Foveal Detection and Approximation for Singularities

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Abstract

Projections in a foveal space at u approximate functions with a resolution that decreases proportionally to the distance from u. Such spaces are defined by dilating a finite family of foveal wavelets, which are not translated. Their general properties are studied and illustrated with spline functions. Orthogonal bases are constructed with foveal wavelets of compact support and high regularity. Foveal wavelet coefficients give pointwise characterization of non-oscillatory singularities. An algorithm to detect singularities and choose foveal points is derived. Precise approximations of piecewise regular functions are obtained with foveal approximations centered at singularity locations.

1 Introduction

The distribution of photoreceptors on the retina is not uniform. The visual acuity is greatest at the center of the retina where the density of receptors is maximum. When moving apart from the center, the resolution decreases nearly proportionally to the distance from the retina center [3, 13]. The high resolution visual center is called the *fovea*. Active vision strategies compensate the non-uniformity of visual resolution with eye saccades, which move successively the fovea over regions of a scene with a high information content. This multiresolution sensor has the advantage of providing high resolution information at selected locations, and a large field of view, with relatively little data. Several approaches have been proposed to model foveal approximations, including the use of a logmap transform [15] or locally translated and dilated wavelets [1, 2]. This paper introduces the notion of foveal approximation spaces and constructs orthogonal bases to compute approximations of signals that have isolated singularities. Singularities are detected and characterized from a multiscale foveal energy measurement.

Foveal spaces are defined in Section 2 with a finite family of generating functions which are dilated. A foveal space centered at a point u can approximate efficiently a function that has an isolated singularity at u if high order polynomials belong to this space. Such foveal spaces can be constructed from a wavelet basis of $\mathbf{L^2}(\mathbb{R})$

by selecting only the wavelets that are in the cone of influence of $t = 0$. However, to reproduce high order polynomials, it requires to use a large number of generating functions. The remaining of the paper studies foveal spaces constructed with only two generating functions. Section 3 shows that such foveal spaces include wavelets with an arbitrary number of vanishing moments, and that one can build an orthogonal basis by dilating two foveal wavelets. Examples with polynomial splines are studied in Section 4. Section 5 is devoted to the construction of pairs of foveal wavelets of compact support whose dilations reproduce polynomials and define orthogonal bases of a foveal spaces. Minimum support wavelets are characterized in Section 5.1 and regular orthogonal foveal wavelets of larger support are computed in Section 5.2. The detection and characterization of singularities with such pairs of foveal wavelets is studied in Section 6.1. Foveal signal approximations are calculated using foveal points at the detected singularity locations. Upper bounds of the approximation error are computed in Section 6.2.

Notation: For any function θ we write

$$\bar{\theta}(t) = \theta(t)\,\mathrm{sign}(t) \quad , \quad \theta^-(t) = \theta(t)\,\mathbf{1}_{(-\infty,0]} \quad , \quad \theta^+(t) = \theta(t)\,\mathbf{1}_{[0,+\infty)} \ ,$$

$$\frac{d^k\theta(t)}{dt^k} = \theta^{(k)}(t)$$

$$\theta_j(t) = 2^{-j/2}\,\theta(2^{-j}t) \ \text{ and } \ \theta_{j,u}(t) = 2^{-j/2}\,\theta(2^{-j}(t-u)) \ .$$

We denote by $\hat{h}(\omega) = \sum_{j=-\infty}^{+\infty} h[j]\,\mathrm{e}^{-ij\omega}$ the Fourier series of a discrete sequence $h[j]$.

2 Foveal Approximations

2.1 Definition and Examples

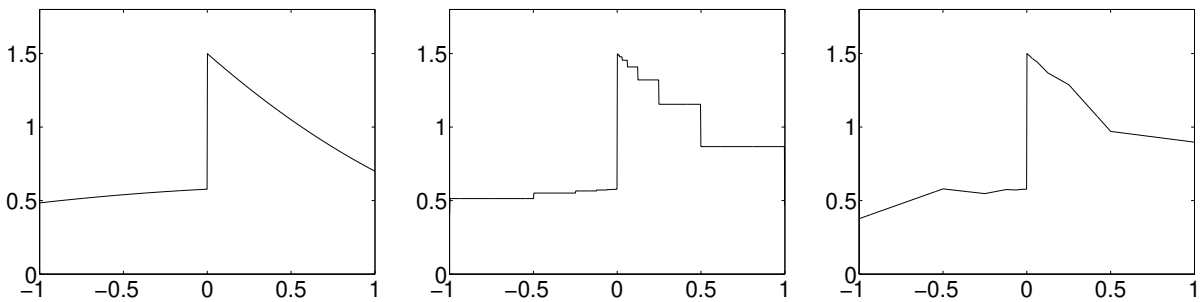
A foveal approximation of $f \in \mathbf{L^2}(\mathbb{R})$ has a resolution which decreases linearly with the distance to the center located at some abscissa u. It is obtained by an orthogonal projection onto a foveal space $\mathbf{V}_u$. For example, the piecewise constant foveal approximation space $\mathbf{V}_u$ is the set of functions which are constant on the intervals $[u - 2^{j+1}\,,\, u - 2^j)$ and $(u + 2^j\,,\, u + 2^{j+1}]$ for any $j \in \mathbb{Z}$. The foveal approximation of f is the orthogonal projection $P_{\mathbf{V}_u} f$ which is equal to the average of f on any of these interval. One can also define a piecewise linear foveal approximation space $\mathbf{V}_u$ of all functions that are continuous and linear on the intervals $[u - 2^{j+1}\,,\, u - 2^j)$ and $(u + 2^j\,,\, u + 2^{j+1}]$. Figure 1 shows an example of foveal approximations. The following definition generalizes these examples.

Definition 1 *A space $\mathbf{V}_0 \subset \mathbf{L^2}(\mathbb{R})$ is a foveal approximation at 0 if there exists a finite number M of generating functions $\{\psi^m\}_{1\le m\le M}$ in $\mathbf{L^2}(\mathbb{R})$ such that $\{\psi_j^m\}_{1\le m\le M, j\in\mathbb{Z}}$ is a Riesz basis of $\mathbf{V}_0$ and*

$$f \in \mathbf{V}_0 \Rightarrow \ f\,\mathbf{1}_{[0,+\infty)} \in \mathbf{V}_0 \ . \tag{1}$$

A foveal space $\mathbf{V}_u$ at u is defined by

$$f(t) \in \mathbf{V}_u \Leftrightarrow \ f(t+u) \in \mathbf{V}_0 \ . \tag{2}$$



(a) (b) (c)

Figure 1: (a): Original function $f(t)$. (b): Piecewise constant foveal approximation of f in the neighborhood of $t = 0$. (c): Piecewise linear foveal approximation.

The existence of a dyadic scaling invariant basis $\{\psi_j^m\}_{1 \leq m \leq M, j \in \mathbb{Z}}$ implies the foveal space is invariant by dyadic dilation:

$$\forall j \in \mathbb{Z} \quad , \quad f(t) \in \mathbf{V}_0 \Leftrightarrow \; f(2^j t) \in \mathbf{V}_0 \; .$$

The property (1) also means that

$$f \in \mathbf{V}_0 \Rightarrow \; f - f\,\mathbf{1}_{[0,+\infty)} = f\,\mathbf{1}_{(-\infty,0]} \in \mathbf{V}_0 \; . \tag{3}$$

The space $\mathbf{V}_0$ may therefore be decomposed in two orthogonal subspaces $\mathbf{V}_0^-$ and $\mathbf{V}_0^+$ of functions in $\mathbf{V}_0$ having their support included respectively in $(-\infty, 0]$ and $[0, +\infty)$. Clearly $\mathbf{V}_0 = \mathbf{V}_0^- \oplus \mathbf{V}_0^+$. The projection of f in $\mathbf{V}_0$ thus defines independent approximations of the restrictions of f in the left and right sides of 0. This extends to any point u.

The piecewise constant foveal approximation is generated by translating a Haar wavelet:

$$\psi(t) = \begin{cases} 1 & \text{if } t \in [0, 1/2) \\ -1 & \text{if } t \in [1/2, 1) \\ 0 & \text{if } t < 0 \text{ or } t \geq 1 \end{cases} \; . \tag{4}$$

The two wavelets $\psi^1(t) = \psi(t+1)$ and $\psi^2(t) = \psi(t)$ have a support respectively equal to $[-1, 0]$ and $[0, 1]$. One can easily verify that the family $\{\psi_j^1 \, , \, \psi_j^2\}_{j \in \mathbb{Z}}$ is therefore an orthogonal basis of the foveal space of functions in $\mathbf{L^2}(\mathbb{R})$ that are constant on each interval $[-2^{j+1} \, , \, -2^j)$ and $(2^j \, , \, 2^{j+1}]$ for $j \in \mathbb{Z}$.

A piecewise linear foveal approximation is constructed with a linear hat function:

$$\psi(t) = \begin{cases} x & \text{if } t \in [0, 1) \\ 2 - x & \text{if } t \in [1, 2) \\ 0 & \text{if } t < 0 \text{ or } t \geq 2 \end{cases} \; . \tag{5}$$

The resulting wavelets $\psi^1(t) = \psi(t+1)$ and $\psi^2(t) = \psi(t)$ have a support respectively equal to $[-2, 0]$ and $[0, 2]$. One can verify that the space $\mathbf{V}_0$ generated by the family $\{\psi_j^1 \, , \, \psi_j^2\}_{j \in \mathbb{Z}}$ is the space of linear splines, that are continuous and linear on each interval $[-2^{j+1} \, , \, -2^j)$ and $(2^j \, , \, 2^{j+1}]$ for $j \in \mathbb{Z}$. Section 4 devoted to foveal splines proves that $\{\psi_j^1 \, , \, \psi_j^2\}_{j \in \mathbb{Z}}$ is a Riesz basis of $\mathbf{V}_0$.

Foveal approximations can be extended to multiple fovea, which means that f is approximated given an increasing sequence of fovea points $\{u_n\}_{n \in \mathbb{Z}}$, with $u_n \leq u_{n+1}$, which could also be finite. For this purpose, it

is useful to limit the maximum scale of a foveal approximation space. We write $\mathbf{V}_{u,J} \subset \mathbf{V}_u$ the space obtained by limiting the maximum scale of generating functions to 2^J and which is therefore generated by the Riesz basis

$$\left\{\psi_j^m\right\}_{1\leq m\leq M\ ,\ -\infty<j\leq J}\ .$$

For each u_n, we shall choose a maximum scale 2^{J_n} and define a multiple fovea approximation of f as the orthogonal projection $P_{\mathbf{V}}f$ in the space

$$\mathbf{V} = \oplus_{n=-\infty}^{+\infty} \mathbf{V}_{u_n,J_n}\ .$$

If all ψ^m have a support included in $[-K,K]$ then all $f \in \mathbf{V}_{u,J}$ have a support included in $[u-K2^J\ ,\ u+K2^J]$. If

$$2^{J_n} \leq \min\Big(\frac{u_n-u_{n-1}}{2K}\ ,\ \frac{u_{n+1}-u_n}{2K}\Big) \tag{6}$$

then functions in $\mathbf{V}_{u_n,J_n}$ and $\mathbf{V}_{u_m,J_m}$ have disjoint support for $n \neq m$, so the spaces $\{\mathbf{V}_{u_n,J_n}\}_{n\in\mathbb{Z}}$ are orthogonal.

Figure 2 gives examples of piecewise linear foveal approximation from multiple fovea, whose locations are obtained with the singularity detection procedure of Section 6.1. In Figure 2b, the maximum scales J_n satisfy (6) so that each singularity are separately approximated. In Figure 2c, the maximum scales are defined by

$$2\min\Big(\frac{u_n-u_{n-1}}{2}\ ,\ \frac{u_{n+1}-u_n}{2}\Big) \leq 2^{J_n} \leq 4\min\Big(\frac{u_n-u_{n-1}}{K}\ ,\ \frac{u_{n+1}-u_n}{K}\Big) \tag{7}$$

in order to approximate f over its whole support, without holes between the successive singularities.

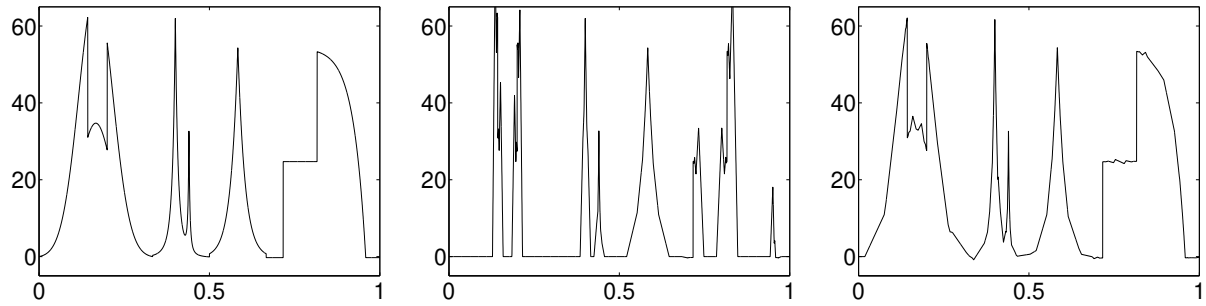


Figure 2: (a): Original function $f(t)$. (b): Piecewise linear foveal approximation from fovea u_n located at $0.14, 0.2, 0.4, 0.44, 0.58, 0.72, 0.82, 0.96$ with maximum scales J_n satisfying (6). (c): The maximum scales J_n are defined by (7).

2.2 From Orthogonal Wavelets to Foveal Approximations

Foveal approximations with translated orthogonal wavelets have been studied in [2] and by Vetterli and Dragotti [14] who aggregate these coefficients to define *footprints* for singularities. In this section, we show that foveal approximations spaces can indeed be constructed by selecting appropriate families of wavelets in a wavelet orthonormal basis of $\mathbf{L^2}(\mathbb{R})$.

We saw in (4) that a piecewise constant foveal approximation space is generated by a family of Haar orthogonal wavelets. This example is generalized by considering any orthogonal wavelet basis of $\mathbf{L^2}(\mathbb{R})$. One can construct wavelets ψ having a compact support centered in $1/2$: $[1/2-K\,,\,1/2+K]$, and such that if $\psi_{j,n}(t) = 2^{-j/2}\psi(2^{-j}t-n)$ then $\{\psi_{j,n}\}_{(j,n)\in\mathbb{Z}^2}$ is an orthonormal basis of $\mathbf{L^2}(\mathbb{R})$ [11]. We denote by $\mathcal{C}_0$ the cone of influence of 0 defined as the set of indexes (j,n) of all wavelets $\psi_{j,n}$ whose support include the point $t=0$. Clearly

$$\mathcal{C}_0 = \left\{ (j,n) \;:\; j\in\mathbb{Z} \;\text{ and }\; -1/2-K \leq n \leq K-1/2 \right\} .$$

At each scale 2^j, there are $M = 2\lfloor K-1/2\rfloor + 2$ such wavelets. Let $m_0 = \lfloor 3/2+K\rfloor$, we denote the M generating wavelets

$$\psi^m(t) = \psi(t-m+m_0) \;\text{ for } 1\leq m\leq M \;,$$

so that $\{\psi_{j,n}\}_{(j,n)\in\mathcal{C}_0} = \{\psi_j^m\}_{1\leq m\leq M\,,\,j\in\mathbb{Z}}$.

Proposition 1 *The space $\mathbf{V}_0$ generated by the translated orthogonal wavelets $\{\psi_j^m\}_{1\leq m\leq M, j\in\mathbb{Z}}$ in the cone of influence of 0 is a foveal space.*

Proof: Clearly $\{\psi_j^m\}_{1\leq m\leq M, j\in\mathbb{Z}}$ is an orthogonal family, hence a Riesz basis of $\mathbf{V}_0$. It remains to verify the property (1). Let $f\in\mathbf{V}_0$. The function $f\mathbf{1}_{[0,+\infty)} \in \mathbf{L^2}(\mathbb{R})$ can be decomposed over the orthogonal basis $\{\psi_{j,n}\}_{(j,n)\in\mathbb{Z}^2}$. If $(j,n)\notin\mathcal{C}_0$ then the support of $\psi_{j,n}$ is either included in $[0,+\infty)$ in which case

$$\langle f\mathbf{1}_{[0,+\infty)}, \psi_{j,n}\rangle = \langle f, \psi_{j,n}\rangle = 0$$

or the support of $\psi_{j,n}$ is included in $(-\infty,0]$ in which case $\langle f\mathbf{1}_{[0,+\infty)}, \psi_{j,n}\rangle = 0$. So $f\mathbf{1}_{[0,+\infty)}$ is orthogonal to all $\psi_{j,n}$ for $(j,n)\notin\mathcal{C}_0$ and hence can be decomposed over the wavelets $\{\psi_{j,n}\}_{(j,n)\in\mathcal{C}_0}$ and thus belongs to $\mathbf{V}_0$. ■

The following proposition proves that even-though f may be singular at u, if it is regular on the left and right hand-side of u then $f(t) - P_{V_u} f(t)$ is small in the neighborhood of u and is uniformly regular if the wavelet ψ is regular. The uniform regularity is measured with uniform Hölder exponents. We say that the restriction of f to $[a,b]$ is uniformly Hölder $\alpha\geq 0$ if there exists $K>0$ such that for all $u\in[a,b]$ there exists a polynomial $q_u(t)$ of degree $m=\lfloor\alpha\rfloor$ with

$$\forall t\in(a,b) \;\;,\;\; |f(t)-q_u(t)| \leq K\,|t-u|^\alpha \;. \tag{8}$$

Proposition 2 *Let $\mathbf{V}_0$ be a foveal space constructed from translated orthogonal wavelets $\{\psi_j^m\}_{1\leq m\leq M, j\in\mathbb{Z}}$ with a mother wavelet ψ which has p vanishing moments. If the restrictions of f to $(-\infty,u]$ and to $[u,+\infty)$ are uniformly Hölder $\alpha<p$ then*

$$|f(t) - P_{\mathbf{V}_u} f(t)| = O(|t-u|^\alpha) \;. \tag{9}$$

Moreover, if ψ is $\mathbf{C}^p$ then $f(t)-P_{\mathbf{V}_u}f(t)$ is uniformly Hölder α on $\mathbb{R}$.

