (of trajectories), both physical and mathematical: the first brick of an enormous physico-mathematical edifice of our making. No surprise then, a surprise still affecting Kontsevich and Zalamea; we are left with great admiration for such a majestic, but very reasonable, mathematical construction. Similarly, the linguist is not surprised if, when we talk, we understand each other: language was born with dialogue, through the practice of mutual understanding and communication. The linguist surely admires a great poem which, with words, introduces a different worldview or an original intelligibility of humans, without ontological miracles but merely with the strength of the words' meaning, a co-constituted product of our human community. Alongside myths, poems and tragedies – rich in human experience, in human, concrete and lived praxis as well as in metaphysics – we have been able to propose the structures of mathematics with their invariants and transformations, rich in those glances and gestures which organize the real, as well as rich in metaphysical nuance – starting with Euclid's line, a limit notion resulting from a dialogue with the Gods. Mathematics is written in natural language, it is a language and a gaze upon the world, *at* and *from* the limit of the world ("mathematics is the science of the infinite" as Weyl (1932) writes).

However, we only see perspectives, albeit coherent and profound ones; points of view on fragments of the world, we organize and make accessible small corners of it. And as soon as that small (but oh so important) brick concerning physico-mathematical genericity and specificity is removed, as happens in the analysis of the living, we find ourselves in trouble. Yet it is nothing unsurmountable: we just have to work on it with the same freedom and secular independence of thought, action, construction and exchange proper of the founding fathers of the physico-mathematical, abandoning the ambition of finding the theoretical or mathematical answer "already there", written by God in the language of already-existing mathematized physics.

> Referring to Peirce, Zalamea too highlights the progressive *constituting* of knowledge of the world:
>
> we see how the 'world' consists in a series of data/structures (Peircean firstness), registers/models (Peircean secondness) and transits/functors (Peircean thirdness), whose progressive interlacing into a web not only allows us to better understand the world, but which constitutes it in its very emergence. (237)

The important thing is to break out, even in foundational analysis, from "an 'absolute mathematics', a mathematics at rest, in the style of Russell" and proceed towards "a 'relative mathematics', a mathematics in motion, in the style of Grothendieck" (240). The entire work of contemporary mathematics, carefully recounted by Zalamea, aimed at the production of

> remarkable invariants … *without any need of being anchored in an absolute ground*. We will therefore take up a revolutionary conception which has surfaced in contemporary mathematics in a *theorematic manner*: the register of *universals capable of unmooring themselves from any 'primordial' absolute, relative universals regulating the flow of knowledge*. (242)

Developing the theme of "relative universals", Zalamea introduces Freyd's "allegories": abstract categories of relations, exposed in diagrammatic terms via representations that obviously "a functional, set-theoretic reading would fail to detect" (243). I want to stress that, in general, categorial diagrams are not "equivalent" to the equations to which they can be formally reduced: the diagrams indeed highlight symmetries that are merely implicit, invisible, in the equations; they need "extracting", just as Noether's theorems extract symmetries from the equations of physics.

> Freyd shows how, starting from pure type theories with certain structural properties (regularity, coherence, first-order, higher-order), one can uniformly construct, by means of a controlled architectonic hierarchy, free categories that reflect the given structural properties in an origin (regular categories, pre-logoi, logoi, and topoi). (243)

In this way, all the invariants of logico-relational transformations – beyond the particular variants of any specific logico-mathematical domain – are expressed in a maximally synthetic and abstract way. As usual, the analysis of transformations, of preserved structural invariants, and of variants (which can however have a "local" sense) is at the heart of mathematics, and this is confirmed by the logical-foundational spirit of Freyd's work. Referring to the latter, and taking his moves from the Yoneda Lemma, Zalamea uses the occasion to explain, as I mentioned above, that pre-sheaves categories can be considered as the general locus of the "continuity" wherein every discrete category can be embedded. Like Thom, one comes to the conclusion that the continuum "underlies" (is an archetype) for the discrete as well (Thom argues that a discrete set is nothing but a collection of singularities in a continuum).

Without necessarily according ontological priority to the one or the other, I would like to observe that, in the natural sciences, the discrete and the continuum organize the world differently, and this can be demonstrated: by analyzing the different role of symmetries and their breakings, which these mathematical structures, when employed for theoretical organization or simulation, accentuate and project upon physical and biological processes (see Longo and Montévil 2014a).

Having passed through a technically pertinent close-up of the reverse mathematics of Friedman and Simpson, Zalamea demonstrates how the work of Zilber contributed to giving a Grothendieckian understanding of the model theory of Tarskian tradition (Chang, Keisler): no more "logic + universal algebra" but "algebraic geometry + fields" (Shelah, Hrushovski, Zilber, Hodges). With Zilber we have "the emergence of 'groups everywhere' – invisible at first, but lying in the depths ('archetypes')" (256). A kind of "renaissance" and generalization of Erlangen's program, as Zalamea rightly notes.

An analogous motto allows us to grasp a central element of Gromov's contribution to geometry: "'smoothing' and 'globalization' that are tied to the notion of *metrics everywhere*" (259). Then Zalamea hints, with fine synthetic and analytic skill (that is, with great command of language and pertinent mathematical references, as always), to the work of Gromov on "partial differential relations, on "symplectic varieties", and on hyperbolic groups (259) – a work enriched by a certain sensitivity, proper of the French-Russian school, to the play between geometric insight, analytic virtuosism and physical applicability. Introducing pseudoholomorphic curves and seeking the

> invariants of those curves, Gromov shows that the spaces modulo the curves are compact, and that it is therefore possible to work out a natural theory of homology, which leads to the Gromov-Witten invariants; in the last instance, the new invariants allow us, on the one hand, to distinguish an entire series of hitherto unclassifiable symplectic varieties, and, on the other, help to model unexpected aspects of string theory. (262)

Once again, the analysis of the invariants and the transformations preserving them – relativizing the movement between a structure to another – is at the core of Gromov's work on Riemaniann manifolds, within a program of "geometrical group theory" described as the project aiming at "characteriz[ing] finitely generated groups, modulo quasi-isometries, which is to say, modulo 'infinitesimal' deformations of Lipschitz-type distances" (264).

