From Completeness to Incompleteness

Questions and Outline for the Summer Immersion 2026

I– Questions addressed to G. Longo on the relation between the ‘Pythagorean Imperative’ and Completeness

1. **Does your take on the ‘Pythagorean Imperative’ imply a kind of drive towards completeness? Either formal, in the sense of complete theories, or scientific, in the sense of a unified science?**

2. **If so, does the irony of the drive toward completeness arrive precisely in the failure to incorporate the incompleteness theorems of Gödel into a scientific or mathematical-logic practice that includes numbers?**

3. **Are there other ironies of Pythagoreanism-Completeness that emerge when considering other major discoveries of mathematical logic such as:**
    a. **Tarski’s undefinability theorem**
    b. **The Church-Turing Thesis**

4. **Are you satisfied with the definitions of completeness sometimes called ‘standard’ today?** For example: logical (are all valid formulas provable?), descriptive (model theoretic), and proof procedures (are all the formulas of a theory decidable? Recursive?, etc.).

II – General Questions on Gödel’s Completeness/Incompleteness:

1. What is the tipping point (redline) for undecidability? Gödel’s *Incompleteness theorems* apply to both Peano arithmetic **(N,+,•)** and Robinson arithmetic **(N,+,•)** (i.e. arithmetic of natural numbers with addition and multiplication). On the other hand, Gödel’s theorems do not apply to Presberger arithmetic **(N,+)** (natural numbers with addition only) or Skolem arithmetic **(N, •)** (natural numbers with multiplication only). What is it about the combination of addition and multiplication that yields Gödel’s results? (This is somewhat surprising given that multiplication is just repeated addition.) More generally, where is the tipping point for this undecidability result?

1.Do these problems lead to crucial algebra problems? For example, the full signature (structure) for the language of both **PA** and **Q** is **{0,≤ , +,•}** and their intended models are the same **(N,+,•).** Yet, there are two separate signatures for Skolem **{1,≤, •}** and Pressburger **{0,≤,+}**.

Gödelian Table

Arithmetics	Signature	Model	Algebraic theory	Logical Status
Pressberger	{A,0,≤,+}	(N,0,≤,+)	monoid	complete/decidable
Skolem	{A,1,≤,·}	(N,1,≤,·)	monoid	complete/decidable
Robinson	{A,0,1, ≤,+,·}	(N,0,1, ≤,+,·)	semi-ring	incomplete/decidable
Peano	{A,0,1,≤,+,·}	(N,0,1,,≤,+,·)	semi-ring	incomplete/decidable

Here, 'A' is a parameter for the variation of the domain. The red line marks the turning point between completeness-decidable and incompleteness-undecidable. See 3(c) for further alignment.

2. What makes Gödel numbering possible/impossible? Gödel uses unique prime factorization to set up his effectively calculable injective map between formulas and natural numbers. But, apparently, the possibility of constructing such a map does not necessarily rely on this property of natural numbers (there are other methods). What are some of these more modern methods of 'Godel numbering'? And what is their relation to unique prime factorization (if any)? More generally still: what exactly about numbers allows this mapping to be carried out? And at what point does this mapping fail?

Are questions (1) and (2) just two different ways of asking the same question?

3. On the algebraic formulation of Gödel's incompleteness: To what algebraic structure does Gödel's incompleteness result apply? What are the structural algebraic features that give rise to incompleteness? What would be an algebra of incompleteness?

(a) If the completeness of classical propositional logic can be expressed in terms of the structural properties of Boolean rings (e.g. existence of maximal ideals via Stone's representation theorem), is there any analogous structural algebraic property that account for incompleteness (e.g. non-existence of maximal ideals; see below c)?

(b) Is there any relation between incompleteness and division problems in the algebra? (Particularly, unique prime factorization, used in Gödel's construction, seems to relate to the properties of divisibility).

i/ If Gödel incompleteness arises from the interaction of addition and multiplication in a **semi-ring** where additive inverses are absent, then a further question concerns what happens in the interaction of addition and multiplication in a **ring –** or rng **–** where multiplicative inverses (division) are absent? What are the implications for Gödel numbering? Call this a second order Gödel extension or *Hyper-Gödelian* move.

ii/ restoration of inverses: Passing from semirings to rings restores additive inversion, while passing from rings to fields restores multiplicative inversion. What is this movement relative to the problem of completeness/incompleteness?

Hyper-Gödelian Table

	Signature	**Model**	**Algebraic Theory**	**Logical Status**
Gödelian	{A,0,1, ≤,+,•}	(N,0,1, ≤,+,•)	Semi-Ring (additive inverse absent)	Incomplete/undecidable
Threshold	{A,0,+, –, •}	(R,0, +, –, •)	Ring (multiplicative unit absent)	Incomplete/undecidable ?
Hyper-Gödelian	{A,0,1, +, –, •}	{Z,0,1, +, –, •}	Ring (multiplicative inverse absent)	Higher-order incompleteness/undecidability ?
	{A,0,S,+,–,•}	(D,0,S,+,–,•)	Modified Ring	Higher-order incompleteness/undecidability ?
	{A,0,1,+,–,•,÷ }	{F,0,1,+,–,•,÷ }	Field (Galois)	?