Proof: The proof is done for $u=0$ and is adapted for any u by translating all functions. Let $r = f - P_{\mathbf{V}_0} f$. Since $f = \sum_{j,n}\langle f,\psi_{j,n}\rangle\,\psi_{j,n}$ and $P_{\mathbf{V}_0}f = \sum_{(j,n)\in\mathcal{C}_0}\langle f,\psi_{j,n}\rangle\,\psi_{j,n}$, we have

$$r(t) = \sum_{(j,n)\notin\mathcal{C}_0} \langle f,\psi_{j,n}\rangle\,\psi_{j,n}(t) \;.$$

For all $(j,n) \in \mathcal{C}_0$ we have $|n+1/2| \leq K$ so there exists $\gamma > 0$ such that if $(j,n) \notin \mathcal{C}_0$ then $|n+1/2| \geq K+\gamma$. Since the support of $\psi_{j,n}$ is $[2^j(n+1/2-K), 2^j(n+1/2+K)]$ if $(j,n) \notin \mathcal{C}_0$ then $\psi_{j,n}(t) = 0$ either if $|t| \leq 2^j\gamma$ or if $|n+1/2-2^{-j}t| \geq K$ and there are at most $\lceil 2K \rceil$ indices n for which $\psi_{j,n}(t) \neq 0$ at each scale 2^j. Over its support $|\psi_{j,n}(t)| \leq \|\psi\|_\infty 2^{-j/2}$. Since the support of $\psi_{j,n}$ is included either in $(-\infty, 0]$ or in $[0, +\infty)$ in which f is uniformly Hölder α, it has been proved [11] that there exists $A > 0$ such that $|\langle f, \psi_{j,n}\rangle| \leq A\, 2^{(\alpha+1/2)j}$. It results that

$$r(t) \leq \sum_{j \leq \log_2(|t|/\gamma)} (2K+1)\, A\, 2^{(\alpha+1/2)j}\, \|\psi\|_\infty\, 2^{-j/2} \leq \frac{(2K+1)\, A\, \|\psi\|_\infty}{1-2^{-\alpha}\,\gamma^\alpha}\, |t|^\alpha$$

which proves (9) for $u = 0$.

If ψ is $\mathbf{C}^p$ then it has been proved [11] that r is uniformly Hölder α on $\mathbb{R}$ if and only if there exists $A > 0$ such that for all $(j,n) \in \mathbb{Z}^2$ $|\langle r, \psi_{j,n}\rangle| \leq A\, 2^{(\alpha+1/2)j}$. But this is indeed the case since $|\langle r, \psi_{j,n}\rangle| = 0$ if $(j,n) \in \mathcal{C}_0$ and we saw that $|\langle r, \psi_{j,n}\rangle| \leq A\, 2^{(\alpha+1/2)j}$ if $(j,n) \notin \mathcal{C}_0$, so r is uniformly Hölder α over $\mathbb{R}$. ■

Daubechies [5] proved that if ψ generates a wavelet orthonormal basis of $\mathbf{L^2}(\mathbb{R})$ and has p vanishing moments then its support size satisfies $K \geq p - 1/2$ and minimum support wavelets satisfy $K = p - 1/2$. In this case there are $M = 2p$ generating functions, and for $p = 1$ it corresponds to the $M = 2$ Haar wavelets previously studied. To obtain wavelets which are $\mathbf{C}^p$ it is however necessary to increase much more the support size K and hence the number M of generating functions. In the remaining of the paper, we study foveal approximations constructed with only $M = 2$ generating functions and which have same approximation performance as foveal approximations derived from compactly supported Daubechies wavelets which are $\mathbf{C}^p$ and have p vanishing moments.

3 Only Two Generating Functions

Foveal approximations with only two generating functions are constructed from a single even window whose left and right parts are dilated. Section 3.1 studies the properties of the resulting foveal approximation and constructs wavelets in the foveal space which generate orthogonal bases. For discrete sequences, similar foveal approximations can be defined, and Section 3.2 gives conditions to construct discrete orthogonal bases.

3.1 From Windows to Foveal Approximations

This section constructs foveal approximation spaces $\mathbf{V}_0 = \mathbf{V}_0^+ \oplus \mathbf{V}_0^-$ with two generating functions whose dilations define Riesz bases respectively of $\mathbf{V}_0^-$ and $\mathbf{V}_0^+$. These generating functions are obtained from a single even window $\phi(t)$ such that $\int \phi(t)\, dt \neq 0$, with $\phi^- = \phi\, \mathbf{1}_{(-\infty,0]}$ and $\phi^+ = \phi\, \mathbf{1}_{[0,+\infty)}$. If the two families $\{\phi_j^-\}_{j\in\mathbb{Z}}$ and $\{\phi_j^+\}_{j\in\mathbb{Z}}$ are Riesz bases of the spaces $\mathbf{V}_0^-$ and $\mathbf{V}_0^+$ that they generate then $\mathbf{V}_0 = \mathbf{V}_0^+ \oplus \mathbf{V}_0^-$ is a foveal approximation space in the sense of Definition 1. The main issue will be to build orthogonal foveal wavelet bases of $\mathbf{V}_0$. We begin with a theorem which gives a necessary and sufficient condition to obtain a Riesz basis by dilating a single function.

Theorem 1 *Let $\theta \in \mathbf{L^2}(\mathbb{R})$ and $\mathbf{V}$ be the closed space generated by $\{\theta_j\}_{j\in\mathbb{Z}}$. Let $h_\theta[j] = \langle \theta\,,\, \theta_j\rangle$. The family*

$\{\theta_j\}_{j\in\mathbb{Z}}$ is a Riesz basis of $\mathbf{V}$ if and only if

$$A = \inf_{\omega\in[-\pi,\pi]} \hat{h}_\theta(\omega) > 0 \quad \textit{and} \quad B = \sup_{\omega\in[-\pi,\pi]} \hat{h}_\theta(\omega) < +\infty \ . \tag{10}$$

The constant A and B are the Riesz bounds.

Proof: The Riesz basis property means that there exists $A, B > 0$ such that $f \in \mathbf{V}$ if and only if it can be written

$$f = \sum_{j=-\infty}^{+\infty} c[j]\, \theta_j$$

with

$$A \sum_{j=-\infty}^{+\infty} |c[j]|^2 \leq \|f\|^2 \leq B \sum_{j=-\infty}^{+\infty} |c[j]|^2 \ . \tag{11}$$

The Riesz bounds correspond to the maximum A and minimum B which satisfy (11). They are respectively equal to the infimum and supremum of the spectrum of the Gram matrix $G = \{g_{j,l}\}_{(j,l)\in Z^2}$ of the family $\{\theta_j\}_{j\in\mathbb{Z}}$, defined by $g_{j,l} = \langle \theta_j \, , \, \theta_l \rangle$ [5, 11]. A change of variable in the inner product integral proves that

$$\langle \theta_j \, , \, \theta_l \rangle = \langle \theta \, , \, \theta_{j-l} \rangle = h_\theta[j-l] \ .$$

The Gram matrix thus corresponds to a convolution operator whose spectrum is given by the Fourier series $\hat{h}_\theta(\omega)$. The Riesz bounds are therefore specified by (10). ■

Let $h_0[j] = \langle \phi, \phi_j \rangle$. Theorem 1 implies that $\{\phi_j^- \ , \ \phi_j^+\}_{j\in\mathbb{Z}}$ is a Riesz basis of the space $\mathbf{V}_0$ it generates if and only if

$$A = \frac{1}{2} \inf_{\omega\in[-\pi,\pi]} \hat{h}_0(\omega) > 0 \ \text{ and } \ B = \frac{1}{2} \sup_{\omega\in[-\pi,\pi]} \hat{h}_0(\omega) < +\infty \ . \tag{12}$$

Indeed, the family $\{\phi_j^- \ , \ \phi_j^+\}_{j\in\mathbb{Z}}$ is a Riesz basis of $\mathbf{V}_0$ if and only if $\{\phi_j^-\}_{j\in\mathbb{Z}}$ and $\{\bar{\phi}_j^+\}_{j\in\mathbb{Z}}$ are Riesz bases of $\mathbf{V}_0^-$ and $\mathbf{V}_0^+$, but since ϕ is even

$$\langle \phi^- \, , \, \phi_j^- \rangle = \langle \phi^+ \, , \, \phi_j^+ \rangle = \frac{1}{2} \langle \phi \, , \, \phi_j \rangle \ .$$

If ϕ is bounded and integrable then one can verify $|h_0[j]| = O(2^{-|j|/2})$ so $\hat{h}_0(\omega)$ is bounded on $[-\pi, \pi]$ and is $\mathbf{C}^\infty$. The existence of B is thus always guaranteed. We know that $\hat{h}_0(\omega) \geq 0$ because it is the spectrum of the Gram matrix (positive, symmetric) of $\{\phi_j\}_{j\in\mathbb{Z}}$. However, further conditions are required so that $\hat{h}_0(\omega)$ does not vanish. For example, if

$$\phi(t) = \begin{cases} 1 & \text{if } |t| \leq 1 \\ 2 & \text{if } 1 < |t| \leq 2 \\ 0 & \text{if } |t| > 2 \end{cases}$$

then a direct calculation shows that $\hat{h}_0(\pi) = 0$ so ϕ does not define a Riesz basis. On the contrary, if $\phi(t) = \mathbf{1}_{[-1,1]}$ then

$$\hat{h}_0(\omega) = \frac{2}{|1 - \sqrt{2}\, \mathrm{e}^{-i\omega}|^2} > 0 \tag{13}$$

so it yields a Riesz basis. One can verify that in this case $\mathbf{V}_0$ is the piece-wise constant foveal approximation, also generated by Haar wavelets.

We now show that the foveal space $\mathbf{V}_0$ includes regular wavelets with vanishing moments, that also generate Riesz bases. For this purpose, we decompose $\mathbf{V}_0$ in two subspaces $\mathbf{V}_0^e$ and $\mathbf{V}_0^o$ of even and odd functions respectively generated by $\{\phi_j\}_{j\in\mathbb{Z}}$ and $\{\bar\phi_j = \phi_j^+ - \phi_j^-\}_{j\in\mathbb{Z}}$. Clearly $\mathbf{V}_0^e$ and $\mathbf{V}_0^o$ are orthogonal and $\mathbf{V}_0 = \mathbf{V}_0^e \oplus \mathbf{V}_0^o$. Generally $\phi(0) \neq 0$ so $\bar\phi$ is discontinuous at $t = 0$. To remove this discontinuity we define

$$\psi^1(t) = \Big(\phi(t) - \phi(2t)\Big)\,\mathrm{sign}(t) = \bar\phi(t) - \bar\phi(2t)\ . \tag{14}$$

It is an odd function and $\psi^1(0) = 0$. If $\phi(t)$ is $\mathbf{C}^n$ with $\phi^{(k)}(0) = 0$ for $k \leq n$ then ψ^1 is also $\mathbf{C}^n$. Let us denote $\psi^0(t) = \phi(t)$. The following proposition constructs even and odd wavelets with arbitrary number of vanishing moments in the foveal space $\mathbf{V}_0$.

Proposition 3 *For any $m \geq 0$ we define by induction*

$$\psi^{m+2}(t) = \psi^m(t) - 2^{m+1}\,\psi^m(2t)\ \in \mathbf{V}_0\ . \tag{15}$$

If $0 \leq l < m$ then $\int t^l\,\psi^m(t)\,dt = 0$.

Proof: We prove that ψ^m has m vanishing moments, by induction on m. It is clearly verified for $m = 0$ and $m = 1$. Suppose now that $\psi^m(t)$ has m vanishing moments, and let us prove that $\psi^{m+2}(t)$ has $m + 2$ vanishing moments. Let us compute

$$\int t^l\,\psi^{m+2}(t)\,dt = \int t^l\,\psi^m(t)\,dt - 2^{m+1}\int t^l\,\psi^m(2t)\,dt\ .$$

For $l < m$ the induction hypothesis implies that two right integrals vanish so $\int t^l\,\psi^{m+2}(t)\,dt = 0$. If $l = m$ the change of variable $t' = 2t$ in the second integral on the right hand-side proves that $\int t^m\,\psi^{m+2}(t)\,dt = 0$. Since ψ^0 and ψ^1 are respectively even and odd, we derive from (15) that ψ^m is even if m is even and is odd if m is odd. It result that $\int t^{m+1}\,\psi^{m+2}(t)\,dt = 0$ because ψ^{m+2} and t^{m+1} have an opposite parity. This proves that ψ^{m+2} has $m + 2$ vanishing moments, which finishes the induction proof. ■

For any $m \geq 0$ the wavelet $\psi^m \in \mathbf{V}_0$ has m vanishing moments and its support is equal to the support of ϕ. If m is even then it is an even function which has the same regularity as ϕ and if m is odd then it is an odd function which has the same regularity as ψ^1. To verify that one can construct a Riesz basis with such foveal wavelets, the following proposition analyzes the properties of a change of basis in a foveal space.

Proposition 4 *Let $\theta \in \mathbf{L^2}(\mathbb{R})$ and*

$$\psi = \sum_{j=-\infty}^{+\infty} c[j]\,\theta_j \quad \textit{with} \quad c[j] \in \mathbf{l^2}(\mathbb{Z})\ .$$

If $h_\theta[j] = \langle\theta\,,\,\theta_j\rangle$ and $h_\psi[j] = \langle\psi\,,\,\psi_j\rangle$ then

$$\hat h_\psi(\omega) = |\hat c(\omega)|^2\,\hat h_\theta(\omega)\ . \tag{16}$$

Suppose that $\{\theta_j\}_{j\in\mathbb{Z}}$ is a Riesz basis of $\mathbf{V}$. Then $\{\psi_j\}_{j\in\mathbb{Z}}$ is a Riesz basis of $\mathbf{V}$ if and only if

$$\inf_{\omega\in[-\pi,\pi]} |\hat c(\omega)| > 0 \quad \textit{and} \quad \sup_{\omega\in[-\pi,\pi]} |\hat c(\omega)| < +\infty\ . \tag{17}$$

Proof: Let us compute

$$h_\psi[j] = \langle \psi \, , \, \psi_j \rangle = \langle \sum_{l=-\infty}^{+\infty} c[l]\, \theta_l \, , \, \sum_{l'=-\infty}^{+\infty} c[l']\, \theta_{j+l'} \rangle$$
$$= \sum_{l=-\infty}^{+\infty} \sum_{l'=-\infty}^{+\infty} c[l]\, c[l']\, \langle \theta_l \, , \, \theta_{j+l'} \rangle \; .$$

Since

$$\langle \theta_l \, , \, \theta_k \rangle = \langle \theta \, , \, \theta_{k-l} \rangle = h_\theta[k-l]$$

we derive that

$$h_\psi[j] = c \star \tilde{c} \star h_\theta[j]$$

with $\tilde{c}[j] = c[-j]$. Computing the Fourier transform of this equality gives (16).

Let us prove that the space $\mathbf{W}$ generated by $\{\psi_j\}_{j\in\mathbb{Z}}$ is equal to $\mathbf{V}$. Clearly $\mathbf{W} \subset \mathbf{V}$ since $\psi \in \mathbf{V}$ and hence $\psi_j \in \mathbf{V}_0$. Conversely let us prove that $\theta \in \mathbf{W}$. Let $c^{-1}[j] \in \mathbf{l^2}(\mathbb{Z})$ be such that $\hat{c}^{-1}(\omega) = 1/\hat{c}(\omega)$ with $\hat{c}(\omega)$ satisfying (17). Then

$$\sum_{l=-\infty}^{+\infty} c^{-1}[l]\, \psi_l = \sum_{l=-\infty}^{+\infty} c^{-1}[l] \sum_{j=-\infty}^{+\infty} c[j]\, \theta_{j+l}$$
$$= \sum_{j'=-\infty}^{+\infty} \theta_{j'} \sum_{l=-\infty}^{+\infty} c^{-1}[l]\, c[j'-l] = \sum_{j'=-\infty}^{+\infty} \psi_{j'}\, c^{-1} \star c[j'] \; = \theta$$

because $c^{-1} \star c[j'] = \delta[j']$. So $\theta \in \mathbf{W}$ and hence $\theta_j \in \mathbf{W}$ for any $j \in \mathbb{Z}$, which proves that $\mathbf{V} \subset \mathbf{V}$. So $\mathbf{W} = \mathbf{V}$.

Theorem 1 proves that $\{\psi_j\}_{j\in\mathbb{Z}}$ is a Riesz basis of $\mathbf{W} = \mathbf{V}$ if and only if

$$\inf_{\omega\in[-\pi,\pi]} \hat{h}_\psi(\omega) > 0 \;\; \text{and} \;\; \sup_{\omega\in[-\pi,\pi]} \hat{h}_\psi(\omega) < +\infty \; . \tag{18}$$

Since $\{\theta_j\}_{j\in\mathbb{Z}}$ is a Riesz basis of $\mathbf{V}$, Theorem 1 proves that this property is valid for $h_\theta(\omega)$ and we derive from (16) that (18) is satisfied if and only if (17) is verified. ■

The following proposition applies this result to the foveal wavelets ψ^m to prove that they generate Riesz bases of the foveal space. We denote $\bar{\psi}^m(t) = \psi^m(t)\,\mathrm{sign}(t)$,

$$\psi_j^m(t) = 2^{-j/2}\, \psi^m(2^{-j}t) \;\; \text{and} \;\; h_m[j] = \langle \psi^m \, , \, \psi_j^m \rangle \; .$$

Proposition 5 *Let $\mathbf{V}_0^e$ and $\mathbf{V}_0^o$ be the even and odd foveal spaces generated respectively by $\{\phi_j\}_{j\in\mathbb{Z}}$ and $\{\bar{\phi}_j\}_{j\in\mathbb{Z}}$. The wavelet family $\{\psi_j^m\}_{j\in\mathbb{Z}}$ is a Riesz basis of $\mathbf{V}_0^e$ if m is even and of $\mathbf{V}_0^o$ if m is odd.*

Proof: Since $\psi^m = \psi^{m-2} - 2^{m-3/2}\, \psi_{-1}^{m-2}$, Theorem 4 proves in (16) that

$$\hat{h}_{m+2}(\omega) = \hat{h}_m(\omega)\, |1 - 2^{m+1/2} \mathrm{e}^{-i\omega}|^2 \; . \tag{19}$$

Let us consider first the case of even foveal wavelets:

$$\hat{h}_{2m}(\omega) = \hat{h}_0(\omega) \prod_{l=0}^{m-1} |1 - 2^{l+1/2} \mathrm{e}^{-i\omega}|^2 \; . \tag{20}$$

Since

$$0 < \prod_{l=0}^{m-1} (1 - 2^{l+1/2}) \leq \prod_{l=0}^{m-1} |1 + 2^{l+1/2} \mathrm{e}^{-i\omega}| \leq \prod_{l=0}^{m-1} (1 + 2^{l+1/2})$$

applying Theorem 4 proves that $\{\psi_j^{2m}\}_{j\in\mathbb{Z}}$ is a Riesz basis of $\mathbf{V}_0^e$.