In Chapter 8, Zalamea synthetizes some of the themes touched in the book, in order to propose his own vision of a "transitory ontology". It is a relativizing, yet not *relativist* vision (of either the "weak" or the "anything goes" variety), an Einstenian vs. Newtonian one, at the center of which lie transformations (passages, transits) and pertinent invariants: "the transit of mathematical objects consists in finding suitable invariants (*no longer elementary or classical*) behind that transit" (271). And so Zalamea himself sums up the themes he examined more extensively earlier in the book

> motifs [p.144-146], pcf theory [p.201-202], intermediate allegories [p.245-246], Zilber's extended alternative [p.257], the h-principle [p.263], etc. […] neither absolute foundations nor fixed objects, not everything turns out to be comparable or equivalent, and where we can calculate correlative archeal structures – that is, invariants with respect to a given context and a given series of correlations – which, precisely, allow differences to be detected and reintegrated. (272)

Representation theorems,which Zalamea often mentions in his book, assign a key role to strong and diverse specifications of the notion of group. To emphasise this role, I borrow Zalamea's own list of topics (specifying, in square brackets, where each theme has been considered), always examined with a refined informality that manages to be both complete and informative.

> homology and cohomology groups [p. 142-148, 178-179], Galois groups [p. 150, 155, 225], group actions [p. 162-163, 180-181], Abelian groups [p. 165], homotopy groups [p. 176], algebraic groups [p. 184], the Grothendieck-Teichmüller group [p. 225, 233], Lie groups [p. 223], quantum groups [p. 223], Zilber groups [p. 255-256], hyperbolic groups [p. 264], etc. (272)

This demonstrates a dynamics of "webs incessantly evolving as they connect with new universes of mathematical interpretation. […] This just goes to reinforce the position of Cavaillès, who understood mathematics as gesture" (273). Such are organizational gestures of correlated mathematical universes, correlated by a web of transformations, like the hand gesture that organizes space, gathers, delimits, and transfers, as we can say with Châtelet. This process assumes an historicity that serves to highlight the sense and the relationship of mathematics vis-à-vis the real: mathematics works (where it does work) and has meaning because it is constituted through a human – all too human – praxis. All too human because it is anchored to pre-human invariants, those of our actions in space and time; universal, for us historical and speaking human beings, precisely because pre-linguistic and pre-historical, even though language alone allowed the transformation of "practical" invariants into concepts and structures. And, in language, writing, as Husserl (1970) observed, has further contributed to the process of the stabilization of concepts.

Considering the correlations between groups, symmetries and invariants, in the context of this section on "groups everywhere, metrics everywhere", I would like to mention the role of (animal) memory in the constitution of invariants. Memory is *forgetful*, that is one of its essential properties: we, as animals, forget irrelevant details of an action, of a lived experience. Irrelevant, that is, with respect to the protensive – intentional (conscious) or not – gesture, already done or still to be performed: memory is selective in both its constitution and in its re-activation. This selective choice allows us to undertake once again a given action in a similar but not identical context, to operate another protension or prevision, counting on the relative stability of the world, through changing distances, for example, which we attempt to organize in stable metric evaluations. We do not access memory as we would access a digital hard drive. The protensive gesture, I say with and beyond Cavaillès, *reactivates* memory every time: not in a passive way, but choosing, selecting and constituting new practical invariances, beyond those isolated and selected by memory in its constituting process. Animal memory is reactivated in a protensive manner, or better, it is *re-lived* for a purpose, be it a conscious or non-conscious one, forgetting all that is irrelevant to the present goal: (Edelman and Tononi 2000) argue that, in the act of memory, he brain puts itself in a lived state..

Meaning derives, moreover, from the intentionality, even a pre-conscious one, that inheres in protensive gestures, particularly in a "perturbative" modality. It is that which *interferes* with, and which operates *a friction* upon, the protensive action which acquires, for us as animals, a meaning. And there is no protension without retention. Obviously, then, a digital machine with a perfect memory cannot do mathematics, because it cannot constitute invariants and its associated transformation groups, because a perfect, non-protensive memory does not construct meaning, not even mathematical meaning. At most, the machine can help with formal fragments of proofs, or check, a posteriori, the formalized proof, or parts of it (proof-assistance and proof-checking are burgeoning fields). Only animal memory and its human meaning allow not only the construction of concepts and structures, but *proof* as well, as soon as the latter requires us to propose new concepts and structures, or the employment of ordering or invariance properties which go beyond the given formal system (well-ordering, say, or the genericity of infinitary structures). It is thus that recent results on the concrete incompleteness of formal systems can be interpreted: meaning demonstrably lurks in the proofs of formally unprovable theorems (see Longo 2010, 2011).

Zalamea's transitory ontology

Zalamea insists on employing a terminology of different forms of "ontology" (local, regional, transitory…). Mathematics, between 1950 and 2000, as he adequately demonstrates, proceeded by an analysis of streams, *transits* and deformations of structures, and their limits. A network was therefore built, a web weaving together – via passages and transits, but also dualities and limits – a bewildering variety of constructions. In such a web even Logic and Proof Theory find a new structural significance,

> where pivotal statements in logic such as the Loz theorem for ultraproducts, the completeness theorem for first-order logic, forcing constructions in sets, and theorems of type omissions in fragments of infinitary logic, can all be seen, uniformly, as constructions of generic structures in appropriate sheaves. (284)

Indeed, sheaves constitute a structure of particular interest, very often mentioned in the text. Born with Leray's analysis of indexes and "converings" of differential equations, "sheaves are precisely what help to capture (and glue together) the continuous variation in the fibers." (285, n. 345). Moreover they allow movement between the local and the global. So, thanks to Grothendieck's generalization (sheaves on a Grothendieck topology), they allow the integration of "a profound web of correlations in which aspects both analytic and synthetic, both local and global, and both discrete and continuous are all incorporated" (286). Obviously, the category-theoretical framework is the most fitting for this organization of mathematics. If in the Category of Sets objects are non-structured and non-correlated conglomerates of elements, "category theory studies objects through their external, synthetic behavior, in virtue of the object's relations with its environment" (288). Avoiding set-theoretic absolutes, in Category Theory the notion of "universality", for example, is relativized, becoming a "unicity" relative to given structures, in the given class of morphisms. We have already observed how the analytic/set-theoretic approach leads, perniciously, to the description of every categorial diagram in terms of equations. Now the constructions (co-product, adjuctions, pull-backs…) or the proofs in Category Theory can be based upon, and have a meaning thanks to, symmetries and dualities present in the diagrams, absolutely invisible in the equations. I therefore once again underscore the fundamental contribution of Noether's theorems, which "extract" physical invariants by reading symmetries in the equations (of motion): in the same way that categorial diagrams "extract" meaning out of mathematical correlations, which then become visible and comparable symmetries.[11]