(c) Do designated elements in the algebra play any role with regard to the question of completeness and incompleteness?

i/For example, unlike in Boolean Rings, 1 is not a maximum element in the algebra of the arithmetic of natural numbers.

ii/ Although Boolean rings may contain distinguished constants such as 0 and, when unity is assumed, 1, these constants are not initially introduced as designated semantic values within the algebraic signature itself. In fact, the Boolean ring signature may be written minimally as **BR={+,·}** or **BR={+,·,0}** without requiring the primitive introduction of a maximal Boolean unit (habitually called a RNG: note this breaks the correlation to a Boolean Algebra). In this sense, the ring preserves a formal symmetry insofar as no admissibility element has yet been structurally privileged. By contrast, consider a modified Boolean Ring signature: **BR*={+,·,S,0}** where S is introduced as a parameter for a designated element, while no classical Boolean unit 1 has been fixed. Two questions:

A) The introduction of the unity 1 into the Ring guarantees complementarity, where $\neg x = x + 1$, thus $x + (x+1) = 1$ and $x \cdot (x+1) = 0$. Can this complementarity be called a **syntactic completeness** in the sense that there is an oppositional polarity and closure laws?

B) But the introduction of S at the place of unity produces a structural orientation absent from ordinary Boolean ring signatures. Rather than yielding immediate Boolean closure under complementarity, the structure preserves a distinction between admissibility and valuation. In this sense, the algebra ceases to be closed in the Boolean sense, since polarity is no longer globally determined by a maximal unit element. Can we call this algebra of **syntactic incompleteness** or a perforated Boolean Ring?

d) Can we generalize (cB) above in terms of a Galois-Algebra, thus, bring together both Boole and Galois, in terms of a modern setting of completeness and incompleteness?

4. Are there any semantic arguments implicated in Gödel's theorems: In "Interfaces of Incompleteness" (Longo), the following clarification is made: "Gödel never used, neither in his statements nor in his proofs, the notion of "truth", which is not a formal concept. … The strength of Gödel's work is that it shatters the formalist program from the inside, using formal tools. He uses pure computations of signs without meaning and therefore does not invoke 'transcendental truths'; he presents his arguments by purely formal means." Granting that the truth of the Gödelian sentence G (the one usually interpreted as stating its own unprovability) is established solely on the basis of the properties of the formal system itself—and not by way of meaning or reference to any ontology—the question remains whether Gödel's theorems (which are clearly about the formal system itself) involve any arguments that are semantic in nature. In this respect, a few aspects of his construction come to mind:

(a) Gödel must show that the formal system is indeed a formalization of arithmetic, i.e. that it is true to its intended interpretation.

(b) Gödel numbering must take the formal theory as an object and cipher it as part of arithmetic (in order then to reflect the formal theory into itself). Instead of interpreting the formal theory as descriptive of arithmetic (model-theoritic semantics), doesn't this move interpret part of arithmetic as descriptive of the formal theory (that is, its syntactic properties)? And isn't this procedure semantic in nature (though perhaps not in the usual sense)?

(c) If Gödel's construction does not rely on 'semantic' arguments in the model theoretic sense (i.e. making a reference to an object external to the formal theory), is there nevertheless an enunciative function in the setting up of the formal theory itself, where something like a sentence, an object, a formula, is simply declared or asserted? And this reference to the place-menent of the formal theory itself, where the form becomes the content, is this not what Gödel's incompleteness shows cannot be eliminated? Or perhaps, said otherwise, does Gödel's construction in any way go beyond the form-content opposition (which subtends the syntax-semantics opposition) in situating a notion of structure that is neither formalisms nor models?

(d) Does Gödel's incompleteness, for you, have any relation to the impossibility of eliminating variables or proper names from logic (i.e. some irreducible reference to variables or rigid designators that cannot be eliminated through definite descriptions)?

5. What makes the Gödelian sentence G true or false? In "Interfaces of Incompleteness" (Longo), it is shown that the truth of the undecidable sentence G really follows from the second theorem [PA⊢(Consist→G)]—and not because of its meaning or the principle of excluded middle. But since G is undecidable in the theory, this means that there are non-standard models in which G is false. Doesn't it follow from this that the provable formula (Consist→G) in some sense encodes in itself the intended interpretation (i.e. fixes the model)—precisely because it implies the truth of G (assuming consistency)? In other words, is it not true that changing the model also changes the formula (Consist→G)?

5.1. Given that (Consist→G) is provable despite the fact that G is undecidable, might it not be possible that for every undecidable formula X, either (Consist→X) or (Consist→~X) is provable? If not, why?

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