For odd foveal wavelets, observe first that $\psi^1 = \bar{\phi} - 2^{-1/2}\bar{\phi}_{-1}$ so (16) implies that

$$\hat{h}_1(\omega) = \hat{h}_0(\omega)\,|1 - 2^{-1/2}\mathrm{e}^{-i\omega}|^2 \ . \tag{21}$$

We thus derive from (19) that

$$\hat{h}_{2m+1}(\omega) = \hat{h}_0(\omega) \prod_{l=-1}^{m-1} |1 - 2^{l+1/2}\mathrm{e}^{-i\omega}|^2 \ . \tag{22}$$

Applying Theorem 4 proves that $\{\psi_j^{2m+1}\}_{j\in\mathbb{Z}}$ is a Riesz basis of $\mathbf{V}_0^o$. ■

Since $\mathbf{V}_0 = \mathbf{V}_0^e \oplus \mathbf{V}_0^o$ is an orthogonal decomposition, this proposition implies that for any $m \geq 0$, each of the foveal wavelet families

$$\{\psi_j^m \ , \ \bar{\psi}_j^m\}_{j\in\mathbb{Z}} \ \text{ and } \ \{\psi_j^m \ , \ \psi_j^{m+1}\}_{j\in\mathbb{Z}}$$

are Riesz bases of $\mathbf{V}_0$. A Riesz basis is an orthogonal basis if and only if the upper and lower Riesz bounds are equal. Theorem 1 proves a foveal family $\{\psi_j^m\}_{j\in\mathbb{Z}}$ is orthogonal if and only if $\hat{h}_m(\omega)$ is constant. Using (20) and (22) one can derive a necessary and sufficient condition on $\hat{h}_0$. The following proposition considers the particular cases $m = 1$ and $m = 2$.

Proposition 6 *The following statements are equivalent:*

(i) $\{\psi_j^1\}_{j\in\mathbb{Z}}$ *is orthogonal.*

(ii) $\{\psi_j^2\}_{j\in\mathbb{Z}}$ *is orthogonal.*

(iii) *There exists* $C > 0$ *such that* $\hat{h}_0(\omega) = C\,|1 - 2^{1/2}\mathrm{e}^{-i\omega}|^{-2}$.

(iv) *There exists* $C > 0$ *such that*

$$\forall j \in \mathbb{Z} \ \ , \ \ h_0[j] = C\,2^{-|j|/2} \ . \tag{23}$$

Proof: The first two foveal wavelet families are orthogonal if and only if $\hat{h}_1(\omega)$ and $\hat{h}_2(\omega)$ are constant. We computed in (21)

$$\hat{h}_1(\omega) = \hat{h}_0(\omega)\,|1 - 2^{-1/2}\mathrm{e}^{-i\omega}|^2 = 2^{-1}\,\hat{h}_0(\omega)\,|1 - 2^{1/2}\mathrm{e}^{-i\omega}|^2$$

we thus derive from (20) that $\hat{h}_1(\omega) = 2^{-1}\,\hat{h}_2(\omega)$. Both are constant if and only if $\hat{h}_0(\omega) = C\,|1 - 2^{1/2}\mathrm{e}^{-i\omega}|^{-2}$ for some constant $C > 0$, which means that $h_0[j] = C\,2^{-|j|/2}$. ■

If any of the conditions of this proposition is satisfied then $\{\psi_j^1\}_{j\in\mathbb{Z}}$ is an orthogonal basis of $\mathbf{V}_0^o$ and $\{\psi_j^2\}_{j\in\mathbb{Z}}$ is an orthogonal basis of $\mathbf{V}_0^e$. Hence each of these three families

$$\{\psi_j^1 \ , \ \bar{\psi}_j^1\}_{j\in\mathbb{Z}} \ \ , \ \ \{\psi_j^1 \ , \ \psi_j^2\}_{j\in\mathbb{Z}} \ \ , \ \ \{\psi_j^2 \ , \ \bar{\psi}_j^2\}_{j\in\mathbb{Z}}$$

defines an orthogonal basis of $\mathbf{V}_0$.

For a piecewise constant foveal approximation, $\phi = \mathbf{1}_{[-1,1]}$ and $h_0[j] = 2^{1-|j|/2}$ so the foveal wavelets ψ^1 and ψ^2 define orthogonal bases of $\mathbf{V}_0$. They are shown in Figure 3(a).

Generally, the foveal wavelets ψ^1 and ψ^2 constructed with a window ϕ generate a Riesz basis that is not orthogonal. Let us orthogonalize $\{\psi_j^1\}_{j\in\mathbb{Z}}$ and $\{\psi_j^2\}_{j\in\mathbb{Z}}$ by constructing two new wavelets:

$$\tilde{\psi}^1 = \sum_{j=-\infty}^{+\infty} c[j]\,\psi_j^1 \quad \text{and} \quad \tilde{\psi}^2 = \sum_{j=-\infty}^{+\infty} c[j]\,\psi_j^2 \ . \tag{24}$$

Let $\tilde{h}_m[j] = \langle \tilde{\psi}^m\,,\,\tilde{\psi}_j^m\rangle$. We showed in (16) that $\widehat{\tilde{h}_m}(\omega) = \hat{h}_m(\omega)\,|\hat{c}(\omega)|^2$. The proof of Proposition 6 shows that $\hat{h}_1(\omega) = 2^{-1}\,\hat{h}_2(\omega)$ so $\{\tilde{\psi}_j^1\}_{j\in\mathbb{Z}}$ and $\{\tilde{\psi}_j^2\}_{j\in\mathbb{Z}}$ are orthogonal if and only if there exists $C>0$ such that

$$|\hat{c}(\omega)|^2 = \frac{C}{\widehat{h_1}(\omega)} \ . \tag{25}$$

The resulting families

$$\{\tilde{\psi}_j^1\ ,\ \overline{\tilde{\psi}_j^1}\}_{j\in\mathbb{Z}} \quad , \quad \{\tilde{\psi}_j^1\ ,\ \tilde{\psi}_j^2\}_{j\in\mathbb{Z}} \ \text{ and } \ \{\tilde{\psi}_j^2\ ,\ \overline{\tilde{\psi}_j^2}\}_{j\in\mathbb{Z}}$$

are then orthogonal bases of $\mathbf{V}_0$. Choosing $\hat{c}(\omega)$ which satisfies (25) is a spectral factorization problem that is further analyzed in the context of spline wavelets.

To approximate efficiently functions that are regular in the left and in the right neighborhood of 0 but which may be singular at $t=0$ as in Proposition 2, we impose that polynomials belong to the foveal space $\mathbf{V}_0$. The following proposition gives a necessary and sufficient condition.

Proposition 7 *Let $\mathbf{V}_0$ be a foveal space and $\{\psi_j^1\ ,\ \bar{\psi}^1\}_{j\in\mathbb{Z}}$ be a foveal Riesz basis. For any polynomial $q(t)$ of degree p there exists $b[j]$ such that*

$$\forall t>0 \quad , \quad q(t) = \sum_{j=-\infty}^{+\infty} b[j]\,\psi_j^1(t) \tag{26}$$

if and only if

$$\forall t>0 \quad , \quad \sum_{j=-\infty}^{+\infty} 2^{(k+1/2)j}\,\psi_j^1(t) = a_k\,t^k \quad \textit{with } a_k\neq 0 \textit{ for } 0\leq k\leq p \quad . \tag{27}$$

Proof Since $q(t)$ can be decomposed in the monomial family $\{t^k\}_{0\leq k\leq p}$ clearly (26) is satisfied if (27) holds. Conversely if (26) is satisfied then setting $q(t)=t^k$ yields

$$t^k = \sum_{j=-\infty}^{+\infty} b[j]\,2^{-j/2}\,\psi^1(2^{-j}t)$$

and writing $t=2t'$, we get

$$2^k\,t^k = \sum_{j=-\infty}^{+\infty} b[j]\,2^{-j/2}\,\psi^1(2^{-j+1}t)$$

so eliminating t^k from the last two equations gives

$$\sum_{j=-\infty}^{+\infty} (b[j] - 2^{-k-1/2}b[j+1])\,2^{-j/2}\,\psi^1(2^{-j}t) = 0$$

Since $\{\psi_j^1\ ,\ \bar{\psi}_j^1\}_{j\in\mathbb{Z}}$ is a Riesz basis we derive that $b[j] = 2^{-k-1/2}\,b[j+1]$ for all $j\in\mathbb{Z}$ and hence that $b[j] = 2^{(k+1/2)j}\,b[0]$ which implies (27). ■

3.2 Discrete Foveal Approximations

Foveal approximations constructed with dilated windows are adapted to discrete sequences. Orthonormal bases are constructed with discrete foveal wavelets. In discrete signals, the minimum scale is limited by the sampling interval, that is normalized to 1, and a maximum scale must be introduced to limit computations over a finite support. The discrete lattice $n \in \mathbb{Z}$ is assimilated to a sampling of $\mathbb{R}$ at $t = n - 1/2$. The origin $t = 0$ is thus in the middle of $n = 0$ and $n = 1$. For any discrete signal $\theta[n]$ we write

$$\bar{\theta}[n] = \theta[n]\ \mathrm{sign}(n + 1/2) \quad , \quad \theta^-[n] = \theta[n]\ \mathbf{1}_{(-\infty,1/2]}(n) \quad , \quad \theta^+[n] = \theta[n]\ \mathbf{1}_{[1/2,+\infty)}(n) \quad .$$

Definition 2 *A family of discrete foveal windows $\{\phi_j[n]\}_{0 \le j \le J}$ satisfies for $0 \le j \le J$:*

- $\phi_j[n] = \phi_j[1 - n]$
- *there exists $K > 0$ such that the support of $\phi_j[n]$ is $[-K2^j + 1\ ,\ K2^j]$*
- *there exists $C > 0$ such that $\sum_{n=0}^{+\infty} \phi_j[n] = C\, 2^{j/2}$*

The family $\{\phi_j^-, \phi_j^+\}_{0 \le j \le J}$ is then linearly independent and thus defines a basis of a foveal space $\mathbf{V}_{0,J}$. A foveal space at m is defined by

$$f[n] \in \mathbf{V}_{m,J} \Leftrightarrow f[n + m] \in \mathbf{V}_{0,J} \ .$$

To verify that $\{\phi_j^-, \phi_j^+\}_{0 \le j \le J}$ is linearly independent, observe first that $\{\phi_j^-\}_{0 \le j \le J}$ and $\{\phi_j^+\}_{0 \le j \le J}$ are orthogonal. Clearly $\{\phi_j^-\}_{0 \le j \le J}$ is linearly independent, because for any $l > 0$, ϕ_l^- does not belong to $\mathrm{Span}\{\phi_j^-\}_{0 \le j < l}$ since the support of ϕ_l is strictly larger than the supports of all ϕ_j^- for $j < l$. The same applies to $\{\phi_j^+\}_{0 \le j \le J}$.

Discrete foveal windows can be defined by discretizing a continuous time foveal window $\phi(t)$ of compact support $[-K, K]$:

$$\phi_j[n] = 2^{-j/2} \int_n^{n+1} \phi(2^{-j}t)\, dt \ . \tag{28}$$

Clearly

$$\sum_{n=-\infty}^{+\infty} \phi_j[n] = 2^{-j/2} \int_{-\infty}^{+\infty} \phi(2^{-j}t)\, dt = 2^{j/2} \int_{-\infty}^{+\infty} \phi(t)\, dt \ .$$

At any scale $2^j > 1$, the first foveal wavelet is defined like in (14) by

$$\psi_j^1[n] = \bar{\phi}_j[n] - 2^{-1/2} \bar{\phi}_{j-1}[n] \ . \tag{29}$$

It is antisymmetric about $-1/2$

$$\psi_j^1[-n] = -\psi_j^1[n - 1]$$

and hence has one vanishing moment

$$\sum_{n=-\infty}^{+\infty} \psi_j^1[n] = 0 \ .$$

The second foveal wavelet is defined as in (15) by

$$\psi_j^2 = \phi_j - 2^{1/2} \phi_{j-1} \ . \tag{30}$$

It is symmetric about $-1/2$

$$\psi_j^2[-n] = \psi_j^2[n-1]$$

and has two vanishing moments

$$\sum_{n=-\infty}^{+\infty} \psi_j^2[n] = 0 \quad \text{and} \quad \sum_{n=-\infty}^{+\infty} n\, \psi_j^1[n] = 0 \ .$$

Let $\mathbf{V}_{0,J}^e$ and $\mathbf{V}_{0,J}^o$ be the spaces of even and odd signals generated respectively by $\{\phi_j\}_{0\le j\le J}$ and $\{\bar\phi_j\}_{0\le j\le J}$. These spaces are orthogonal and $\mathbf{V}_{0,J}^e \oplus \mathbf{V}_{0,J}^o = \mathbf{V}_{0,J}$. The following theorem is a discrete equivalent of Proposition 6, which gives necessary and sufficient conditions to obtain orthonormal bases.

Theorem 2 *Let $\mathbf{V}_{0,J}$ be the space generated by foveal windows $\{\phi_j\ ,\ \bar\phi_j\}_{0\le j\le J}$. The following statements are equivalent:*

- *$\{\bar\phi_0\ ,\ \psi_j^1\}_{1\le j\le J}$ is an orthogonal basis of $\mathbf{V}_{0,J}^o$*
- *$\{\psi_j^2\ ,\ \phi_J\}_{1\le j\le J}$ is an orthogonal basis of $\mathbf{V}_{0,J}^e$*
- *there exists $C>0$ such that*

$$\forall j,l \ge 0 \quad , \quad \langle \phi_j\ ,\ \phi_l\rangle = C\, 2^{-|l-j|/2} \ . \tag{31}$$

Proof: With the recursive equations (29) and (30) we verify directly that $\{\bar\phi_0\ ,\ \psi_j^1\}_{1\le j\le J}$ generates $\mathbf{V}_{0,J}^o$ and $\{\psi_j^2\ ,\ \phi_J\}_{1\le j\le J}$ generates $\mathbf{V}_{0,J}^e$. Let us now prove that the orthogonality of $\{\bar\phi_0\ ,\ \psi_j^1\}_{1\le j\le J}$ is equivalent to (31). Let us compute

$$\langle \bar\phi_0\ ,\ \psi_j^1\rangle = \langle \phi_0\ ,\ \phi_j\rangle - 2^{-1/2}\langle \phi_0\ ,\ \phi_{j-1}\rangle \tag{32}$$

and

$$\langle \psi_l^1\ ,\ \psi_j^1\rangle = \langle \phi_l\ ,\ \phi_j\rangle - 2^{-1/2}\langle \phi_l\ ,\ \phi_{j-1}\rangle - 2^{-1/2}\langle \phi_{l-1}\ ,\ \phi_j\rangle + 2^{-1}\langle \phi_{l-1}\ ,\ \phi_{j-1}\rangle \ . \tag{33}$$

If (31) is satisfied then a direct calculation shows that $\langle \bar\phi_0\ ,\ \psi_j^1\rangle$ and $\langle \psi_l^1\ ,\ \psi_j^1\rangle$ are zero and hence that $\{\bar\phi_0\ ,\ \psi_j^1\}_{1\le j\le J}$ is orthogonal. Conversely, let us suppose that these inner products are 0. We prove with an induction on j that for all $0\le l\le j$

$$\langle \phi_j\ ,\ \phi_l\rangle = C\, 2^{-(j-l)/2} \ .$$

For $j=0$ and hence $l=0$ this is trivial for $C=\langle \phi_0\ ,\ \phi_0\rangle$. Suppose that it is true for all $l\le j=n-1$, we are going to verify it for all $0\le l\le j=n$. This is proved by another induction on l. For $l=0$, applying the induction hypothesis on (32) yields

$$\langle \phi_0\ ,\ \phi_n\rangle = 2^{-1/2}\langle \phi_0\ ,\ \phi_{n-1}\rangle = 2^{-n/2}\,\langle \phi_0\ ,\ \phi_0\rangle$$

which verifies the property for $j=n$. Suppose now that it is valid for $l=p-1$. The vanishing inner product (33) implies that

$$\langle \phi_p\ ,\ \phi_n\rangle = 2^{-1/2}\langle \phi_p\ ,\ \phi_{n-1}\rangle + 2^{-1/2}\langle \phi_{p-1}\ ,\ \phi_n\rangle - 2^{-1}\langle \phi_{p-1}\ ,\ \phi_{n-1}\rangle \ .$$

Applying our induction hypothesis for $l\le j=n-1$ and for $l=p-1$ and $j=n$ proves that $\langle \phi_p\ ,\ \phi_n\rangle = C\, 2^{-(n-p)/2}$, which verifies the hypothesis for $l=p$ and $j=n$ and finishes the induction proof.