Zalamea's work aims at moving the web of mathematical structures that have been introduced by contemporary mathematics to the level of epistemological analysis, similarly as we saw the transfer the methodological content of Einstein's invarintentheorie to a foundational approach. That is to say, it aims at the construction of a comparative epistemology, "a sort of epistemological sheaf, sensitive to the inevitable complementary dialectic of variety and unity that contemporary mathematics demands" (296). A mathematical knowledge some of whose highest peaks Zalamea (296) enumerates ("Grothendieck's motifs beneath the variations of cohomologies [p. 144-148] […] Freyd's classifying topoi beneath the variations of relative categories [p. 245-246]"), proceeds between conceptual networks and their deformations "by means of series of iterations in correlative triadic realms: differentiation-integration-invariance, eidos-quidditas-arkhê, abduction-induction-deduction, possibility-actuality-necessity, locality-globality-mediation" (297). The goal is that of "a sort of epistemological 'sheafification', where the local differential multiplicity is recomposed into an integral global unity" (299).

Is this a "foundationalist" epistemological analysis? It surely is, in my opinion, since every epistemology is also an analysis of a network of correlations and an history, a rational reconstruction of a constitutive path, evidencing the network of passages and transits and, in this way, the unity of the construction of knowledge. Of course, such an analysis doesn't propose logical or ontological absolute foundations, since the network is held together thanks to its own structure, but also thanks to its friction upon the world, thanks to the unity of language, thanks to its history – through which it constituted itself – and thanks to the windows of intelligibility that it bestoys upon us. In this sense, to be provocative once again, I would go as far as to say that mathematics helps us to construct objectivity precisely *because* it is contingent, the result of the "history" of a real friction with the world. In

11 We should note that the notions of "scheme" from algebraic geometry, of "frame of locale theory", or of Grothendieck topos, and their properties, are not captured by an approach in terms of "space = set + topology" (or "space = set + structure). For example, from the constructivist point of view, important theorems like Heine-Borel's do not hold in set-theoretic contexts, while they do in adequate, point-free, topos (see Cederquist and Negri 1996). Similarly, constructions based on pull-back, insofar as they are eminently categorial, allow to distinguish the obtained structure from the set of points (when it is not an invariant with respect to the "sets of points" in question). And a pull-back, typically, has a meaning – a visible meaning – only if we can appreciate its symmetries: the construction itself is given by a duality (a symmetry) upon diagrams.

this history we need to include that cognitive rooting, all the way back to its pre-human form, at which I hinted before when considering the role of memory in the constitution of invariants. Zalamea briefly refers to another interesting and technically deeper "cognitive" analysis, correlated to Gestalt, with which Petitot (2008), and Citti and Sartri (2013) describe the visual brain, *neurogeometry*. In the construction of the world (in its friction with it) the brain, always active and plastic, structures itself in a way that can be grasped geometrically, thanks to complex symplectic structures. The brain organizes the world through vision by imposing contours, correlating points with the regularity of minimal forms, relative geodetics, and reading and imposing symmetries.

These kind of analyses, like those I mentioned above vis-à-vis memory, are not operations of cognitive "reduction", but rather tend to highlight the possible initial steps of a constitutive path through which our communicating community has assembled conceptual mountains – in a contingent, because historical, way. An alien friend of mine, from the Sirius system, has no corporeal symmetry and interacts with her ecosystem thanks to *zuzrbs*, and organizes her world on the basis of a fundamental regularity that we cannot appreciate, but that may nevertheless be singled out, the *tzsuxu*. It is another gaze, another epistemically efficacious perspective, one perhaps compatible with ours (or even able to unify microphysics and astrophysics, still, for us, objects of incompatible descriptions). Another light is thus shone upon the universe, of which we see little more than the humble tick, whose Umwelt is so adroitly described by Von Uexküll (1934), a tick who has been successfully coping with the universe for far longer than we have.

Zalamea, instead, insists much on

> the hypothesis of a continuity between the world of phenomena, the world of mathematical (quasi-)objects associated with those phenomena, and the world of the knowledge of those objects – which is to say, the hypothesis of a continuity between the phenomenal, the ontic and the epistemic … From an epistemological point of view, the distinct perspectives are nothing other than breaks in continuity. (304-5)

I will leave it to the reader to adjudicate whether or not it is possible to move "with continuity" between our two points of view, and with mutual enrichment. As for myself, I will insist, in the next section, on the "critical transition" between these worlds, which needs to be analyzed in terms of physical measure, or ways of access to phenomena. I have indeed spoken of the constitution of invariants that lies at the heart of the construction of (physico-mathematical) knowledge, in continuity with *action* upon the world, yet not with the world in itself.

I am in complete agreement with the project of a "geometricization of epistemology […] that would help us to overcome (or, at least, to complement) the 'logicization of epistemology' undertaken throughout the twentieth century" (307). The distinction between "principles of proof" and "principles of (conceptual) construction" (in Bailly and Longo 2011) and the comparative analysis of the two sets of principles in mathematics and physics first, and in biology, is precisely aimed at overcoming (complementing) the monomaniacal (if profound and fertile) approach to Proof Theory as the only locus for the foundations of mathematics. And this "geometry of epistemology" consists, *in primis*, in a Grothendieck-Lawvere-style geometrization of logic (but one that also follows from Girard and his geometry of proof [2001, 2007]). A project analogous to the *geometrization of physics,* from Poincaré's geometry of dynamical systems to the enormous work that goes from Riemann to Einstein and Weyl in physics and from Gromov and Connes in quantum mechanics. We speak, therefore, of the construction of "mathematico-philosophico-metaphorical" tools which, as Châtelet puts it (paraphrased by Zalamea) in his historical study of the nineteenth century,

> in this search for a continuous articulation, include 'dialectical balances', 'diagrammatic cuts', 'screwdrivers', 'torsions', and 'articulating incisions of the successive and the lateral', which is to say, an entire series of gestures attentive to movement and which 'inaugurate dynasties of problems' and correspond to a certain fluid electrodynamics of knowing. (309)

Merleau-Ponty speaks of a "*glissment du savoir*", in both space and time: the epistemological challenge is to structure and organize such knowledge, to give meaning to the moves of both space and time in an historical and human sense of knowledge, and consequently fostering the creation of new perspectives, including new scientific perspectives.

