The interval analysis of multilinear expressions

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The problem

Interval analysis

Input:
- a polynomial arithmetic expression $E$
- the range of every variable $x_i \in E$

Output:
- the range of expression $E$

Context:
- static analysis of source code
Example instance

Example

Compute the range of:

\[ E = x \times (y - z) + z \]

knowing that:

\[ \begin{align*}
    x & \in X = [ \underline{X}, \overline{X} ] = [0, 1] \\
    y & \in Y = [ \underline{Y}, \overline{Y} ] = [0, 2000] \\
    z & \in Z = [ \underline{Z}, \overline{Z} ] = [0, 2000]
\end{align*} \]
Historic technique

Standard Interval Arithmetics (SIA) - [Moore ’66]

Example

\[ X \times (Y - Z) + Z = [0, 1] \times ([0, 2000] - [0, 2000]) + [0, 2000] = \]
\[ = [0, 1] \times [0 - 2000, 2000 - 0] + [0, 2000] = \]
\[ = [0, 1] \times [-2000, 2000] + [0, 2000] = \]
\[ = [-2000, 2000] + [0, 2000] = \]
\[ = [-2000, 4000] \]
**Drawback**: expressions with *multiple occurrences of the same variable*

**Example**

\[
\text{range}_{SIA}(E) = [-2000, 4000] \supseteq [0, 2000] = \text{range}(E)
\]

*Dependency Problem*: different occurrences of the same variable are abstracted by independent intervals
Multilinear expressions

Expressions that are linear w.r.t. every variable.

**Example**

\[ E = x \times (y - z) + z \]

Meaningful class because all expressions in it share an interesting property . . .
Theorem

Let $f : \mathbb{R}^n \to \mathbb{R}$ be a multilinear function. If $f$ has a local minimum or a local maximum then $f$ is constant.

Corollary

Let $f : \mathbb{R}^n \to \mathbb{R}$ be a multilinear function. The lower and upper bounds of $f$ in the hypercube

$$H = [a_1, b_1] \times [a_2, b_2] \times \cdots \times [a_n, b_n]$$

occur at the vertices of $H$. 
Vertex Evaluation Technique (VE)

VertexEvaluation:

1. find the hypercube’s vertices
2. evaluate the expression in each of these points
3. keep the minimum and maximum values: $min(E)$ and $max(E)$
4. set $range_{VE}(E) := [min(E), max(E)]$

Computational complexity: $O(n \cdot 2^{2n})$

Gaganov, *Computational complexity of the range of a polynomial in several variables*, [‘85] shows that this problem is NP-hard.
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Generalization

Two steps:
1. reduce a generic expression to a multilinear one
2. apply VertexEvaluation

Example

\[ E = x^5 - x^3z + xy - xz + z \]

1\textsuperscript{st} step: \( u \) replaces \( x^3 \) (\( u \in U = X^3 \))

\[ E' = ux^2 - uz + xy - xz + z \]

2\textsuperscript{nd} step: \( v \) replaces \( x^2 \) (\( v \in V = X^2 \))

\[ E'' = uv - uz + xy - xz + z \]

N.B. reduction strategy is not unique!
Proposition

*In the general case:*

\[
\text{range}(E) \subseteq \text{range}_{VE}(E)
\]

*because we lose some information on dependence.*

Example

\[
ux^2 - uz + xy - xz + z \leadsto uv - uz + xy - xz + z
\]

*in the right-hand side there is no dependence between \(x\) and \(v\)!*
Computational complexity: \( O(n(1 + d/2) \cdot 2^{2n(1+d/2)}) \)

Pros:
- multilinear case → exact range
- we pay an exponential cost at compile time

Cons:
- no guarantee that it returns a sharper range
Two possible sources of imprecision:

- over-approximating the range of exponential terms as $x^n$
- poor handling of dependence

Tradeoff between these two aspects.

Example expression:

$$E = x^3 - x^2y + y$$
Miné’s method gives:

\[ x^3 - x^2 y + y \mapsto [a, b]^3 + (-[a, b]^2 + [1, 1])y \]

our technique gives:

\[ x^3 - x^2 y + y \mapsto ux - uy + y \text{ with } u \in U = X^2 \]

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Comparison with Miné’s symbolic methods - 2

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Conjecture

In the case of positive intervals our technique attains a sharper range.

...still to be proven ...for the moment we can show that the following result holds:

Proposition

When all variables’ ranges are positive:

\[ \text{range}_{VE}(E) \subseteq \text{range}_{Miné}(E) \]
Conclusions

Contribution:

- an alternative technique:
  - multilinear case $\rightarrow$ exact range
  - general case $\rightarrow$ over-approximation
  - soundness proof

- tool:
  - to be used @ Magneti Marelli
  - static analysis of C code

For the future:

1. impact of different reduction strategies
2. estimate of the error introduced by the chosen strategy
3. classification of expressions w.r.t. precision
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