To prove that the orthogonality of $\{\psi_j^2\ ,\ \phi_J\}_{1\leq j\leq J}$ is equivalent to (31) we compute

$$\langle\psi_l^2\ ,\ \psi_j^2\rangle = \langle\phi_l\ ,\ \phi_j\rangle - 2^{1/2}\langle\phi_l\ ,\ \phi_{j-1}\rangle - 2^{1/2}\langle\phi_{l-1}\ ,\ \phi_j\rangle + 2\langle\phi_{l-1}\ ,\ \phi_{j-1}\rangle \tag{34}$$

and

$$\langle\phi_J\ ,\ \psi_j^2\rangle = \langle\phi_J\ ,\ \phi_j\rangle - 2^{1/2}\langle\phi_J\ ,\ \phi_{j-1}\rangle\ . \tag{35}$$

If (31) is satisfied then these inner product are zero which proves that $\{\psi_j^2\ ,\ \phi_J\}_{1\leq j\leq J}$ is orthogonal. Conversely, if this family is orthogonal we prove that for $j\leq l\leq J$

$$\langle\phi_j\ ,\ \phi_l\rangle = C\,2^{-(l-j)/2}\ ,$$

with a double induction on j and l, but in this case for j decreasing from J to 0. For $j=J$ and hence $l=J$ this is verified for $C=\langle\phi_J\ ,\ \phi_J\rangle$. Suppose that it is true for all $l\geq j=n$, we are going to verify it for all $J\geq l\geq j=n-1$. This is proved by another induction on l. For $l=J$, by applying the induction hypothesis on (35) we get

$$\langle\phi_J\ ,\ \phi_J\rangle\,2^{-(J-n)/2} = \langle\phi_J\ ,\ \phi_n\rangle = 2^{1/2}\,\langle\phi_J\ ,\ \phi_{n-1}\rangle$$

which verifies the property. Suppose now that it is valid for $l=p$. The vanishing inner product (33) implies that

$$\langle\phi_{p-1}\ ,\ \phi_{n-1}\rangle = -2^{-1}\langle\phi_n\ ,\ \phi_p\rangle + 2^{-1/2}\langle\phi_n\ ,\ \phi_{p-1}\rangle + 2^{-1/2}\langle\phi_{n-1}\ ,\ \phi_p\rangle\ .$$

Applying our induction hypothesis for $l\geq j=n$ and for $l=p$ and $j=n$ proves that $\langle\phi_{p-1}\ ,\ \phi_{n-1}\rangle = C\,2^{(n-p)/2}$, which verifies the hypothesis for $l=p-1$ and $j=n-1$ and finishes the proof. ■

Since $\mathbf{V}_{0,J}^e$ and $\mathbf{V}_{0,J}^o$ are orthogonal complements in $\mathbf{V}_{0,J}$, if (31) is satisfied then each of the following three families

$$\{\bar\phi_0\ ,\ \psi_j^1\ ,\ \phi_0\ ,\ \bar\psi_j^1\}_{1\leq j\leq J}\quad ,\quad \{\psi_j^2\ ,\ \phi_J\ ,\ \bar\psi_j^2\ ,\ \bar\phi_J\}_{1\leq j\leq J} \tag{36}$$

and

$$\{\bar\phi_0\ ,\ \psi_j^1\ ,\ \psi_j^2\ ,\ \phi_J\}_{1\leq j\leq J} \tag{37}$$

is an orthogonal basis of $\mathbf{V}_{0,J}$.

If (31) is not satisfied then we can orthogonalize $\{\bar\phi_0\ ,\ \psi_j^1\}_{1\leq j\leq J}$ and $\{\psi_j^2\ ,\ \phi_J\}_{1\leq j\leq J}$ with a Gram-Schmidt orthogonalization. Suppose that we begin with the second one: $\tilde\psi_1^2=\psi_1^2$ and for any $1<j\leq J$

$$\tilde\psi_j^2 = \psi_j^2 - \sum_{l=1}^{j-1}\frac{\langle\psi_j^2\ ,\ \tilde\psi_l^2\rangle}{\|\tilde\psi_l^2\|^2}\,\tilde\psi_l^2$$

with

$$\tilde\phi_J = \phi_J - \sum_{l=1}^{J}\frac{\langle\phi_J\ ,\ \tilde\psi_l^2\rangle}{\|\tilde\psi_l^2\|^2}\,\tilde\psi_l^2\ .$$

The resulting family $\{\tilde\psi_j^2\ ,\ \tilde\phi_J\}_{1\leq j\leq J}$ is an orthogonal basis of $\mathbf{V}_{0,J}^e$ and $\tilde\psi_j^2$ has 2 vanishing moments for any $j\leq J$. The support of $\tilde\psi_j^2$ is equal to the support of ψ_j^2 and hence to $[-K2^j+1\ ,\ K2^j]$, and the support of $\tilde\phi_J$ is equal to the support of ϕ_J and hence to $[-K2^J+1\ ,\ K2^J]$. Similarly, let for any $1<j\leq J$

$$\tilde\psi_j^1 = \psi_j^1 - \frac{\langle\psi_j^1\ ,\ \bar\phi_0\rangle}{\|\bar\phi_0\|^2}\bar\phi_0 - \sum_{l=1}^{j-1}\frac{\langle\psi_j^1\ ,\ \tilde\psi_l^1\rangle}{\|\tilde\psi_l^1\|^2}\,\tilde\psi_l^1\ .$$

The resulting family $\{\bar{\phi}_0\ ,\ \tilde{\psi}_j^1\}_{1\leq j\leq J}$ is an orthogonal basis of $\mathbf{V}_{0,J}^o$ and the support of $\tilde{\psi}_j^1$ is equal to the support of ψ_j^1 and hence to $[-K2^j+1\ ,\ K2^j]$. From these orthogonal basis of $\mathbf{V}_{0,J}^e$ and $\mathbf{V}_{0,J}^o$ we can construct three orthogonal bases of $\mathbf{V}_{0,J}$ as in (36) and (37).

4 Foveal Splines

Foveal approximations with two generating functions are constructed with polynomial splines. It generalizes the piecewise constant foveal approximations. We denote $\phi^0 = \mathbf{1}_{[-1,1]}$ and define for any $p > 0$:

$$\phi^p(t) = \int_{-\infty}^{t} \Big(\phi^{p-1}(2x) - \phi^{p-1}(x)\Big)\, \mathrm{sign}(x)\, dx\ . \tag{38}$$

The following properties are easily verified by induction on p. The window ϕ^p is a polynomial spline of degree p whose support is equal $[-1,1]$. Moreover, its derivative $\phi^{p\prime}(t) = 0$ if $|t| < 2^{-p}$ so $\phi^p(t)$ is constant on $[-2^{-p}\ ,\ 2^{-p}]$. If $t < 2^{-p}$ then $\phi^{p\prime}(t) \geq 0$ and if $t > 2^{-p}$ then $\phi^{p\prime}(t) \leq 0$, so $\phi^p(t)$ does not change sign, and $\phi^p(t) \geq 0$ because $\phi^0(t) \geq 0$.

The first foveal wavelet is

$$\psi^{1,p}(t) = \bar{\phi}^p(t) - \bar{\phi}^p(2t)\ . \tag{39}$$

The support of $\psi^{1,p}$ is $[-1, 2^{-p-1}] \cup [2^{-p-1}, 1]$, and $\psi^{1,p}(t) \geq 0$ for $t \geq 0$ whereas $\psi^{1,p}(t) \leq 0$ for $t \leq 0$. Inserting (38) gives

$$\phi^p(t) = -\int_{-\infty}^{t} \psi^{1,p-1}(x)\, dx \tag{40}$$

and

$$\psi^{1,p}(t) = \mathrm{sign}(t) \int_{t}^{2t} \psi^{1,p-1}(x)\, dx\ . \tag{41}$$

For any $m \geq 2$, the foveal wavelet $\psi^{m,p}$ with m vanishing moments, calculated with (15), has a support included in $[-1,1]$. Figure 3 gives examples of such foveal wavelets for $p = 0, 1, 2$. The following theorem proves that ϕ^p defines a Riesz basis of a polynomial spline space which is specified.

Theorem 3 *For any $p \geq 1$, the spline window family $\{\phi_j^p\ ,\ \bar{\phi}_j^p\}_{j\in\mathbb{Z}}$ is a Riesz basis of the space $\mathbf{V}_0^p \subset \mathbf{L^2}(\mathbb{R})$ of functions that are $\mathbf{C}^{p-1}$ on $\mathbb{R} - \{0\}$ and equal to a polynomial of degree p on $[-2^{j+1}\ ,\ -2^j)$ and on $(2^j\ ,\ 2^{j+1}]$, for any $j \in \mathbb{Z}$.*

Proof Let $h_{0,p}[j] = \langle \phi^p\ ,\ \phi_j^p\rangle$ and $h_{1,p}[j] = \langle \psi^{1,p}\ ,\ \psi_j^{1,p}\rangle$. We saw in (12) that $\{\phi_j^p\ ,\ \bar{\phi}_j^p\}_{j\in\mathbb{Z}}$ is a Riesz basis of the space $\mathbf{V}_0^p$ it generates if $\hat{h}_{0,p}(\omega)$ is bounded and is strictly positive. It is equivalent to show this property on $\hat{h}_{1,p}(\omega)$ because (21) proves that $\hat{h}_{1,p}(\omega) = \hat{h}_{0,p}(\omega)\, |1 - 2^{-1/2}\mathrm{e}^{-i\omega}|^2$.

Since the support of $\psi^{1,p}$ is $[-1, -2^{-p-1}] \cup [2^{-p-1}, 1]$, the sequence $h_{1,p}[j]$ has a support included in $[-p, p]$ so $\hat{h}_{1,p}(\omega)$ is bounded and is $\mathbf{C}^\infty$. Suppose that $\inf_\omega \hat{h}_{1,p}(\omega) = 0$. Since $\hat{h}_1^p(\omega)$ is continuous and 2π periodic, there must exist ξ such that $\hat{h}_1^p(\xi) = 0$. We first prove that it implies that $f = \sum_{j=-\infty}^{+\infty} \mathrm{e}^{ij\xi}\psi_j^{1,p} = 0$ by showing that $\int |f(t)|^2 dt = 0$, and then show that it leads to a contradiction. Since $f(t) = -f(-t)$ it is sufficient to restrict ourself to $t \geq 0$. Suppose that there exists $k \in \mathbb{Z}$ such that $\int_{2^k}^{2^{k+1}} |f(t)|^2 dt = C > 0$. Since the support

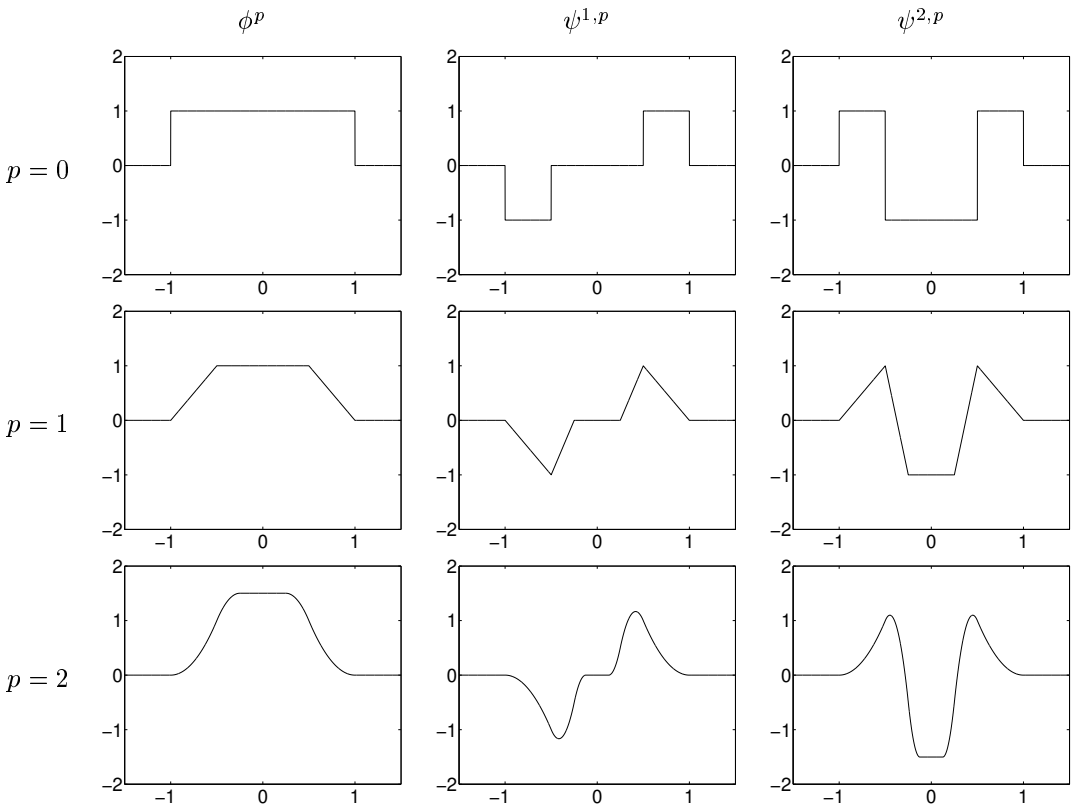


Figure 3: Foveal spline wavelets of order p, for $p = 0, 1, 2$.

of $\psi^{1,p}$ is $[-1, -2^{-p-1}] \cup [2^{-p-1}, 1]$ for $t \in [2^{-l}, 2^{l-p}]$ we have $f(t) = \sum_{j=-l}^{l} \mathrm{e}^{ij\xi}\, \psi_j^{1,p}(t)$ and hence

$$\left\| \sum_{j=-l}^{l} \mathrm{e}^{ij\xi} \psi_j^{1,p} \right\|^2 \geq \int_{2^{-l}}^{2^{l-p}} |f(t)|^2\, dt = \sum_{j=-l}^{l-p-1} \int_{2^k}^{2^{k+1}} 2^{j-k}\, |f(2^{j-k}t)|^2\, dt\ .$$

One can also verify that $2^{1/2}\, f(2t) = \mathrm{e}^{i\xi}\, f(t)$ so

$$\left\| \sum_{j=-l}^{l} \mathrm{e}^{ij\xi} \psi_j^{1,p} \right\|^2 \geq (2l-p) \int_{2^k}^{2^{k+1}} |f(t)|^2\, dt = (2l-p)\, C\ . \tag{42}$$

Moreover, since $\hat{h}_{1,p}(\xi) = 0$, for any $l \in \mathbb{Z}$

$$\langle \sum_{j'=-l}^{l} \mathrm{e}^{ij'\xi}\, \psi_{j'}^{1,p}\, ,\ \sum_{j=-\infty}^{+\infty} \mathrm{e}^{ij\xi} \psi_j^{1,p} \rangle = \sum_{j'=-l}^{l} \sum_{j=-\infty}^{+\infty} \mathrm{e}^{-i(j-j')\xi} h_{1,p}[j-j'] = 0\ ,$$

and hence

$$\left\| \sum_{j=-l}^{l} \mathrm{e}^{ij\xi} \psi_j^{1,p} \right\|^2 = - \sum_{j'=-l}^{l} \sum_{|j|>l} \mathrm{e}^{-i(j-j')\xi} h_{1,p}[j-j']\ .$$

Since $h_{1,p}[k] = 0$ for $|k| > p$, we derive that

$$\left\| \sum_{j=-l}^{l} \mathrm{e}^{ij\xi} \psi_j^{1,p} \right\|^2 \leq 2\, p^2 \sup_{j \in \mathbb{Z}} |h_{1,p}[j]|$$

which contradicts (42) for l sufficiently large. It results that that $\int |f(t)|^2 dt = 0$ and since $f(t)$ is continuous for $p > 0$

$$\forall t \in \mathbb{R} \quad , \quad f(t) = \sum_{j=-\infty}^{+\infty} \mathrm{e}^{ij\xi}\, \psi_j^{1,p}(t) = 0 \; . \tag{43}$$

We differentiate this identity using (41) which shows that

$$\frac{d\psi^{1,p}(t)}{dt} = \mathrm{sign}(t) \left(2\, \psi^{1,p-1}(2t) - \psi^{1,p-1}(t) \right) \; .$$

It results that

$$\frac{df(t)}{dt} = \sum_{j=-\infty}^{+\infty} \mathrm{e}^{ij\xi}\, 2^{-j} 2^{-j/2} \frac{\psi^{1,p}(2^{-j}t)}{dt} = (\mathrm{e}^{i\xi}\, 2^{-1/2} - 1) \sum_{j=-\infty}^{+\infty} \mathrm{e}^{ij\xi}\, 2^{-j}\, \psi_j^{1,p-1}(t) \; .$$

Further derivatives give

$$\frac{d^p f(t)}{dt^p} = \prod_{k=1}^{p} (\mathrm{e}^{i\xi}\, 2^{-k+1/2} - 1) \sum_{j=-\infty}^{+\infty} \mathrm{e}^{ij\xi}\, 2^{-pj}\, \psi_j^{1,0}(t) = 0 \; ,$$

which is wrong since $\psi^{1,0} = \mathbf{1}_{[1/2,1]} - \mathbf{1}_{[-1,-1/2]}$. It results that there does not exists ξ such that $\hat{h}_{1,p}(\xi) = 0$ and hence that $\{\phi_j^p \, , \, \bar{\phi}_j^p\}_{j\in\mathbb{Z}}$ and $\{\psi_j^{1,p} \, , \, \bar{\psi}_j^{1,p}\}_{j\in\mathbb{Z}}$ are Riesz bases of $\mathbf{V}_0^p$.