To sum up, consider that in mathematics, in Zalamea's words

> the notion of sheaf, in a very subtle manner, combines the analytic and the synthetic, the local and the global, the discrete and the continuous, the differential and the integral [p. 285-288]. In this way, the 'sheafification' of the analysis/synthesis polarity generates a new web of epistemological perspectives. (319)

Zalamea presents his Platonism accordingly: not static, but processual and methodological, so that "the definitions of mathematics, in reality, define methods; in no way do they define existent things or simple properties inherent in such things" (330). This outlook mirrors my own stance on the matter, and it is precisely that which allows us to pose the problem to what extent such methods are to be preserved and to what extent they are to be enriched or modified, when moving to the interface between mathematics and biology (Longo and Montévil 2014) – and to what extent our attempts of theoretical objectification of the living can still be inscribed within this framework. The notion of "mobility of the base" to which Zalamea refers, is close to the vision of objectivity and effectiveness of mathematical construction upon which I insist, insofar as it is the result of a phylogenesis *and* of a human history: "as the Platonic mobile base suggests, neither invention nor discovery are absolute; they are always correlative to a given flow of information, be it formal, natural or cultural" (333). Which "base" changes should be operated in order to move from the interface between mathematics and physics to that between mathematics and biology? From the epistemological point of view, but also from that of an original scientific construction, we are not interested in an ontology of the "transcendence" of mathematical objects, but rather in their "transcendental constitution", as the phenomenologist would have it – that is, their constituting through (and a "transit" upon) the praxis of life and knowledge internal to mathematics and often (an in a particularly fecund manner) located in the interface with other forms of knowledge.

By posing the question of the relationship between mathematics and biology, therefore, I do not exclude a certain autonomy of pure mathematics and of its effects on the world. I want to stress, however, that mathematics has always nourished itself on new interfaces, on new problems to which new theoretical answers needed to be formulated. Thus, the "fluid electrodynamics of knowing" can take us very far from the original frictions, and an innovative metaphysics can further fluidify this exchange – just think of the role that the philosophies of Nicola Cusano and Giordano Bruno, as well as the practices of the painters of Italian perspective, played in helping us to think the mathematical infinite and, in general, to conceive of new symbolic constructions of science and mathematics (see Petitot 2004; Longo 2011b; Angelini and Lupacchini 2013).

Regarding the relationship between culture, arts and mathematics, and their capacity to interact through the creation of "perspectives" and points of view, Zalamea borrows Deleuzian themes, and quotes at length an art historian, Francastel. On these themes I want to remember Arasse, a disciple of Francastel and historian of painting, from whose more refined analysis of the aesthetico-epistemological role of Italian perspective I suggest we draw precious insights regarding the play between the (local) detail and (global) sense of a painting, the interaction between painting and knowing artistic subject (see Arasse 1999, 2009; S. Longo 2014), as well as the sense of the (mathematical) infinite in renaissance painting.

> The breath of aesthetics permeates mathematical creativity on at least two levels, as detonator and as regulator. Referring to the artistic imagination, Valéry writes in his Cahiers: 'Imagination (arbitrary construction) is possible only if it's not forced. Its true name is deformation of the memory of sensation' […] We have seen how contemporary mathematics systematically studies deformations of the representations of concepts. […] The visions of 'cohomologies everywhere' in Grothendieck [p. 146], of 'groups everywhere' in Zilber [p. 256], or 'metrics everywhere' in Gromov [p. 259], ultimately answer to a new aesthetic sensibility, open to contemplating the local variations of (quasi-)objects through global environments of information transformation. The aesthetic regulation that allows the invasion of cohomologies, groups or metrics be calibrated is decisive. (372-3)

Number and the Question of Measure

When three stones are lying on the ground and a volcano spits out other two stones, neither the number 3, nor the number 2, nor the concept of sum are there – there are some stones on the ground, and that's it. These will be five stones for the *practical action* of whatever being decides to cut them apart from their background, as we do (unlike the tick, for example).

When a lion, in a group of three or four, hears five or six distinct roars in the distance, it prudently changes course, in order to avoid an uneven conflict – or so the ethologists tell us. The lion "isolates" an *invariant of praxis,* a praxis wherein memory helps it to compare different active experiences, from vision, hear and smell. However, the lion does not possess the *concept* of number, it merely builds – but this itself is no mean feat – an invariant of action.

When we make the difficult, and very human, gesture of an open hand with five outstretched fingers symbolizing a numerical correspondence, and we refer to it in language, we are giving ourselves the concept, furtther stabilized in writing. Number is not already "inscribed in the world", not even in the discrete material of the stones on the ground, not before they are isolated from their background – pragmatically as many animals know how to, as well as in mythical-theoretical manner, through language, as we have learned how to.

Number is not to be located in the biological rhythms that regulate the time of the living either (Chaline 1999; Longo and Montévil 2014). What is however interesting is the association that Brouwer makes between the construction of the concept of number and the "two-ness" of temporal discreteness: that moment which passes by and becomes another (Brouwer 1975) in the discrete succession of a musical rhythm, the rhythm of the living, a proposal that evokes the Pythagorean intuition of number and music. This picture is incomplete though: only a plurality of active experiences permits the constitution of an invariant, of that which does not change in the transformation of one experience into another. The rhythm that organizes time into the discrete, the "small counting" (the comparing and counting of small quantities) which we share with many animals (see Dehaene 1998), the spatial ordering of different objects, together with the sense of movement associated with order (Berthoz 1997) – all of these precede and contribute to the constitution of the (practical and conceptual) invariant, being different active experiences. The passage, the transit, the transformation of one into the other are necessary in order to produce the invariant. All Pythagoreanism, holding number as intrinsic in the world, is misplaced: a brain, embedded in its preferred ecosystem – the body of a human, historical and dialogical being – is needed, along with a plurality of praxis from which to distill an invariant in memory and *then* produce (in language) number, in order to stabilize a concept resulting from a practical invariance with a long evolutionary history.

Such constituted invariance comes into play even more when it comes to analyzing processes and dynamics, where one needs to remember that in physics and, a fortiori, in biology there is nothing but dynamics. We need then to measure this or that observable pertinent of the selected process, a theoretical proposal, also fixing a moment of measure, and decide a beginning and an end of the process – a far more complex act than that of counting five stones. So measure *necessarily* is, because of physical principles, an *interval*. Thermic and gravitational fluctuations, as well as quantum non-commutativity, do not allow us to associate a number with their dynamics and with the pertinent observables, but only approximations, changeable intervals. There is no intrinsic number in no physical process: it is we, through the difficult gesture of measurement, who associate numbers with certain dynamics, as couples, extremes of rational intervals, as concepts and as writing, constructed in language. And then, with an eminently mathematical passage to the limit, one which took 2.500 years to be achieved in relative completeness, we have proposed numbers without jumps nor gaps, the Cantorian continuum, one of many possible continua where the intervals of measure could converge.

The mediation or interface between mathematics and the world requires the selection of a frame of reference and measurement, the production of a number which is not in the world but which must be extracted or proposed in order to organize the world. In some cases a structure, a geometry, can organize the world “without numbers”, so to speak. That’s precisely what happened in the various facets of the “geometrization of physics”, of which I spoke above – from Riemann to Poincaré and Einstein, from Weyl to Connes – structures that were somehow derived, as I said, from the problem of measurement (ruler and compass, rigid body, Heisenberg's non commutative algebras). This method can also be found, for example, in the symplectic geometrization of the visual cortex (see Petitot 2008; Citti and Sarti 2013). But like the others, even this organizational proposal, a proposal of intelligibility that justifies the co-constitution of Gestalt *with* and *within* the world, must then allow us to analyze fluxes, to study functionalities and the dynamics of vision, analogously to physical processes. And so geometry too requires numerical measure, with all the characteristics I mentioned, as does every access to the structures of geometrized physics – with its difficulties and limits: classical, relativistic and quantum (and in this case, biological).

The flat (unidimensional) computationalists who see algorithms and numerical calculi as coinciding with the world should first reply to the provocative question I addressed to the Pythagoreans, (see Longo and Paul 2010 for a formulation of it) since they seem not to care about the issue of whether the fundamental constants of physics are computable real numbers. How unfortunate that Planck’s h is not a whole number, with G and c whole multiples of it! Is that God playing tricks on us? And these “constants” (approximated invariants of measure and theory) are present in all the significant equations, those that define the alleged “computable functions” of physical processes. We also suggested to fix $h = 1$, a legitimate move, modulo some transformation in the metric of energy or time, but then the computationalists are not able to compute G or c as exact real numbers, stuck, like everyone else, in the interval of the new measure. In a global perspective and with “some art” (or artefacts, see Duff et al., 2002), one may set h, G and c to 1, in their dimensions, but then the problem moves to the dimensionless constants, such as the constant of matter $\alpha = e^2/hc$: its common approximation is 1/137, but … is it?

If I were to go out on a limb, I would bet that the fundamental constants are “random real numbers” *à la* Martin-Löf (see Calude 2002), that is, strongly uncomputable real numbers, since they have a Lebesgue measure of 1 (“probability” 1) in every interval of the reals. It should be said that “randomness”, for real numbers, is a notion that has a meaning only to the infinite limit: these incomputable reals are therefore an asymptotic *jeu de hazard,* an infinitary dice game, available to God alone – and this capable of convincing even Einstein.

I defined the partisans of the “computational world” as “unidimensional”, since the question of dimension is at the heart of their flattening of knowledge. A first way of being in the world and of constructing the intelligibility of the world with other disciplines, indeed, is to appreciate its “dimensionality”, in the entire semantic richness of the word. To begin with, it should be observed that everything changes, in biology but also in physics, with the Cartesian dimension. From Poisson’s equations of heat, a standout case, to all physical and biological processes, the spatial dimension within which a process is analyzed is fundamental: its fixing precedes every theoretical analysis – it functions as its condition of possibility, we should say with Kant. In general, the choice of a Category or of a Topos and their embedding in a relative universe of Categories, with transits, functors, and “natural” transformations to move from one to the other, is a fundamental theoretical passage.

Consider the poverty, in speaking about the world, of a Category, that of Sets, as an alleged ultimate universe of fundamentals of intelligibility, where the set R of the reals is isomorphic with R^n: the dimension being irrelevant for the analysis. Or, even worse, the parody of a universe postulated by the computationalists: the Category of discrete sets and computable functions, where N is isomorphic with N^n. These isomorphisms are essential to the theories in question: in the first case they allow us to speak of cardinality, in the second they allow the definition of Universal Machine, one of Turing's great ideas, which led to the production of compilers and operative systems of informatics. Personally, I have found technical work in this latter Category, and its Types (see Rogers 1967; Barendregt 1984; Girard et al. 1989; Odifreddi 1989, 1999) very interesting, as explained for example in Longo and Moggi 1984. The second Category is also well correlated to the first one, once some algebra is added to it (see Longo 1983). Computability and Types, from Church to Girard, are at the origin of – and are still capable of giving mathematical sense to – the extraordinary machines we have invented; we need, however, to always try and offer correspondences between their category and others of different nature (see Asperti and Longo 1991).

Yet there are still those who want to analyze the Universe, the brain, and the organism (the latter being codified by the discrete structure of DNA) by remaining within N and its finite, isomorphic powers. Now, the minimal structure one needs to assume in order to correlate mathematics and the world is a topological invariance, that of dimension. So, if we consider, on R, the so-called "natural" topology, that of intervals, the structure forbids the absurd isomorphisms mentioned above: an isomorphism between two topologically open sets of two different spaces forces the same dimension of these spaces, which is then a topological invariant. This is a simple but beautiful correlation between topology and physical measure, since natural topology derives from classical physical measure, an interval. This allows us to come back to what I mentioned above about measure, and how such a topological invariance has no meaning upon the discrete, where the access is exact, absolute, and far from any form of measure and access to physical and biological processes. When we hope to ground the intelligibility of the world upon one-dimensional, codifiable mathematical universes, as the strings of bits that codify an image on a computer screen, we break the symmetries that make the world intelligible (Longo and Montévil 2014; 2014a).

Synthetically, one could say that that which is geometric, and therefore *a fortiori* categorial, is "sensitive to coding": form, structure, the diagrammatic Gestalt, and organicity are not invariants of coding, their entire sense is lost by coding, as instead are information or digital computation, where independence of coding is their mathematical strength. It is therefore licit to claim that *no* physical process computes (Longo 2009). In order to build one such process, the digital computer, we had to invent the alphabet, modern logic from Boole to Frege, Hilbertian formalisms, and Turing's and Gödel's formidable codings. We thus individuated a new fundamental invariant, the notion of computable function, independent from the formal system. We had to inscribe these calculations, codify them in a machine with discrete states, and make the latter stable and insensitive to the codings and fluctuations I mentioned above, forcing an electromagnetic dynamics into the discrete, channeling it into an exact interface. So every process in digital machines can be iterated in an identical manner, via the implementation, on structures of discrete data, of "term-rewriting systems", i.e. systems of alphabetic writing and rewriting, the most general form of computability (see Bezem et al. 2003). This is a massive amount of science and engineering, which includes the Lambda Calculus, with and without types (see Barendregt 1984; Barendregt *et al*. 2013) to which we gave, with many others, a geometrical significance in adequate Topos, bringing them back to bear upon that geometrical organization I insisted upon, far from the monomaniacal obsession with the computable discrete. This has been a part of the network of constructed relations, the synthetic movement of thought which lies at the heart of the construction of mathematical knowledge, rich in concrete and historical friction with the world.