Let us now prove that $\mathbf{V}_0^p$ is equal to the space $\mathbf{W}_p$ of finite energy functions that are polynomial of degree p on $[-2^{j+1} \, , \, -2^j]$ and on $[2^j \, , \, 2^{j+1}]$ and are $\mathbf{C}^{p-1}$ on $\mathbb{R} - \{0\}$. Since $\psi^{1,p} \in \mathbf{W}_p$ clearly $\mathbf{V}_0^p \subset \mathbf{W}_p$. Conversely, now prove that $\mathbf{W}_p \subset \mathbf{V}_0^p$. Let us decompose $\mathbf{W}_p = \mathbf{W}_p^- \oplus \mathbf{W}_p^+$ and $\mathbf{V}_0^p = \mathbf{V}_0^{p-} \oplus \mathbf{V}_0^{p-}$, where $\mathbf{W}_p^-$ and $\mathbf{V}_0^{p-}$ are composed of functions whose support are included in $(-\infty, 0]$, where as $\mathbf{W}_p^+$ and $\mathbf{V}_0^{p-}$ are composed of functions whose support are included in $[0, +\infty)$. We are first going to prove by induction on p that $\mathbf{W}_p^- \subset \mathbf{V}_0^{p-}$. For $p = 0$ this result is clear. Suppose now that it is true for $p-1$ and let us consider $f \in \mathbf{W}_p^-$. For $n \in \mathbb{Z}$, let $q_n(t)$ be the polynomial of degree p such that $q_n(t) = f(t)$ for $t \in [-2^{-n+1}, -2^{-n}]$. Let us define

$$f_n(t) = \begin{cases} f(t) & \text{if } t < -2^{-n} \\ q_n(t) & \text{if } t \in [-2^{-n}, 0) \\ 0 & \text{if } t \geq 0 \end{cases} \; . \tag{44}$$

The function f_n is $\mathbf{C}^{p-1}$ over $\mathbb{R} - \{0\}$ and hence remains in $\mathbf{W}_p^-$. Let us show that for any $\epsilon > 0$ there exists $n > 0$ such that $\|f - f_n\| \leq \epsilon$. One can prove that there exists $C(p)$ such that for all polynomials $q(t)$ of degree p and all $\Delta \in \mathbb{R}$

$$\int_0^{\Delta} |q(t)|^2 \, dt \leq C(p) \int_{\Delta}^{2\Delta} |q(t)|^2 \, dt \; . \tag{45}$$

To verify this result, we use an orthonormal basis $\{e_i(t)\}_{1\leq i\leq p}$ of polynomials of degree p defined over $[1, 2]$. Observe that $\{\Delta^{-1/2} e_i(\Delta^{-1} t)\}_{1\leq i\leq p}$ is an orthonormal basis of polynomials of degree p on $[\Delta, 2\Delta]$, so

$$\forall t \in [0, 2\Delta] \quad q(t) = \sum_{i=0}^{p} a_i\, \Delta^{-1/2} e_i(\Delta^{-1} t)$$

with $|a_i|^2 \leq \int_{\Delta}^{2\Delta} |q(t)|^2 \, dt$ for $0 \leq i \leq p$. With a change of variable, we show that

$$\begin{aligned} \int_0^{\Delta} |q(t)|^2 \, dt &\leq \left(\max_{0\leq i\leq p} \int_0^1 |e_i(t)|^2 \, dt \right) (p+1)^2 \max_{1\leq i\leq p} |a_i|^2 \\ &\leq C(p) \int_{\Delta}^{2\Delta} |q(t)|^2 \, dt \; . \end{aligned} \tag{46}$$

Since f has a finite energy, there exists n such that

$$\int_{-2^{-n+1}}^{0} |f(t)|^2 \, dt \leq \frac{\epsilon^2}{2C(p)+2} \ . \tag{47}$$

Moreover (44) implies that

$$\|f - f_n\|^2 \leq 2 \int_{2^{-n}}^{0} |f(t)|^2 \, dt + 2 \int_{2^{-n}}^{0} |f_n(t)|^2 \, dt \tag{48}$$

and applying (45) to $\Delta = 2^{-n}$ shows that

$$\int_{2^{-n}}^{0} |f_n(t)|^2 \, dt = \int_{2^{-n}}^{0} |q_n(t)|^2 \, dt \leq C(p) \int_{2^{-n+1}}^{2^{-n}} |q_n(t)|^2 \, dt = \int_{2^{-n+1}}^{2^{-n}} |f(t)|^2 \, dt \ .$$

We thus derive from (47) and (48) that $\|f - f_n\| \leq \epsilon$.

The function $f_n'(t)$ is a polynomial spline of degree $p-1$ which is $\mathbf{C}^{p-2}$ on $(-\infty, 0]$. To prove that it has a finite energy we use the fact that there exists $K(p)$ such that for all polynomials $q(t)$ of degree p and all $\Delta \in \mathbb{R}$

$$\int_{0}^{\Delta} |q'(t)|^2 \, dt \leq K(p) \, \Delta^{-1} \int_{0}^{\Delta} |q(t)|^2 \, dt \ . \tag{49}$$

This is shown with the same approach as for (45) by decomposing $q(t)$ on an orthonormal basis $\{\Delta^{-1/2} e_i(\Delta^{-1} t)\}_{1 \leq i \leq p}$ where $\{e_i(t)\}_{1 \leq i \leq p}$ is an orthonormal basis of polynomials of degree p on $[0,1]$. Calculating $q'(t)$ leads to (49) with an argument similar to (46). Applying this result to $\Delta = 2^{-n}$ and $q(t) = f_n(t - m\Delta)$ for any $m \in \mathbb{Z}$ implies that

$$\int_{-\infty}^{0} |f_n'(t)|^2 \, dt \leq K(p) \, 2^n \int_{-\infty}^{0} |f_n(t)|^2 \, dt \ .$$

So f_n' has a finite energy and hence $f_n' \in \mathbf{W}_{p-1}^-$. The induction hypothesis implies that $f_n' \in \mathbf{V}_0^{p-}$ and hence can be written

$$f_n'(t) = \sum_{j=-\infty}^{+\infty} b_1[j] \left(\psi_j^{1,p-1}(t) - \bar{\psi}_j^{1,p-1}(t) \right) \ . \tag{50}$$

Since f_n' has a finite energy and is a polynomial on any $[m2^{-n}, (m+1)2^{-n}]$ for $m \in \mathbb{Z}$, we easily verify that $\lim_{t \to -\infty} {f'}_n(t) = 0$. Integrating (50) using (40) gives

$$f_n(t) = - \sum_{j=-\infty}^{+\infty} b_1[j] \, 2^j \left(\phi_j^{p-1}(t) - \bar{\phi}_j^{p-1}(t) \right) \ .$$

Since f_n has a finite energy it results that $f_n \in \mathbf{V}_0^{p-}$. But $\|f - f_n\| \leq \epsilon$ and this result can be obtained for any $\epsilon > 0$, so we derive that $f \in \mathbf{V}_0^{p-}$ which verifies our induction hypothesis and hence that $\mathbf{W}_p^- \subset \mathbf{V}_0^{p-}$. Similarly we prove that $\mathbf{W}_p^+ \subset \mathbf{V}_0^{p-}$ and hence that $\mathbf{W}_p \subset \mathbf{V}_0^p$. ■

For $p = 1$, this theorem proves that $\{\phi_j^1 \ , \ \bar{\phi}_j^1\}_{j \in \mathbb{Z}}$ is a Riesz basis of the linear spline space $\mathbf{V}_0^1$ introduced in Section 2.1. By decomposing the linear hat function $\psi(t)$ in (5), and applying Proposition 4 one can also verify that linear hats also generate a Riesz basis of $\mathbf{V}_0^1$.

For $p \neq 0$, the foveal polynomial spline wavelets $\psi^{1,p}$ and $\psi^{2,p}$ are not orthogonal. We apply the orthogonalization procedure of Section 3.1, which yields orthogonal foveal wavelets of compact support whose properties are studied. For $m = 1, 2$, the new foveal wavelets are defined by

$$\tilde{\psi}^{m,p} = \sum_{j=-\infty}^{+\infty} c[j] \, \psi_j^{m,p} \ , \tag{51}$$

and according to (25) they generate an orthogonal basis if and only if $|\hat{c}(\omega)|^{-2} = C^{-1}\widehat{h_{1,p}}(\omega)$. Since the support of $\psi^{1,p}$ is $[-1,-2^{-p-1}] \cup [2^{-p-1},1]$

$$h_{1,p}[j] = \langle \psi^{1,p}\,,\, \psi^{1,p}_j \rangle = 0 \quad \text{if } |j| > p.$$

Since $\hat{h}_{1,p}(\omega)$ is a positive trigonometric polynomial, a lemma by Riesz shows that it can be written

$$\hat{h}_{1,p}(\omega) = C \prod_{l=1}^{q} |1 - r_{l,p}\, \mathrm{e}^{-i\omega}|^{2m_l} = Q_p(\mathrm{e}^{i\omega})$$

where each $r_{l,p}$ for $1 \leq l \leq q$ is a root of multiplicity m_l of the trigonometric polynomial $\hat{h}_{1,p}(\omega)$ and $|r_{l+1,p}| < |r_{l,p}| \leq 1$ for $1 \leq l \leq q$. Theorem 3 proves that $\hat{h}_{1,p}(\omega) > 0$ so necessarily $|r_{l,p}| \neq 1$ for $1 \leq l \leq q$. Let us choose

$$\hat{c}(\omega) = \prod_{l=1}^{p-1} \frac{1}{(1 - r_{l,p}\, \mathrm{e}^{-i\omega})^{m_l}}\ . \tag{52}$$

The inverse Fourier transform of $\hat{c}(\omega)$ can be written:

$$\forall j \leq 0 \quad , \quad c[j] = \sum_{l=1}^{p-1} q_l[j]\, (r_{l,p})^{-j} \quad \text{and} \quad \forall j > 0 \quad c[j] = 0\ , \tag{53}$$

where $q_l[j]$ is a polynomial in j of degree $m_l - 1$.

Since $\psi^{1,p}$ and $\psi^{2,p}$ have a support included in $[-1,1]$ and $c[j] = 0$ for $j < 0$, we derive from (51) that the support of $\tilde{\psi}^{1,p}$ and $\tilde{\psi}^{2,p}$ is also included in $[-1,1]$. When j goes to $-\infty$ the leading term in $|c[j]|$ is $|q_1[j]|\, |r_{1,p}|^{-j}$ because $|r_{1,p}| = \max_l |r_{l,p}| < 1$. Since $q_1[j]$ is a polynomial of degree $m_1 - 1$ and the support of $\psi^{1,p}$ is $[-1,-2^{-p-1}] \cup [2^{-p-1},1]$, we derive from (51) that for $j < 0$

$$\forall t \in [2^j, 2^{j+1}] \quad , \quad |\tilde{\psi}^{1,p}(t)| = O(2^{-j/2}\, |j|^{m_1-1}\, |r_{1,p}|^{-j})$$

and hence that for $|t| \leq 1$

$$|\tilde{\psi}^{1,p}(t)| = O(|\log_2 t|^{m_1-1}\, |t|^{\beta_p}) \quad \text{with} \quad \beta_p = -\log_2 |r_{1,p}| - \frac{1}{2}\ . \tag{54}$$

This exponent β_p can not be improved in the sense that one can find $0 < \gamma < 1$ such that $|\tilde{\psi}^{1,p}(\gamma^j)| \geq C\, |j|^{m_1-1} \gamma^{j\beta_p}$ for $j < 0$. Since $\psi^{2,p}(t)$ is constant for $|t| \leq 2^{-p-1}$, similarly we verify from (51) and (53) that

$$|\tilde{\psi}^{2,p}(t) - \tilde{\psi}^{2,p}(0)| = O(|\log t|^{m_1-1}\, |t|^{\beta_p})\ .$$

Section 6 shows that the approximation performance of foveal wavelet bases depend upon this exponent β_p. The following proposition gives an analytical expression of $\hat{h}_{1,p}(\omega)$ to compute the roots $r_{l,p}$ and hence β_p.

Proposition 8 *For a polynomial spline $\psi^1 = \psi^{1,p}$ of degree p*

$$\hat{h}_{1,p}(\omega) = 2^{-p(p+1)} \sum_{k=0}^{p} \frac{(-1)^k\, (2^{2k} - 2^{-1})}{(k+p+1)!\, (p-k)!} \prod_{\substack{l=0 \\ l \neq k}}^{p} (2^{-1/2}\mathrm{e}^{i\omega} - 2^l)\, (2^{-1/2}\mathrm{e}^{-i\omega} - 2^l)\ . \tag{55}$$

Proof: The first step of the proof shows that $\psi^{1,p}$ satisfies the monomial decomposition formula (27) with

$$a_k = a_{k,p} = \frac{2^{k(k+1)/2}\, H(k)\, H(p-k)}{k!\, (p-k)!} \quad \text{for} \quad 0 \leq k \leq p \tag{56}$$

where $H(l) = \prod_{n=1}^{l}(1-2^{-n})$ for $l \geq 1$ and $H(0) = 1$. To obtain this expression, we prove by induction on p that

$$\forall t > 0 \quad , \quad \sum_{j=-\infty}^{+\infty} 2^{(k+1/2)j}\, \psi_j^1(t) = a_{k,p}\, t^k \tag{57}$$

with

$$a_{k,p} = a_{k-1,p-1}\, \frac{2^k - 1}{k} \ . \tag{58}$$

Property (57) clearly holds for $p = 0$ and hence $k = 0$. Suppose that it is valid for $p-1$ and all $0 \leq k \leq p-1$. For $k = 0$ observe that:

$$\forall t \geq 0 \quad , \quad \sum_{j=-l}^{l} \psi^{1,p}(2^{-j}t) = \phi^p(2^{-l}t) - \phi^p(2^l t)$$

so $\sum_{j=-\infty}^{+\infty} \psi_j^{1,p}(2^{-j}t) = a_{0,p} = \phi^p(0)$. For $1 \leq k < p$ we apply the induction hypothesis:

$$\forall x \geq 0 \ \sum_{j=-\infty}^{+\infty} 2^{jk}\, \psi^{1,p-1}(2^{-j}x) = a_{k,p-1}\, x^k \ .$$

Integrating x between t and $2t$ gives for $t \geq 0$ with (41):

$$\forall t \geq 0 \quad , \quad \sum_{j=-\infty}^{+\infty} 2^{j(k+1)}\, \psi^{1,p}(2^{-j}t) = \frac{a_{k,p-1}(2^{k+1}-1)}{k+1}\, t^{k+1}$$

and hence $a_{k+1,p} = a_{k,p-1}(2^{k+1}-1)/(k+1)$, which verifies (58). Using recursively this induction relation, we obtain

$$a_{k,p} = a_{0,p-k} \prod_{l=1}^{k} \frac{2^l - l}{l} \ . \tag{59}$$

It now remains to compute

$$a_{0,q} = \phi^q(0) = \int_0^{+\infty} \psi^{1,q-1}(t)\, dt \ . \tag{60}$$

By integrating by parts and by inserting (41) we verify that

$$\begin{aligned} \forall k, l \geq 0 \quad , \quad \int_0^{+\infty} t^k\, \psi^{1,l}(t)\, dt &= \int_0^{+\infty} \frac{t^{k+1}}{k+1} \left(\psi^{1,l-1}(t)\, dt - 2\psi^{1,l-1}(2t) \right) dt \\ &= \frac{1 - 2^{-k-1}}{k+1} \int_0^{+\infty} t^{k+1}\, \psi^{1,l-1}(t)\, dt \end{aligned}$$

Since $\psi^{1,0} = \mathbf{1}_{[-1,-1/2]\cup[1/2,1]}$, it results that

$$\int_0^{+\infty} t^k\, \psi^{1,q}(t)\, dt = \prod_{l=k+1}^{k+q} \frac{1-2^{-l}}{l} \int_0^{+\infty} t^{k+q}\, \psi^{1,0}(t)\, dt = \prod_{l=k+1}^{k+q+1} \frac{1-2^{-l}}{l} \ . \tag{61}$$

For $k = 0$, inserting this in (60) together with (59) gives (56).

To compute $\hat{h}_{1,p}(\omega)$ let us now integrate $\psi^{1,p}(t)$ against both sides of the equation

$$\sum_{j=-\infty}^{+\infty} 2^{(k+1/2)j}\, \psi_j^{1,p}(t) = a_{k,p}\, t^k$$

With (61) we obtain

$$\begin{aligned} \sum_{j=-p}^{p} 2^{(k+1/2)j}\, h_{1,p}[j] &= a_{k,p} \int_0^{+\infty} t^k\, \psi^{1,p}(t)\, dt \\ &= \frac{2^{k(k+1)/2}\, H(k+p+1)\, H(p-k)}{(k+p+1)!\,(p-k)!} = b_{k,p} \ . \end{aligned} \tag{62}$$

The Fourier series of $h_{1,p}[j]$ is $\hat{h}_{1,p}(\omega) = Q_p(\mathrm{e}^{i\omega})$ with

$$Q_p(z) = \sum_{j=-p}^{p} h_{1,p}[j]\, z^j = Q_p(z^{-1})$$

and (62) shows that $Q_p(2^{k+1/2}) = b_{k,p}$ for $-p \leq k \leq p$. The polynomial $Q_p(z)$ can therefore be decomposed as a sum of symmetric Lagrange interpolation polynomials

$$Q_p(z) = \sum_{k=0}^{p} b_{k,p}\, \bar{L}_{k,p}(z) \tag{63}$$

where $\bar{L}_{k,p}(z)$ satisfies $\bar{L}_{k,p}(z) = \bar{L}_{k,p}(z^{-1})$ and $\bar{L}_{k,p}(2^{k+1/2}) = \delta[k-p]$ and is thus given by

$$\bar{L}_{k,p}(z) = \frac{\prod_{\substack{l=0 \\ l\neq k}}^{p} (z - 2^{l+1/2})\,(z^{-1} - 2^{l+1/2})}{\prod_{\substack{l=0 \\ l\neq k}}^{p} (2^{k+1/2} - 2^{l+1/2})\,(2^{-k-1/2} - 2^{l+1/2})}\ . \tag{64}$$

Inserting the expression (62) of $b_{k,p}$ in this formula and simplifying the algebraic expression yields (55) by setting $z = \mathrm{e}^{i\omega}$. ■

p	1	2	3	4	5	6	7	8	9	10
β_p	1.497	.8646	.5929	.4439	.3513	.2890	.2445	.2114	.1859	.1657

Table 1: Decay exponent β_p for orthogonal polynomial spline foveal wavelets of order p.

The roots of $\hat{h}_{1,p}(\omega)$ have been computed for $1 \leq p \leq 10$ using (55). All roots are simple so $|\tilde{\psi}^{1,p}(t)| = O(|t|^{\beta_p})$ for $|t| \leq 1$ and Table 1 gives the corresponding values of β_p in (54). Observe that $\beta_p < 1$ for $2 \leq p \leq 10$. The orthogonalized wavelets $\tilde{\psi}^{1,p}$ and $\tilde{\psi}^{2,p}$ in Figure 4 are calculated from the analytical expression of $\hat{c}(\omega)$ in (52) by computing its inverse Fourier transform to get $c[j]$ which is inserted in (51).