To sum up, number and its structures are not already in the world, and neither is it "effective computing", which is nothing but the formal transformation of the writing of number: it is expressed in systems of re-writing, transformations of alphanumeric writing, upon which a machine can operate. Phenomena, in physics in particular, are on the other hand organized by us through non-arbitrary principles of intelligibility, among which conservation and symmetry principles that have dimensions and pose the problem of access and measure. More precisely, I want to recall how the conservation of energy and momentum (that are theoretical symmetries) allow us to write the Hamiltonian, from which to derive, for example, Newton's equations – a specific case but of great historical importance. From these, indeed, we can proceed deducing the orbits with Keplerian properties.

This backward reading of history (starting with Noether-Weyl's symmetries, and going back to Hamilton, Newton, and Kepler) makes us appreciate the beauty and unity of this strongly geometric construction of physico-mathematical knowledge. This holds even if the planets and the Sun are not identifiable with a material point mass, even if the phenomenal continuum is not made of Cantorian points (see Weyl 1987 on this topic) and thermic and gravitational fluctuations make physical trajectories different from mathematical ones, especially when there are two or more planets (Poincaré's problem). The system, then, is chaotic and unpredictable in modest astronomical time-frames (see Laskar 1994). And the mathematics of "negative results", as Poincaré rebutted to Hilbert, makes such phenomena intelligible. Only on a computer screen does a trajectory made of pixels – even the chaotic one of a double pendulum – follow exactly the path dictated by the numerical solutions of an equation and can iterate it exactly – a physical nonsense. The symmetries of a computational model are different from those of the continuum, as we observed (see Longo and Montévil 2014a). So the digital trajectory quickly diverges from that of the mathematical continuum and from the "real" one. Moreover, restarted with the same digital approximation, on the same number, it repeats itself again and again, identical to itself, *in secula seculorum,* something that never happens in physics – and even less in biology, a science of radically non-reversible and non-iterable onto-phylogenetic trajectories, cascades of changes of symmetry: a science of *correlated variations* (Darwin)

Towards Biology: Problems and Conjectures

<u>1- Variation, Continuum</u>

I already talked at some length of the revolutionary role, in contemporary mathematics, of sheaves and pre-sheaves. These allow, in particular, for the construction of a new outlook on variation, on the continuum and on the relation between local and global. It is thus possible to break free of the dictatorship of a continuum *qua* set of points and "punctual" variables which do not make jumps nor sink into gaps – a beautiful construction we owe to Cantor and Dedekind, one of the most profound constructions of mathematics, but very far from the continuum of phenomena. Weyl (1987) has already explained how absurd it is to consider such a mathematical universe as congruent with the phenomenal continuum – the temporal continuum in particular, which is certainly not made of points. It is meaningless, Weyl argued, to isolate in a point a present moment that is not there anymore (as Augustine would have it), even if he admits that, at the time, he was inevitably subordinated to that exact construction of mathematics. Today, we can do better, even though Cantor's and Dedekind's construction is still profoundly entrenched into our mathematical imagery, and it is indeed the common sense of every school-educated person. Attempts (that of Lawvere-Bell for example, see Bell 1998) to introduce the Topos-theoretic vision into university educational programs have had, for now, scarce success.

Perhaps the very general form of variation (or sheafification, as Zalamea puts it) on a continuum not composed of points (and without "enough points", as morphisms of the terminal object upon the one in question) can fall back upon the phenomena and help us make intelligible the "continuous variation" considered in biology, just as complex numbers – imaginary objects of algebra – have helped us to understand microphysics. I said that variation is (one of) the fundamental invariant(s) of biology, and that the mesh of biological and ecosystemic relations channels this variation and forces a permanent determination of the local by the global (and *vice versa*), in a permanent critical transition which, for the time being, resists a general and efficacious mathematization.

It is not obvious how to apply new instruments such as Grothendieck's in a theoretical-biological field, and I personally know of no successful attempt to do so. I have not seen, and I do not know how to bring about, a passage from "set-theoretical punctuality to the actions of non-commutative monoids in Grothendieck topoi" (223–4) as applied to a satisfactory theory of organisms: it may be a job for a next generation. The first obstacle, following our approach, is the genericity of the physico-mathematical object and the specificity of its trajectories. The objects and the transformations in and on the Topos have the physico-mathematical character of genericity and specificity: this is reversed in biology, as we said, with a duality which represents a major conceptual challenge.

What type of categorial, technical, duality can reflect this theoretical duality and produce a new outlook on biological phenomena? I would be wary of shortcuts and of the arrogance of anyone who would master such a beautiful mathematics: the living is an extremely hard subject matter, a difficulty of a different kind than the one faced by the beautiful mathematics we have discussed. We must first appreciate the richness of the Theory of Evolution, the only great theory in biology, as recounted by many great contemporary evolutionists – to observe the complexity of the embryogenesis of a fly's leg, or the possible embryogenetic bifurcations of a zebra-fish – in order to fully understand why the competent and honest experimental biologist is unable to give an answer to 80% of the questions that the theorist poses to her when visiting the lab. This is not the case in physics.

Perhaps another duality can be more easily grasped through new structures. From Hamilton to Schrödinger we have become used to understanding energy as an operator (the Hamiltonian, the Lagrange transformation) and time as a parameter. I hold that this approach, in biology, should be inverted: here time is the fundamental operator, constitutive of the biological object by way of its phylogenetic and ontogenetic history, while energy is nothing but a parameter, as it indeed appears in scaling and allometric equations (see Bailly and Longo 2009; Longo and Montévil 2014). If we clear our mind of the classical schemes in which Hamilton's and Schrödinger's operators – and Pauli's controversial theorem, which partially formalizes the distinct physical role of energy and time, (see Galapon 2002) – are given, we can perhaps begin to see the whole in a new, dual way, as required by the phenomenality of the living – by its historicity, in this case.