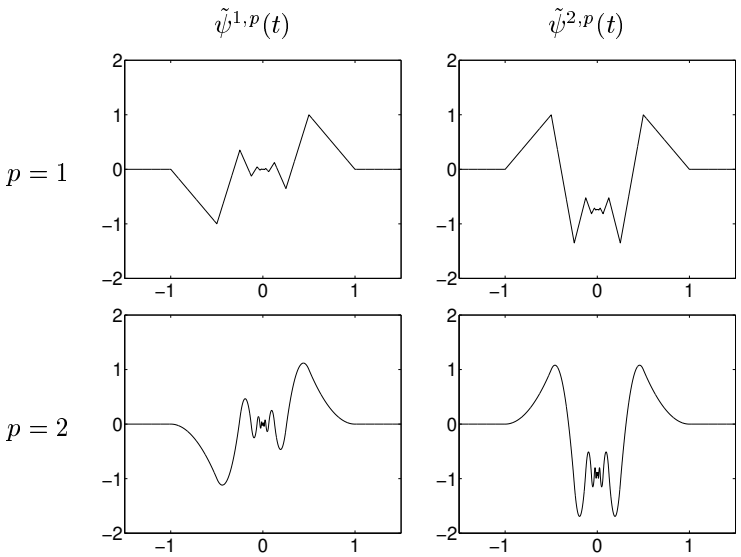


Figure 4: Orthogonal foveal spline wavelets of order $p = 1$ and $p = 2$.

5 Design of Orthogonal Foveal Wavelets

This section optimizes the construction of orthogonal foveal wavelets for foveal approximations derived from a single window ϕ. Section 5.1 constructs foveal wavelets ψ^1 whose dilations define an orthogonal family that reproduces polynomials of degree p and which have a minimum support. These minimum support foveal wavelets are not continuous. Section 5.2 constructs orthogonal foveal wavelets of larger support but which are also regular. These foveal wavelets are used in Section 6.2 to approximate signals having isolated singularities.

5.1 Orthogonal Foveal Wavelets of Minimum Support

Let ψ^1 be such that $\{\psi^1_j\}_{j\in\mathbb{Z}}$ is orthogonal and reproduces polynomials of degree p. It is said to have a minimum support if there is a minimum number of scales 2^j such that the support of ψ^1_j intersects the support of ψ^1. If the support of ψ^1 is included in $[-C_2,-C_1]\cup[C_1,C_2]$ then there are at most $\lceil \log_2(C_2/C_1)\rceil$ scales 2^j for which the support of ψ^1_j intersects the support of ψ^1. The number of intersections is invariant by dilation. Let $\psi^{1,s}(t) = s^{-1/2}\,\psi^1(s^{-1}t)$. For any $s>0$, the family $\{\psi^{1,s}_j\}_{j\in\mathbb{Z}}$ is orthogonal and reproduces polynomials of degree p if and only if $\{\psi^1_j\}_{j\in\mathbb{Z}}$ is orthogonal and reproduces polynomials of degree p. Moreover, ψ^1 has a minimum support if and only if $\psi^{1,s}$ has a minimum support. The support of ψ^1 is thus defined up to a scaling factor, which we normalize by supposing that it is equal to $[-1,1]$. The fact that ψ^1 reproduces a constant is given by:

$$\forall t>0 \quad \sum_{j=-\infty}^{+\infty} \psi^1(2^{-j}t) = a_0 \ .$$

The amplitude of ψ^1 is renormalized by setting $a_0=1$.

The following theorem computes minimum support foveal wavelets using the inverse $M^{-1}=(w_{j,k})_{0\le j,k\le p}$ of the Vandermonde matrix $M=(2^{kj})_{0\le k,j\le p}$, whose coefficients are calculated from Lagrange interpolation polynomials

$$L_{p,k}(x) = \frac{\displaystyle\prod_{\substack{n=0\\ n\neq k}}^{p}(x-2^n)}{\displaystyle\prod_{\substack{n=0\\ n\neq k}}^{p}(2^k-2^n)} = \sum_{j=0}^{p} w_{j,k}\,x^j \ . \tag{65}$$

Theorem 4 *A foveal wavelet ψ^1 has a support in $[-1,-2^{-b}]\cup[2^{-b},1]$, generates an orthogonal family $\{\psi^1_j\}_{j\in\mathbb{Z}}$ and satisfies*

$$\forall t>0 \quad \sum_{j=-\infty}^{+\infty} 2^{-kj}\,\psi^1(2^j t) = a_k\,t^k \quad \textit{with} \quad a_k\neq 0 \quad \textit{for} \quad 0\le k\le p \tag{66}$$

only if $b\ge p+1$. There exists a foveal wavelet of minimum support $[-1,-2^{-p-1}]\cup[2^{-p-1},1]$ with $a_0=1$ if and only if for $1\le k\le p$

$$a_k\sum_{l=1}^{p} a_l\,\frac{1-2^{-k-l-1}}{k+l+1}\,L_{p,l}(2^{-k-1}) + a_k\,\frac{1-2^{-k-1}}{k+1}\,L_{p,0}(2^{-k-1}) =$$
$$1-2^{-p-1}+\sum_{l=1}^{p} a_l\,\frac{1-2^{-l-1}}{l+1}\,L_{p,l}(2^{-1}) \tag{67}$$

and ψ^1 is defined on its support by

$$\forall t \in [2^{-j-1}, 2^{-j}] \quad \psi^1(t) = \sum_{l=0}^{p} 2^{jl}\, w_{j,l}\, a_l\, t^l \quad \textit{for} \quad 0 \leq j \leq p \ . \tag{68}$$

Proof If the support of ψ^1 is included in $[-1, -2^{-b}] \cup [2^{-b}, 1]$ with $b \leq p+1$ then we easily verify that (66) is equivalent to

$$\forall t \in [1/2, 1] \quad , \quad \sum_{j=0}^{p} 2^{kj}\, \psi^1(2^{-j}t) = a_k\, t^k \quad \text{for} \quad 0 \leq k \leq p \ . \tag{69}$$

The solution of this linear system is computed from the inverse $M^{-1} = (w_{j,k})_{0 \leq j,k \leq p}$ of the Vandermonde matrix $M = (2^{kj})_{0 \leq k,j \leq p}$

$$\forall t \in [1/2, 1] \quad , \quad \psi^1(2^{-j}t) = \sum_{l=0}^{p} w_{j,l}\, a_l\, t^l \ . \tag{70}$$

We derive from (65) that $w_{j,p} \neq 0$ for any $0 \leq j \leq p$ so $\psi^1(t)$ is a non-zero polynomial of degree p on each interval $[2^{-j-1}\ , 2^{-j}]$ for $0 \leq j \leq p$, which proves that $b \geq p+1$.

Suppose now that ψ^1 has a support in $[-1, -2^{-p-1}] \cup [2^{-p-1}, 1]$. The family $\{\psi^1_j\}_{j \in \mathbb{Z}}$ is orthogonal if and only if $h_1[j] = \langle \psi^1\ , \psi^1_j \rangle = 0$ for $0 < |j| \leq p$. This is expressed as an equivalent condition on the moments of ψ^1. Since ψ^1 is odd, we derive from (66) that

$$\frac{1}{2} \sum_{j=-p}^{p} 2^{(k+1/2)j}\, h_1[j] = a_k \int_0^{+\infty} t^k\, \psi^1(t)\, dt \ . \tag{71}$$

If $\{\psi^1_j\}_{j \in \mathbb{Z}}$ is orthogonal then $h_1[j] = 0$ for $j \neq 0$ so

$$2\, a_k \int_0^{+\infty} t^k\, \psi^1(t)\, dt = \|\psi^1\|^2 \ . \tag{72}$$

Since $a_0 = 1$ we get

$$a_k \int_0^{+\infty} t^k\, \psi^1(t)\, dt = \int_0^{+\infty} \psi^1(t)\, dt \quad \text{for } 1 \leq k \leq p. \tag{73}$$

Conversely if (73) and (66) are satisfied let us show that $\{\psi^1_j\}_{j \in \mathbb{Z}}$ is orthogonal. Since $h_1[j] = h_1[-j]$ we derive from (71) and (73) that

$$\frac{1}{2} \sum_{j=-p}^{p} h_1[j]\, (2^{(k+1/2)j} + 2^{-(k+1/2)j}) = \int_0^{+\infty} \psi^1(t)\, dt \ . \tag{74}$$

We proved in (62) and (63) that this system of equation characterizes uniquely the coefficients $h_1[j]$ for $0 \leq |j| \leq p$, that can be calculated using coefficients of symmetric Lagrange interpolation polynomials (64). It results that $h_1[0] = \int_0^{+\infty} \psi^1(t)\, dt$ and $h_1[j] = 0$ for $|j| \leq p$ and hence that $\{\psi^1_j\}_{j \in \mathbb{Z}}$ is orthogonal. Since $a_0 = 1$ we have $\sum_{j=-\infty}^{+\infty} \psi^1(2^{-j}t) = 1$ and integrating this equation against $\psi^1(t)$ proves that $\|\psi^1\|^2 = \int_0^{+\infty} \psi^1(t)\, dt$.

The orthogonality condition (73) is rewritten by decomposing the integral over intervals $[2^{-j-1}, 2^{-j}]$

$$\int_0^{+\infty} t^k\, \psi^1(t)\, dt = \sum_{j=0}^{p} \int_{2^{-1}}^{1} 2^{-j(k+1)}\, t^k\, \psi^1(2^{-j}t)\, dt \ .$$

Inserting (70) gives

$$\int_0^{+\infty} t^k\, \psi^1(t)\, dt = \sum_{j=0}^{p} 2^{-j(k+1)} \sum_{l=0}^{p} w_{j,l}\, a_l\, \frac{1 - 2^{-k-l-1}}{k+l+1} \ . \tag{75}$$

But (65) shows that $\sum_{j=0}^{p} 2^{-j(k+1)} w_{j,l} = L_{p,l}(2^{-k-1})$ and we can also verify that $L_{p,0}(2^{-1}) = 2 - 2^{-p}$. Inserting (75) in (73) gives the system of equations (67). Since (73) is equivalent to the orthogonality of $\{\psi_j^1\}_{j\in\mathbb{Z}}$, this finishes the proof. ■

The equations (67) can be considered as a system of p polynomial equations of degree 2 in the p unknown $\{a_k\}_{1\leq k\leq p}$. To each real solution corresponds a unique wavelet ψ^1 specified by (68), whose support and amplitude are normalized. The generalized Bézout theorem proves that the system (67) has at most 2^p solutions. However we are interested in real solutions and we are not guaranteed a priori that there exists such a solution. For $p = 1$ this system is reduced to a single equation of degree 2 which has 2 real solutions. For larger p, a common approach to compute solutions consists in computing a Gröbner basis of the system. However, the computation of a Gröbner basis may be quite expensive.

The system (67) can be evaluated fast: it is sparse, and all equations share the same right-hand side. To solve this system, we used the algorithm of Giusti, Lecerf and Salvy [7], which is able to take advantage of this feature to lower the complexity of the computations, without appealing to Gröbner basis computations. It is based on several theoretical papers by the TERA group [6], and implemented Magma in the Kronecker package [8]. The solutions calculated by Schost [12] with this software are given in Table 2 for $p \leq 5$, and the corresponding functions ψ^1 are shown in Figure 5. To compute ϕ from ψ^1 observe that $\bar{\phi}(t) = \sum_{j=0}^{+\infty} \psi^1(2^j t)$ because $\psi^1(t) = \bar{\phi}(t) - 2\bar{\phi}(2t)$. Since $\sum_{j=-\infty}^{+\infty} \psi^1(2^j t) = a_0 = 1$, for a minimum support wavelet of degree p having a support in $[-1, -2^{-p-1}] \cup [2^{-p-1}, 1]$ it results that

$$\phi(t) = \begin{cases} 1 & \text{if } |t| \leq 2^{-p} \\ \sum_{j=0}^{p-1} \psi^1(2^j t) & \text{if } |t| \in [2^{-p}, 1] \\ 0 & \text{if } |t| > 1 \end{cases}$$

and we derive the second wavelet $\psi^2(t) = \phi(t) - 2\phi(2t)$. Figure 5 shows several examples of functions ϕ and ψ^2 having a minimum support. The solutions have also been calculated for $p = 6, 7, 8$ and there are 2^{p-2} real solutions in each of these cases.

The solutions of the system for $p-1$ and for p equations may a priori be completely different but Table 2 shows that for a degree $p-1$ and a degree p, the corresponding systems (67) may have similar solutions. In particular, some solutions $(a_{k,p-1})_{1\leq k\leq p-1}$ of a system of size $p-1$ and some solutions $(a_{k,p})_{1\leq k\leq p}$ of a system of size p satisfy $a_{k,p} \approx \bar{a}_{k,p-1}$ for $1 \leq k \leq p-1$. As a result, corresponding foveal wavelets $\psi^1(t)$ shown in Figure 5 have nearly the same graph. These similarities come from the fact that the coefficients of the system (67) converge to constant values when p is large. Indeed, a direct calculation using (65) shows that

$$(1 - 2^{-k-l-1}) L_{p,l}(2^{-k-1}) = (-1)^l \, 2^{-l(l-1)/2} \, \frac{H(p+k+1)}{H(l)\, H(k)\, H(p-l)}$$

with $H(l) = \prod_{n=1}^{l} (1 - 2^{-n})$ for $l \geq 1$ and $H(0) = 1$. As a result

$$\lim_{p\to+\infty} (1 - 2^{-k-l-1})\, L_{p,l}(2^{-k-1}) = (-1)^l \, 2^{-l(l-1)/2} \, \frac{1}{H(l)\, H(k)} \, . \tag{76}$$

When increasing the order of the system (67) from $p-1$ to p the coefficients of the first $p-1$ equations are slightly modified and there is one more equation corresponding to $k = p$. This gives a strategy to obtain a solution $(a_{k,p})_{1\leq k\leq p}$ of the system of p equation by approximating it with a solution $(a_{k,p-1})_{1\leq k\leq p-1}$ of

p	k	a_k
1	1	1.38929754034
	1	2.46784531681
2	1	1.42156733273
	2	1.95703524452
	1	4.00184272521
	2	16.87872416048
3	1	1.43896043716
	2	2.00492714857
	3	2.68901769737
3	1	2.40826558195
	2	6.34326400942
	3	24.84883380353
3	1	3.67398277565
	2	12.75294954651
	3	31.26189394313
3	1	6.14883219776
	2	40.34826199541
	3	288.88675880998
4	1	1.44799978962
	2	2.02991856780
	3	2.73773545364
	4	3.55410449056
4	1	3.53110143673
	2	11.38960166084
	3	25.91105763966
	4	33.32116128073
4	1	3.68246985711
	2	15.53391326869
	3	99.52366453588
	4	1226.74836883958
4	1	8.98009426273
	2	87.15871029071
	3	941.93301433661
	4	11501.26012264996

p	k	a_k
5	1	1.45260930654
	2	2.04269295137
	3	2.76270880317
	4	3.59526088078
	5	4.51880240215
5	1	2.11595720318
	2	4.52698150687
	3	10.30310423277
	4	29.56144211827
	5	2659.03726215417
5	1	3.46565528688
	2	10.82788594645
	3	23.92860560391
	4	31.16824359368
	5	30.23989336339
5	1	3.60652678031
	2	14.62460997988
	3	82.74687046438
	4	675.97404696759
	5	257.40941193917
5	1	5.04827983337
	2	23.99657736388
	3	89.23811202948
	4	256.27576397792
	5	17794.31718866944
5	1	5.25348163808
	2	32.41081283406
	3	308.59192628344
	4	5558.08558140915
	5	151469.60567457325
5	1	12.53382883579
	2	171.80290584747
	3	2672.80231927319
	4	48184.47702691261
	5	1013636.87007247459
5	1	8.60450125654
	2	77.52198330520
	3	716.69414671157
	4	5860.19330933748
	5	1722.58764049202

Table 2: Real solutions $\{a_k\}_{1\leq k\leq p}$ of the system (67) calculated for $1 \leq p \leq 5$.

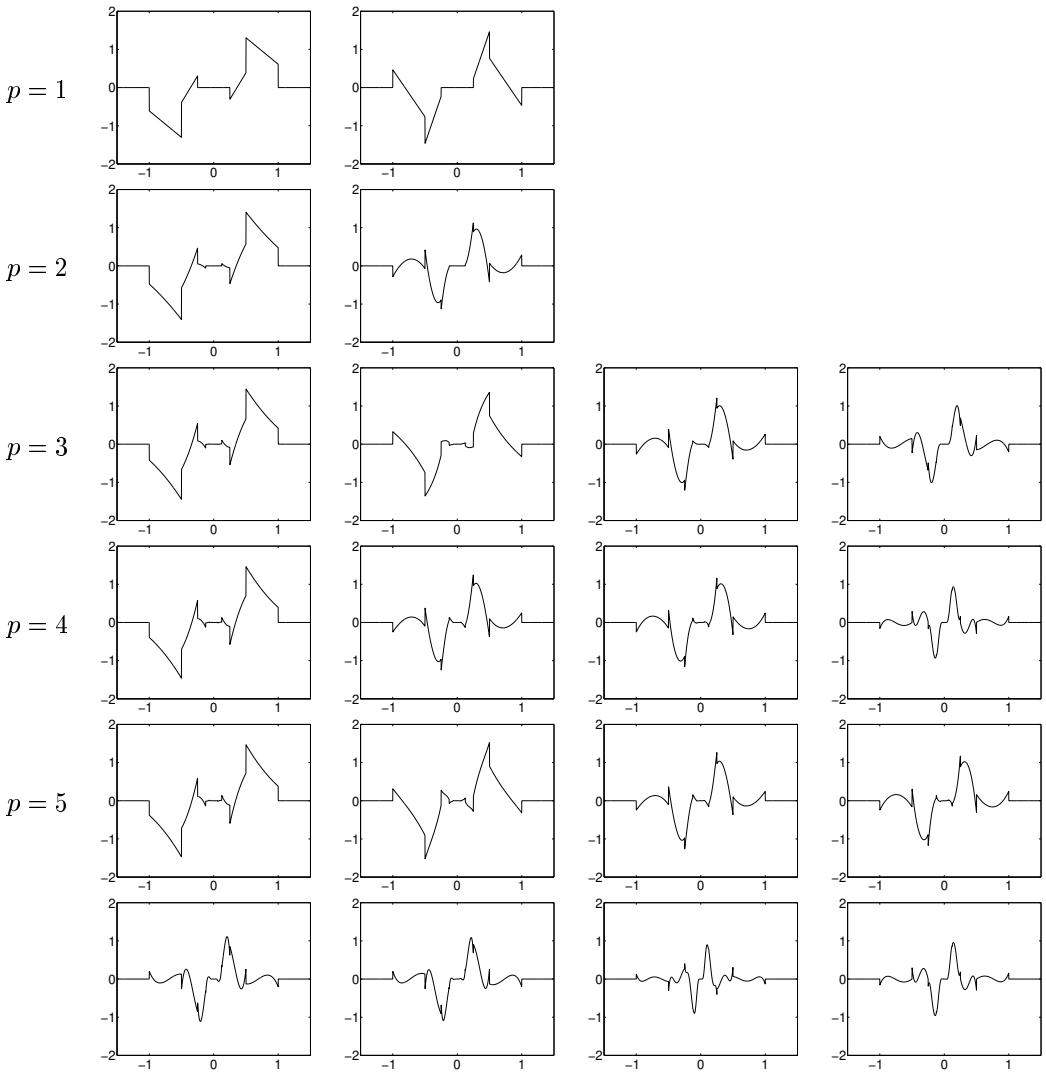


Figure 5: Minimum support wavelet ψ^1 calculated for $1 \leq p \leq 5$ with (68) from the solutions $\{a_k\}_{1 \leq k \leq p}$ given in Table 2, in the same order.