Another theoretical path that needs a new outlook in terms of continuity, density (as the rational numbers in the reals) and of analysis of the local vs. the global is that of "extended criticality" (see Bailly and Longo 2011, Longo and Montévil 2014). Critical Transition Theory, in physics, is an extremely interesting discipline – born within the fold of post-War quantum mechanics yet further developed also in a classical form – for analyzing phase transitions through the application of (quite a bit of) mathematics. The dominant framework, obviously formalized on Cantor-style real numbers, describes the "transition" as punctual, and this punctuality is essential to the methods of Renormalization (see Binney et al. 1992; Laguës and Lesne 2003). These deal with a cascade of models which describe changes of scale and of pertinent objects, with a change of symmetries (both breaking and construction of new ones) at the punctual limit of the transition, where the local appears imbricated with the global. The most familiar examples are the formation of a crystal or of a snowflake, the para-ferromagnetic phase transition, and Ising's transition, all mathematized as punctual transitions.

The criticality of the living, on the other hand, is extended: it is always in a state of "phase transition", in a permanent reconstruction of its internal "symmetries" and in correlation with the environment (see Longo and Montévil 2014). Indeed, in an organism every cellular reproduction has the characteristics of a critical phase transition, for internal reconstruction and of the surrounding tissue. And within the cell itself, molecular cascades pass through critical values which can similarly be seen as phase transitions. The slight modifications that always follow it are part of adaptive biology, including ageing (the increase in metabolic instability, oxidative stress). An organism is somewhat like a snowflake which reconstitutes itself in permanence, partially modifying its symmetries, jointly to the correlations with the ecosystem. In short, an organism is not merely a process, a dynamics, but is always in an (extended) state of critical transition, permanently reconstituting local and global "symmetries". An interval of criticality can give some idea, as I am trying to convey it, but the density that would be necessary to describe it cannot be the "point by point" density of a segment of Cantor's line in respect to every pertinent parameter – or if it is, it is only so in an inadequate manner. In any case, renormalization methods cannot be applied, as such, to a classical interval of criticality. A reasonable objective could be that of replacing the Cantorian interval with the variation in/of a point-less (pre)sheaf, thus giving a representation of density adequate to renormalization, suitably extended.

2 - Measure

I have already discussed the crucial role and the theoretical and experimental richness of measure in physics, the sole form of access we have to the world (including perceptual "measure"), an interface between mathematics and phenomena. In biology the situation is even more complex. In the first instance, a difference must be drawn between "in vitro" and "in vivo", a difference which has no meaning in physics. Moreover, over the last few years we have seen the development of refined techniques of three dimensional cultures: cells or tissue fragments from an organism are developed in collagen suspensions from the same organism, giving rise to matrixes or parts of tissues impossible to observe in traditional and "bidimensional" Petri dishes. Thus both observation and measure are profoundly changed, as if (but not quite as if) we were somewhere in between the "in vivo" and the "in vitro".

In any case, the duality I examined between generic and specific, between biology and physics, radically changes the meaning of a measure. The biological object is not an invariant either of experience or of a theory, unlike the mathematical and physical object. It is specific and historical and, to a greater or lesser degree, individuated. Of course, the individuation of a monocellular organism or of a single cell in a tissue is minimal compared to that of a primate. And yet a cellular culture is prepared, by biologists, with a full awareness of the history of cells: cells from a given tissue are labelled, and the descendents are distributed with the utmost care throughout the world in order to reflect, collectively, on the iterability of an experiment in reference to the history (i.e. the specificity) of each cell or tissue. The same goes for lab rats, labelled and traced along families as offspring of a same couple, so that they will have a common, or at least known, phylogenetic history.

In an ongoing project, between laboratory experience and theory, Mael Montévil is working on a theoretical analysis of what he calls the "controlled symmetrization" of the biological object factually practiced in laboratories, in order to deal with its specificity and to make it as "generic" as possible. One of the consequences of biological specificity is that the Gaussian distribution of a measure does not have the same meaning that it does in physics. For example in physics, in general, deviations from or situations marginal to the Gaussian can be seen as noise and decrease, relatively speaking, with the increase of the total number of samples. In biology "deviations" are "specific cases" that can have great significance for (cellular) differentiation and speciation, and increase as the number of samples grows: enlarging the samples from one population of cells, or rats (or of humans) to another may radically change the response (to a therapy, say), a major experimental and theoretical challenge. Only the "control" (the normal cell, or rat, used as control), an unknown notion in physics, can help us understand the significance of a variation, which is biological variability. And I want to insist that variability, in biology, is not noise: it is at the origin of diversity and therefore of the biological resilience of an individual, a population or a species – and that this takes place even in a population with a small number of individuals: even in a population of a few thousands, individual diversity contributes to evolutionary stability.

Which mathematical instruments should we use, or create, starting with *contemporary mathematics* – that is to say, going beyond mere systems of (at best non-linear) equations, and statistical methods invented at the end of the 1800s? When Connes proposed non-commutative geometry he stood on the shoulders of early 1900s giants. A highly refined theoretical work then transferred the problem of quantum measure to Heisenberg's matrix calculus, correlated with Weyl's algebras and Hilbert's spatial continua, both used by Schrödinger for his equation. As in relativity theory, or perhaps even more so, the problem of measure had produced an imposing theoretical edifice. This is certainly not the case in biology, where practically no theory, as far as I know, accompanies or guides extremely stringent experimental protocols, whose originality and rigour are truly astounding for the theorist who happens to visit the laboratory.

In short, I believe that it is necessary to first clarify what "to measure" *means* before being able to imagine a process of co-construction of mathematics and biology in a way vaguely comparable to what took place between mathematics and physics in the last four centuries. The physicalist who denies the existence of a properly biological problem, or the Pythagorean who claims that "number is already there", should look elsewhere. To associate a number with five stones, six roars or five fingers, i.e. to build an invariant, is a long historical process. To associate it with a physical or biological process is a task which lies at the heart of experimental work, and represents a major theoretical challenge, in biology even more than in physics.

Conclusions on Zalamea's Book and Grothendieck's unifying views

14.3.2. For mathematicians, logical axioms delimit a playground. But

which game are we going to play next?
7.4.1. Desire, and the resistance of the object, are what mathematicians ordinarily use to distinguish mathematics from logic.
7.5.1. Grothendieck is rather like the Freud of epistemology.
(Lochak, 2015).