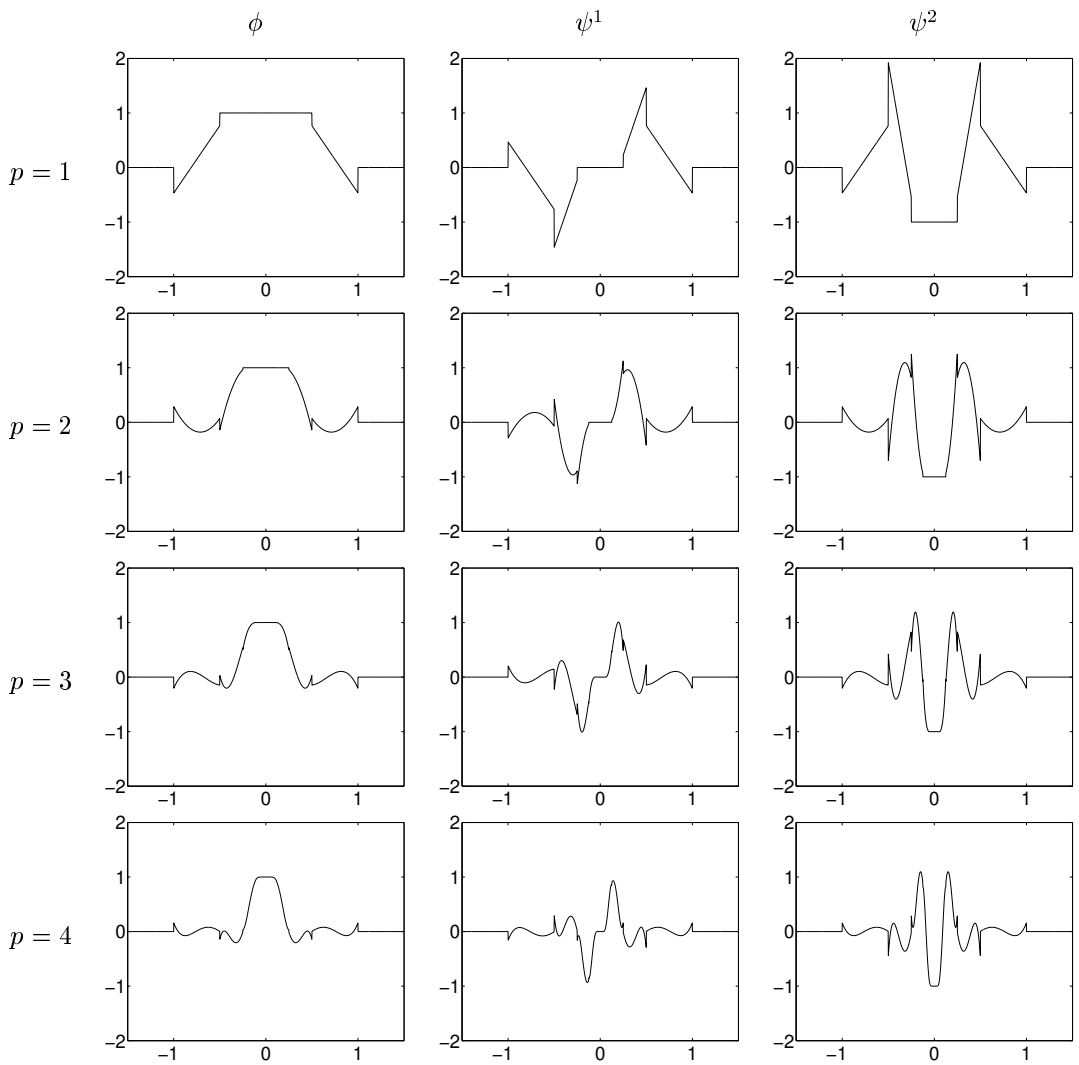


Figure 6: Examples of foveal windows ϕ and foveal wavelets ψ^2 corresponding to a minimum support foveal wavelet ψ^1, selected for $1 \leq p \leq 4$ from the solutions shown in Figure 5.

the system of $p-1$ equations and computing the closest exact solution with a local search. The solution is initialized to $x_{k,p} = a_{k,p-1}$ for $1 \leq k \leq p-1$ and $x_{p,p}$ is computed from these values as a solution of the second degree equation corresponding to the last equation for $k = p$ of the system (67):

$$x_{p,p} \sum_{l=1}^{p} x_{l,p} \frac{1-2^{-p-l-1}}{p+l+1} L_{p,l}(2^{-p-1}) \quad + \quad x_{p,p} \frac{1-2^{-p-1}}{p+1} L_{p,0}(2^{-p-1}) = \\ 1 - 2^{-p-1} + \sum_{l=1}^{p} x_{l,p} \frac{1-2^{-l-1}}{l+1} L_{p,l}(2^{-1}).$$

The vector $(x_{l,p})_{1 \leq l \leq p}$ is then used as an initial point of a Newton search to find an exact solution of the system. This Newton search converges only if the initial point is sufficiently close to a the solution of the system. This is the case in number cases. Using this technique, a solution for the system of degree $p = 20$ has been computed by propagating progressively the solution of the system whose values $(a_{k,p})_{1 \leq l \leq p}$ are given first in Table 2 for $1 \leq p \leq 5$, and for which $a_{1,p} \approx 1.4$. It seems that the system (67) has solutions for any $p \geq 0$ since some solutions such as this first solution does propagate with little perturbations when p increases, but this is only a conjecture.

5.2 Regular Orthogonal Foveal Wavelets

Foveal wavelets of minimum support are discontinuous piecewise polynomial functions. To construct more regular foveal wavelets, it is necessary to extend the support size. We construct orthogonal foveal wavelets ψ^1 whose support are included in $[-1, -2^{-p-2}] \cup [1, 2^{-p-2}]$, which reproduce polynomials of degree p, and which have an arbitrary degree of regularity. The following theorem generalizes Theorem 4 by constructing larger support orthogonal foveal wavelets which are piecewise polynomials plus an orthogonal perturbation which guarantees that the wavelets are regular. For the interval $I = [1/2, 1]$ we denote

$$\langle f \, , \, g \rangle_I = \int_{1/2}^{1} f(t)\, g(t)\, dt \quad \text{and} \quad \|f\|_I^2 = \int_{1/2}^{1} |f(t)|^2 \, dt \; .$$

Theorem 5 *Let ψ^1 be a foveal wavelet having a support included in $[-1, -2^{-p-2}] \cup [2^{-p-2}, 1]$. The family $\{\psi_j^1\}_{j \in \mathbb{Z}}$ is orthogonal and satisfies*

$$\forall t > 0 \quad \sum_{j=-\infty}^{+\infty} 2^{kj}\, \psi^1(2^{-j} t) = a_k\, t^k \quad \textit{with} \;\; a_k \neq 0 \;\; \textit{for} \;\; 0 \leq k \leq p \tag{77}$$

and $a_0 = 1$ if and only if there exists a polynomial $q(t)$ of degree at most p whose moments $\langle q \, , t^k \rangle_I$ on $I = [1/2, 1]$ satisfy for $1 \leq k \leq p$

$$a_k \left(\sum_{l=1}^{p} a_l \frac{1-2^{-k-l-1}}{k+l+1} L_{p,l}(2^{-k-1}) + \frac{1-2^{-k-1}}{k+1} L_{p,0}(2^{-k-1}) - \right. \tag{78}$$

$$\left. \langle q \, , t^k \rangle_I \left(2^{-(p+1)(k+1)} - \sum_{j=0}^{p} 2^{-j(k+1)} L_{p,j}(2^{p+1}) \right) \right) =$$

$$\sum_{l=1}^{p} a_l \frac{1-2^{-l-1}}{l+1} L_{p,l}(2^{-1}) + 1 - 2^{-p-1} + \langle q \, , \, 1 \rangle_I \left(2^{-(p+1)} - \sum_{j=0}^{p} 2^{-j} L_{p,j}(2^{p+1}) \right)$$

and $\eta(t)$ with $p+1$ vanishing moments on $I=[1/2,1]$ such that

$$\sum_{l=0}^{p} w_{0,l}\, a_l\, \langle q\,,t^l\rangle_I = (-1)^p\, 2^{p(p+1)/2}\left(\|q\|_I^2+\|\eta\|_I^2\right)\ . \tag{79}$$

The resulting ψ^1 is defined for $0\leq j\leq p$ by

$$\forall t\in[2^{-j-1},2^{-j})\quad \psi^1(t)=\sum_{l=0}^{p} 2^{jl}\, w_{j,l}\, a_l\, t^l - L_{p,j}(2^{p+1})\left(q(2^jt)+\eta(2^jt)\right) \tag{80}$$

and

$$\forall t\in[2^{-p-2},2^{-p-1})\quad \psi^1(t)=q(2^{p+1}t)+\eta(2^{p+1}t)\ . \tag{81}$$

The wavelet ψ^1 is uniformly Hölder $k+1$ with $k\in\mathbb{N}$ if and only if $\eta(t)$ is uniformly Hölder $k+1$ on $[1/2,1]$ and for $0\leq n\leq k$

$$\frac{d^nq(1/2)}{dt^n}+\frac{d^n\eta(1/2)}{dt^n} \;=\; 0 \tag{82}$$

$$\sum_{l=n}^{p} w_{0,l}\,\frac{l!}{(l-n)!}\,a_l-(-1)^p\,2^{p(p+1)/2}\left(\frac{d^nq(1)}{dt^n}+\frac{d^n\eta(1)}{dt^n}\right) \;=\; 0\ . \tag{83}$$

Proof: For any $0\leq j\leq p+1$ we can write

$$\forall t\in[1/2,1)\ \ ,\ \ \psi^1(2^{-j}t)=q_j(t)+\eta_j(t) \tag{84}$$

where $q_j(t)$ is a polynomial of degree at most p and $\eta_j(t)$ is orthogonal to any polynomial of degree p on $I=[1/2,1]$, which means that it has $p+1$ vanishing moments. We write $q_{p+1}(t)=q(t)$ and $\eta_{p+1}(t)=\eta(t)$. Since ψ^1 has a support included in $[-1,-2^{-p-2}]\cup[2^{-p-2},1]$, the equality (77) restricted to $[1/2,1]$ and projected on the space of polynomials of degree p and on its orthogonal complement yields the following two conditions:

$$\forall t\in[1/2,1)\ \ ,\ \ \sum_{j=0}^{p}2^{kj}\,q_j(t)=a_k\,t^k-2^{k(p+1)}\,q(t)\ \ \text{for } 0\leq k\leq p \tag{85}$$

and

$$\forall t\in[1/2,1)\ \ ,\ \ \sum_{j=0}^{p}2^{kj}\,\eta_j(t)=-2^{k(p+1)}\,\eta(t)\ \ \text{for } 0\leq k\leq p\ . \tag{86}$$

Inverting these linear systems with the inverse $M^{-1}=(w_{j,k})_{0\leq j,k\leq p}$ of the Vandermonde matrix $M=(2^{kj})_{0\leq k,j\leq p}$ gives

$$q_j(t)=\sum_{l=0}^{p} w_{j,l}\,a_l\,t^l-q(t)\,\sum_{l=0}^{p} w_{j,l}\,2^{l(p+1)} \tag{87}$$

and

$$\eta_j(t)=-\eta(t)\,\sum_{l=0}^{p} w_{j,l}\,2^{l(p+1)}\ . \tag{88}$$

Since $w_{j,l}=w_{l,j}$, we derive from (65) that $\sum_{l=0}^{p} w_{j,l}\,2^{l(p+1)}=L_{p,j}(2^{p+1})$. Inserting (87) and (88) in in (84) proves that (80) and (81) are equivalent to (77).

Since ψ^1 has a support included in $[-1,-2^{-p-2}]\cup[2^{-p-2},1]$, the orthogonality of $\{\psi^1_j\}_{j\in\mathbb{Z}}$ is equivalent to impose that $h_1[j]=0$ for $1\leq j\leq p+1$. For $j=p+1$, using the orthogonality of $\eta(t)$ and $\eta_0(t)$ with respect

to polynomials of degree p on $[1/2,1]$, as well as (87) and (88), this condition becomes

$$\begin{aligned} 0 = \int_0^{+\infty} \psi^1(t)\,\psi^1(2^{-p-1}t)\,dt &= \int_{1/2}^1 \psi^1(t)\,\psi^1(2^{-p-1}t)\,dt \\ &= \int_{1/2}^1 q_0(t)\,q(t)\,dt + \int_{1/2}^1 \eta_0(t)\,\eta(t)\,dt \\ &= \int_{1/2}^1 \left(\sum_{l=0}^p w_{0,l}\,a_l\,t^l - L_{p,0}(2^{p+1})\,q(t)\right) q(t)\,dt - L_{p,0}(2^{p+1}) \int_{1/2}^1 |\eta(t)|^2\,dt\ . \end{aligned}$$

This equality can thus be written

$$\sum_{l=0}^p w_{0,l}\,a_l\,\langle q\ ,\ t^l\rangle_I = L_{p,0}(2^{p+1})\left(\|q\|_I^2 + \|\eta\|_I^2\right)\ .$$

With (65), we verify that $L_{p,0}(2^{p+1}) = (-1)^p\,2^{p(p+1)/2}$ from which we get (79).

Knowing that $h_1[p+1] = 0$ and $a_0 = 1$ the orthogonality of $\{\psi_j^1\}_{j\in\mathbb{Z}}$ is equivalent to $h_1[j] = \langle\psi^1\ ,\ \psi_j^1\rangle = 0$ for $0 < |j| \leq p$, and like in (73) we prove that it is the case if and only if

$$a_k \int_0^{+\infty} t^k\,\psi^1(t)\,dt = \int_0^{+\infty} \psi^1(t)\,dt\ . \tag{89}$$

Decomposing the left integral over intervals $[2^{-j-1}, 2^{-j}]$ yields

$$\int_0^{+\infty} t^k\,\psi^1(t)\,dt = \sum_{j=0}^{p+1} \int_{2^{-1}}^1 2^{-j(k+1)}\,t^k\,\psi^1(2^{-j}t)\,dt\ .$$

Inserting (80) and (81) gives

$$\int_0^{+\infty} t^k\,\psi^1(t)\,dt = \sum_{j=0}^p 2^{-j(k+1)} \sum_{l=0}^p w_{j,l}\,a_l\,\frac{1-2^{-k-l-1}}{k+l+1} + \langle q\ ,\ t^k\rangle_I \left(2^{-(p+1)(k+1)} - \sum_{j=0}^p 2^{-j(k+1)}\,L_{p,j}(2^{p+1})\right)\ .$$

Since $\sum_{j=0}^p 2^{-j(k+1)} w_{j,l} = L_{p,l}(2^{-k-1})$ and $L_{p,0}(2^{-1}) = 2 - 2^{-p}$, inserting this equation in the system (89) gives the system (78), which is therefore equivalent to the orthogonality of the $\{\psi_j^1\}_{j\in\mathbb{Z}}$.