I hope I have managed to give the reader an appreciation of how the immense shadow of Grothendieck dominates Zalamea's book. A French mathematician, the son of internationalist revolutionaries, migrating throughout all political turmoils in Europe between the Russian revolution of 1905 and the Second World War, Grothendieck comes to France when twelve years old, while the latter war was raging. He first lived with his mother, and then in hiding. His life is as original as his mathematics (see Lochak 2015). Without going into the – mostly dramatic – details of the first, it is interesting to note how Grothendieck is the only one of eleven French winners of a Field Medal, who have had their university studies in France, to have neither studied nor taught at the ENS in Paris, yet another touch of originality.

Following Grothendieck, Zalamea's book gives priority to the structures of mathematics, to their transformations and deformations, and to the construction of meaningful invariants. Taking this focus on structures, invariants and transformations as the way to do philosophy of mathematics – the philosophical sheafication I mentioned above – we move away from set theoretical, logicist and formalist absolutes (still grounded on the myth of the "discrete" and the "finite" as absolutes) programmatically outside of the world.

We should however add that Grothendieck's work goes beyond these speculations on symmetries, invariants, and transformations. He had an exquisitely refined sense of the "purity" of a mathematical definition. He was able to avoid, arguably as no mathematician before him could, every "contingency" in the structures and proofs he proposed. All his notions intrinsically encapsulate, so to speak, the maximal invariance of a concept, to the extent that there is no need to prove it, by identifying the adequate transformations: they are intrinsic to the definitions [12]. Grothendieck's approach unifies remote constructions in mathematics, by proposing invariants surprisingly shared by groups, topological spaces, manifolds of different sorts (differential, geometric …), and by constructing, as "bridging" notions, new mathematical structures. It is more than a unification by generality, as the new objects proposed have an autonomous and robust, profound mathematical structuring. This allows to "circulate" in mathematics and to propose and transfer common mathematical meaning to apparently unrelated mathematical constructions. As Grothendieck observed, sheaves on suitably changing sites allow to circulate between continuous and discrete structures, beyond the "the founding aporia" of mathematics, to put it in Thom's terms.

As Zalamea's book reminds us with regard to physics, yet pushing beyond Zalamea's arguments, it seems to me that the fundation of mathematics must take nourishment from the dialogue with the theoretical foundations of

12 A typical example is the notion of "étale topology". It is defined on a category as a category, whose objects are morphisms on which schemes act (as morphisms): the topology thus is given in a relational way, which forces its right level of invariance. The notion of Topos as well is given in a "category-theoretic" way: they are sheaves on sites (a small category with a covering) .

other disciplines. Not only in the dimension of historical analysis, but also in the positive work of scientific creation, where epistemology becomes entangled with the analysis of the construction of knowledge. This construction is the result of a protensive gesture which organizes the world, rich with desire for (knowledge of) the real and constitutive of the mathematical object through which it can be made intelligible; a real which resists and channels mathematical invention, together with its history. The analysis of this protensive gesture, and of its historicity, is part of epistemological reflection, qua analysis of a construction *in fieri*. The wandering of mathematical work beyond any relation with the natural sciences is yet an essential component of this construction, even more so if it gives rise to new spaces for creation, new correlations and abstract structures – like **Set** or a new category of pre-sheaves. The mistake is to take one of these creations and "put it back", as ultimate foundation, as a kind of Cantorian paradise outside the world. In doing so, one loses the meaning of the whole edifice, a network of relations of intelligibility, by absurdly turning it upside down and making it stand on (perhaps unidimensional) feet of clay. I am not here insisting on the exigency of "foundations" as locus of certainty, but rather on the necessity of the analysis of conceptual and cognitive roots, of structures of sense as *correlations*, tracing their constitutive and historical path (broadly construed, as to include its pre-human dimension). This project is far from pursuing those "unshakable certainties" sought by Hilbert in a time of great non-Euclidean uncertainties: on the contrary, there is nothing more uncertain than the cognitive foundations of mathematics – as uncertain as any biological or pre-human dynamics, as uncertain as a physical measure. However, drawing upon a plurality of correlations of knowledge, an historical epistemology of the interface between disciplines construes them as mutually supportive, as epistemological and epistemic webs: networks of meaning where the meaning of one helps us understand and constitute the other. An epistemology, moreover, that helps us discern, in an original, critical and ever-renewed way, the road to be built ahead, which is what matters most. Grothendieck unifying methodology, within mathematics, based on the construction of new, often complex, but deep structures, is a remarkable example also in the foundational analysis and the practice in other disciplines: reduction, say, rarely applies, while unification by new, difficult theories marked the growth of science[13]. Science is not the progressive occupation of the real by known tools, in a sort of fear of the novelty, but the difficult construction of new theoretical frames, objects and structures for thought, conceptual bridges or even enlightening dualities, such the specificity of the biological vs. the genericity of the inert, with its major consequences for a close analysis of measurement, as hinted above.

My analysis has been inevitably superficial and incomplete; even Zalamea's large book is incomplete when it comes to the richness of contemporary mathematical invention. Zalamea's style, informal and philosophical, may irritate some readers, due to what could be considered as frequent flights of rhetorical fancy. Personally, I find it an extremely efficacious way to express the enthusiasm that such mathematical abundance deserves. As for rigour, when it comes to those fields in which I can claim some technical competence (Types, Categories and Topos, … Girard, Lawvere…) it all seemed to me to be presented in a coherent and pertinent way, within the limits imposed by

13 Newton unified Galilean falling apples and planetary movements, by inventing brand new mathematics and theories. Similarly, Boltzmann unified mechanics and thermodynamics at the asymptotic limit of the ergodic hypothesis and the thermodynamic integral. Connes aims at the unification of quantum and relativistic fields by a reinvention of (differential) geometry.

the limited space dedicated to the numerous themes transversally touched by Zalamea, who demonstrates an outstanding breadth of knowledge.

I would like, finally, to commend the two associated publishing houses that published this volume: Urbanomic and Sequence Press. In this as in other publications – as for example the forthcoming English retranslation of Châtelet's book (an extremely hard work as Cavazzini, who recently translated it in Italian, knows all too well) – they certainly seem to favour the creation of a critical space, by promoting originality, and offering an alternative to debates as well-established as they are sclerotized in an oscillation between this or that Scylla and Charybdis, even when the latter approach would promise immediate success and, therefore, an high Impact Factor – a factor that is having a very negative impact on science, Longo 2014.

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