Since $\psi^1(t) = q(2^{p+1}t) + \eta(2^{p+1}t)$ over $(2^{-p-2}, 2^{-p-1})$ and $q(t)$ is a polynomial, it is uniformly Hölder $k+1$ on this interval if and only if $\eta(t)$ is uniformly Hölder $k+1$ on $[1/2,1]$. It result from (80) that ψ^1 is uniformly Hölder $k+1$ on each interval $(2^{-j-1}, 2^{-j})$ for $0 \leq j \leq p+1$. Given this property, one can verify that a necessary and sufficient condition to guarantee that ψ^1 is uniformly Hölder $k+1$ on its support is that ψ^1 is $\mathbf{C}^k$ at the junction points 2^{-j} for $1 \leq j \leq p+1$. The conditions (82) and (83) can be rewritten

$$\lim_{\substack{t\to 2^{-p-2}\\ t>2^{-p-2}}} \frac{d^n\psi^1(t)}{dt^n} = \lim_{\substack{t\to 1\\ t<1}} \frac{d^n\psi^1(t)}{dt^n} = 0 \quad \text{for } 0 \leq n \leq k\ .$$

Since $\psi^1(t) = 0$ for $t < 2^{-p-2}$ and for $t > 1$ these conditions are equivalent to imposing that $\psi^1(t)$ is $\mathbf{C}^k$ at $t = 2^{-p-2}$ and at $t = 1$. Let us show that this is sufficient to insure that ψ^1 is $\mathbf{C}^k$ at any 2^{-j} for $0 \leq j \leq p+1$. As a consequence of (77) we have for $0 \leq k \leq p$

$$\forall t \in [3/8, 3/4] \quad \sum_{j=0}^p 2^{kj}\,\psi^1(2^{-j}t) = a_k\,t^k - 2^{-k}\psi^1(2t) - 2^{k(p+1)}\psi^1(2^{-p-1}t)\ .$$

Since $M = (2^{kj})_{0 \leq k,j \leq p}$ is invertible, we can write $\psi^1(2^{-j}t)$ as a linear combination of $\psi^1(2t)$ and $\psi^1(2^{-p-1}t)$. But $\psi^1(t)$ is $\mathbf{C}^k$ at $t = 2^{-p-2}$ and $t = 1$, so $\psi^1(t)$ is also $\mathbf{C}^k$ at 2^{-j-1} for any $0 \leq j \leq p$. ■

If ψ^1 has a minimum support $[-1, -2^{-p-1}] \cup [2^{-p-1}, 1]$ and has a normalized amplitude so that $a_0 = 1$ then Theorem 4 proves that it is entirely characterized by the moment parameters $\{a_k\}_{1 \leq k \leq p}$. Theorem 5 shows that increasing the support introduce new degrees of freedom that depend upon $\psi^1(t)$ on $[2^{-p-2}, 2^{-p-1}]$, which is decomposed in a polynomial part $q(t)$ and an orthogonal complement $\eta(t)$. The p equations (78) specify the p moments $\{\langle q\,,\, t^k\rangle_I\}_{1 \leq k \leq p}$ of q as a function of the $p+1$ variables $\{\langle q\,,\, 1\rangle_I\,,\, a_k\}_{1 \leq k \leq p}$. The polynomial $q(t)$ of degree p is thus entirely defined by these $p+1$ variables, and the condition (79) can therefore be rewritten as a non-linear equation that relates $\|\eta\|_I$ to these $p+1$ variables. Finally, (82) and (83) imposes boundary conditions on the derivatives of $\eta(t)$ to obtain a regular wavelet ψ^1. Since $q(t)$ is a polynomial of degree p, if $k > p$ then for $p < n \leq k$ these conditions are reduced to

$$\frac{d^n \eta(1/2)}{dt^n} = \frac{d^n \eta(1)}{dt^n} = 0\ .$$

The main difficulty is to find values for the $p+1$ variables $\{\langle q\,,\, 1\rangle_I\,,\, a_k\}_{1 \leq k \leq p}$ so that there exists $\eta(t)$ which satisfy all these conditions. The following proposition derives such values from the minimum support wavelets of Theorem 4 and constructs regular orthogonal foveal wavelets.

Proposition 9 *Let ψ^1 be an orthogonal foveal wavelet whose support is in $[-1, -2^{-p-1}] \cup [2^{-p-1}, 1]$, which reproduces polynomials of degree p with constants $\{a_k\}_{1 \leq k \leq p}$, for $p \leq 100$. There exist orthogonal foveal wavelets $\bar{\psi}^1$ of support in $[-1, -2^{-p-2}] \cup [2^{-p-2}, 1]$, which are $\mathbf{C}^\infty$ and which reproduce polynomials of degree p with the same constants $\{a_k\}_{1 \leq k \leq p}$.*

Proof: We first show that there exists $\eta(t)$ with $p+1$ vanishing moments which satisfies (79), (82) and (83) for $k = \infty$ if

$$A = (-1)^p\, 2^{-p(p+1)/2} \sum_{l=0}^{p} w_{0,l}\, a_l\, \langle q\,,\, t^l\rangle_I - \|q\|_I^2 > 0\ . \tag{90}$$

A family of solutions $\eta(t)$ is explicitly constructed. We first adjust the boundaries of η by defining for any $1/8 > \gamma > 0$ a $\mathbf{C}^\infty$ window $g_\gamma(t)$ such that

$$g_\gamma(t) = \begin{cases} 1 & \text{if } t \in [1/2, 1/2+\gamma] \\ 0 & \text{if } t \in [1/2+2\gamma, 1-2\gamma] \\ 1 & \text{if } t \in [1-\gamma, 1] \end{cases}$$

and which is monotonous in the transition bands $[1/2+\gamma, 1/2+2\gamma]$ and $[1-2\gamma, 1-\gamma]$. The function

$$\eta_{0,\gamma}(t) = \begin{cases} -g_\gamma(t)\, q(t) & \text{if } t \in [1/2, 1] \\ g_\gamma(t)\Big(2^{-1} \sum_{l=0}^{p} w_{0,l}\, a_l\, t^l - q(t)\Big) & \text{if } t \in [3/2, 1] \end{cases}$$

is $\mathbf{C}^\infty$ and satisfies (82) and (83). To cancel the first $p+1$ moments of $\eta_{0,\gamma}$ we use a $\mathbf{C}^\infty$ function θ of support in $[0,1]$ with $\int_0^1 \theta(t)\, dt \neq 0$. The k^{th} order derivative $\theta^{(k)}$ has k vanishing moments on $[0,1]$. Observe that the $p+2$ functions

$$\phi_k(t) = \theta^{(k)}\left(4(p+2)\Big(t - \frac{5}{8} - \frac{k}{4(p+2)}\Big)\right) \quad \text{for } 0 \leq k \leq p+1$$

have disjoint supports in $[5/8, 7/8]$. Let us define

$$\eta_{k+1,\gamma}(t) = \eta_{k,\gamma}(t) - \frac{\langle \eta_{k,\gamma}\,,\, t^k \rangle_I}{\langle \phi_k\,,\, t^k \rangle_I}\, \phi_k(t) \quad \text{for } 0 \leq k \leq p-1\ .$$

Clearly $\eta_{p,\gamma}$ has $p+1$ vanishing moments while still satisfying (82) and (83). Since $\lim_{\gamma \to 0} \|\eta_{0,\gamma}\|_I = 0$ we verify that $\lim_{\gamma \to 0} \|\eta_{p,\gamma}\|_I = 0$ and hence that there exists $\gamma > 0$ such that $\|\eta_{p,\gamma}\|_I < A$. The function

$$\eta(t) = \eta_{p,\gamma} - \frac{\sqrt{A - \|\eta_{p,\gamma}\|_I}}{\|\phi_{p+1}\|_I}\, \phi_{p+1}(t)$$

satisfies $\|\eta\|_I = A$ and thus (79), it has $p+1$ vanishing moments and verifies the boundary conditions (82) and (83) for $k = \infty$.

Let us now consider a solution $\{a_k\}_{1 \leq k \leq p}$ of the system (67) of Theorem 4 and which therefore specifies a minimum support foveal wavelet. It remains to prove that there exists $\langle q, 1 \rangle_I$ such that (90) is satisfied, where the moments $\langle q, t^k \rangle$ for $0 < k \leq p$ are defined by the system (78). Inserting the equations (67) in this system yields an equivalent set of equations:

$$a_k \langle q\,, t^k \rangle_I \left(2^{-(p+1)(k+1)} - \sum_{j=0}^{p} 2^{-j(k+1)}\, L_{p,j}(2^{p+1}) \right) = \langle q\,, 1 \rangle_I \left(2^{-(p+1)} - \sum_{j=0}^{p} 2^{-j}\, L_{p,j}(2^{p+1}) \right)\ . \tag{91}$$

To prove that there exists $\langle q\,, 1 \rangle_I$ such that (90) is satisfied, observe that $\|q\|_I^2$ is quadratic functional of its moments $\{\langle q, t^k \rangle\}_{0 \leq k \leq p}$ and it results from (91) that there exists $\lambda > 0$ such that $\|q\|_I^2 = \lambda\, |\langle q\,, 1 \rangle_I|^2$. Inserting this in (90) together with (91) gives an equivalent condition

$$A = \xi_p\, \langle q, 1 \rangle_I - \lambda\, |\langle q, 1 \rangle_I|^2 > 0 \tag{92}$$

with

$$\xi_p = (-1)^p\, 2^{-p(p+1)/2} \sum_{l=0}^{p} w_{0,l}\, \frac{2^{-(p+1)} - \sum_{j=0}^{p} 2^{-j}\, L_{p,j}(2^{p+1})}{2^{-(p+1)(k+1)} - \sum_{j=0}^{p} 2^{-j(k+1)}\, L_{p,j}(2^{p+1})}\ . \tag{93}$$

If $\xi_p \neq 0$ then (92) is satisfied for $\langle q, 1 \rangle_I = \eta\, \xi_p$ with $1/\eta > \lambda$. This is verified numerically for $p \leq 100$. ■

This proposition constructs $\mathbf{C}^\infty$ orthogonal foveal wavelets that reproduce polynomials, by adding a small perturbation to the minimum support wavelet, which cancels the discontinuities of its derivatives at the points 2^{-j} for $0 \leq j \leq p$. Such a small perturbation exists if the constant ξ_p in (93) is non-zero. This was verified numerically for $p \leq 100$ but we conjecture that this is true for any $p \neq 0$. Although all derivatives are continuous, they may have relatively brutal variations at the transition points 2^{-j}. To construct orthogonal foveal wavelets whose derivatives vary more smoothly, in the following, we minimize a Sobolev semi-norm of degree $k > 0$:

$$S_k(\psi^1) = \int_{-\infty}^{+\infty} \left| \frac{d^k \psi^1(t)}{dt^k} \right|^2 dt\ . \tag{94}$$

To remove a singularity that is Hölder $\alpha < p+1$ and obtain a residual which is Hölder $p+1$, besides the reproduction of polynomials of degree p, Theorem 8 requires ψ^1 to be uniformly Hölder $p+1$. Let $\mathcal{F}_p$ be the set of all foveal wavelets ψ^1 having a support in $[-1, -2^{-p-2}] \cup [2^{-p-2}, 1]$, which are uniformly Hölder $p+1$, which reproduce polynomials of degree p and yield an orthogonal family $\{\psi_j^1\}_{j \in \mathbb{Z}}$. Theorem 5 characterizes in (80) and (81) any such wavelets with a polynomial $q(t)$ and a perturbation $\eta(t)$. The polynomial $q(t)$ is itself defined by the $p+1$ parameters $\{\langle q\,, 1 \rangle_I\,, a_k\}_{1 \leq k \leq p}$ through the system (78). To compute a wavelet ψ^1 in $\mathcal{F}_p$

which minimizes the Sobolev semi-norm of degree $k = p+1$, the following proposition gives a parameterized expression of $\eta(t)$.

Proposition 10 *Let $\psi^1 \in \mathcal{F}_p$ be a foveal wavelet corresponding to $\{\langle q\,,\,1\rangle_I\,,\,a_k\}_{1\leq k\leq p}$ and a perturbation $\eta(t)$. For any $\eta(t)$ which minimizes $S_{p+1}(\psi^1)$ there exist $\lambda \geq 0$, $(\mu_l)_{0\leq l\leq p}$ and $(d_l)_{0\leq l\leq p}$ such that*

$$\eta(t) = \sum_{l=0}^{p} \mu_l\, t^l + \sum_{l=1}^{2p+2} d_l\, \mathrm{e}^{r_l\, t} \tag{95}$$

with $r_l^{2p+2} = (-1)^{p+1}\,\lambda$ for $1 \leq l \leq 2p+2$. Moreover

$$S_{p+1}(\psi^1) = \left(\sum_{j=0}^{p+1} 2^{-j+1}\, L_{p,j}(2^{p+1})\right) \int_{1/2}^{1} \left|\sum_{l=1}^{2p+2} d_l\, r_l^{p+1}\, \mathrm{e}^{r_l\, t}\right|^2 dt\ . \tag{96}$$

Proof: Dividing the Sobolev integral (94) over the dyadic intervals $[2^{-j-1}, 2^{-j}]$ and inserting (80) and (81) together with the fact that ψ^1 is odd gives

$$S_{p+1}(\psi^1) = \int_{-\infty}^{+\infty} \left|\frac{d^{p+1}\psi^1(t)}{dt^{p+1}}\right|^2 dt = 2\sum_{j=0}^{p+1} 2^{-j}\, L_{p,j}(2^{p+1}) \int_{1/2}^{1} \left|\frac{d^{p+1}\eta(t)}{dt^{p+1}}\right|^2 dt\ . \tag{97}$$

Minimizing $S_{p+1}(\psi^1)$ is thus equivalent to minimizing $\int_{1/2}^{1} \left|\frac{d^{p+1}\eta(t)}{dt^{p+1}}\right|^2 dt$. Theorem 5 imposes that $\eta(t)$ has $p+1$ vanishing moments

$$\int_{1/2}^{1} \eta(t)\, t^l\, dt = 0 \quad \text{for} \quad 0 \leq l \leq p \tag{98}$$

that it satisfies the boundary conditions (82) and (83) as well as (79):

$$\int_{1/2}^{1} |\eta(t)|^2\, dt = \sum_{l=0}^{p} (-1)^p\, 2^{-p(p+1)/2}\, w_{0,l}\, a_l \int_{1/2}^{1} q(t)\, t^l\, dt - \int_{1/2}^{1} |q(t)|^2\, dt = C\ . \tag{99}$$

If $\{\langle q\,,\,1\rangle_I\,,\,a_k\}_{1\leq k\leq p}$ and hence $q(t)$ correspond to $\psi^1 \in \mathcal{F}_p$ then the right hand-side of (99) is positive. A solution of the minimization of $S_{p+1}(\psi^1)$ together with all these constraints is a saddle point of the Lagrangian

$$\mathcal{L}\Big(\eta, \lambda, \{\lambda_l\}_{0\leq l\leq p}\Big) = \int_{1/2}^{1} \left|\frac{d^{p+1}\eta(t)}{dt^{p+1}}\right|^2 dt + \lambda\Big(C - \int_{1/2}^{1} |\eta(t)|^2\, dt\Big) + \sum_{l=0}^{p} \lambda_l \int_{1/2}^{1} \eta(t)\, t^l\, dt\ ,$$

where λ is a positive Lagrange multiplier. A saddle point satisfies

$$(-1)^{p+1}\, \frac{d^{2p+2}\eta(t)}{dt^{2p+2}} - 2\lambda\, \eta(t) + \sum_{l=0}^{p} \lambda_l\, t^l = 0 \tag{100}$$

with the boundary conditions (82) and (83), where λ and the λ_l are Lagrange multipliers associated to the constraints (98) and (99). The solution $\eta(t)$ can therefore be written as in (95). Inserting this in (97) yields (96). ■

Let us now explain how to reduce the global minimization of $S_{p+1}(\psi^1)$ over $\mathcal{F}_p$ to an optimization of the $p+1$ parameters $\{\langle q\,,\,1\rangle_I\,,\,a_k\}_{1\leq k\leq p}$. This optimization is performed under the constraint that the Lagrange multiplier λ of $\eta(t)$ takes a prescribed value, which specifies the roots r_l in (95). Since η has $p+1$ vanishing moments

$$\langle \eta\,,\,t^l\rangle_I = \int_{1/2}^{1} \eta(t)\, t^l\, dt = 0 \quad \text{for} \quad 0 \leq l \leq p\ .$$

Together with the boundary conditions (82) and (83), we get a system of $3(p+1)$ linear equations, from which we compute the $3(p+1)$ constants $\{\mu_l\}_{0\leq l\leq p}$ and $\{d_l\}_{1\leq l\leq 2p+2}$ of $\eta(t)$ as a function of our $p+1$ parameters $\{\langle q\,,\,1\rangle_I\,,\,a_k\}_{1\leq k\leq p}$. The condition (79) on $\eta(t)$ can now be rewritten as a rational equation that relates these $p+1$ parameters. The Sobolev semi-norm $S_{p+1}(\psi^1)$ in (96) is also a rational function of these parameters. The minimization of $S_{p+1}(\psi^1)$ under the constraint (79) can therefore be written as the solution of a system of polynomial equations. The real solutions of this system are computed and we keep the one that minimizes $S_{p+1}(\psi^1)$. This minimum is a function of the initial parameter $\lambda \geq 0$. A global minimum is obtained by finding the λ which yields the smallest minimum. The overall computation, thus involves a loop over the parameter λ and for each λ a calculation of the solutions of a system of $p+1$ polynomial equations. All calculations have been performed by LePennec and Schost [12] with the package Kronecker in Magma [8].

For $p=1$, for each λ, the minimization of $S_2(\psi^1)$ is obtained through a system of 2 equations of degrees up to 7 with 2 unknown that yields 32 solutions. Among these solutions the one that minimizes $S_2(\psi^1)$ is selected, and a minimization is then performed over λ. For $p=1$, the resulting ψ^1 is shown in Figure 7 with the corresponding ϕ and ψ^2. Observe that ϕ is positive, which is a useful property used by Theorem 6 for the characterization pointwise Hölder exponent from the asymptotic decay of foveal coefficients at fine scales. For $p=2$, for a fixed λ the minimization of $S_3(\psi^1)$ yields a system of 3 equations of degrees up to 12 with 3 unknown, which can have up to 232 real solutions. The calculation of these solutions is computationally highly intensive [12]. Figure 7 shows first a solution corresponding to the one obtained with $p=1$, which also leads to a positive $\phi(t)$. For $p=2$, there exists another foveal wavelet shown at the bottom of Figure 7, whose Sobolev semi-norm $S_3(\psi^1)$ is smaller. However, the resulting ϕ changes sign. The parameters $\{\langle q\,,\,1\rangle_I\,,\,a_k\}_{1\leq k\leq p}$ corresponding to all these foveal wavelets are in Table 3.

	$\langle q\,,\,1\rangle_I$	a_1	a_2
$p=1$	-.030988103	1.594961259113511617 26	
$p=2$	-0.0099565418	4.1928848382905952295497	17.586488446503838048
$p=2$	0.00613082569	1.6588558551518189845313	2.6682731679354740948

Table 3: Parameters $\{\langle q\,,\,1\rangle_I\,,\,a_k\}_{1\leq k\leq p}$ corresponding to the optimized foveal wavelets shown in Figure 7.

6 Singularities

Foveal wavelets are particularly well adapted to detect singularities of non-oscillatory functions. When centering the foveal wavelets ψ^1 and ψ^2 on an isolated singularity, Section 6.1 shows that the foveal wavelet coefficients can characterize its Hölder regularity. A procedure is derived to detect non-oscillatory singularities from discrete signals. Section 6.2 proves that if the pair of foveal wavelets reproduce polynomials of sufficiently high order then transition singularities are reconstructed from foveal coefficients with a small residual error, like in Proposition 2 for translated Daubechies wavelets.

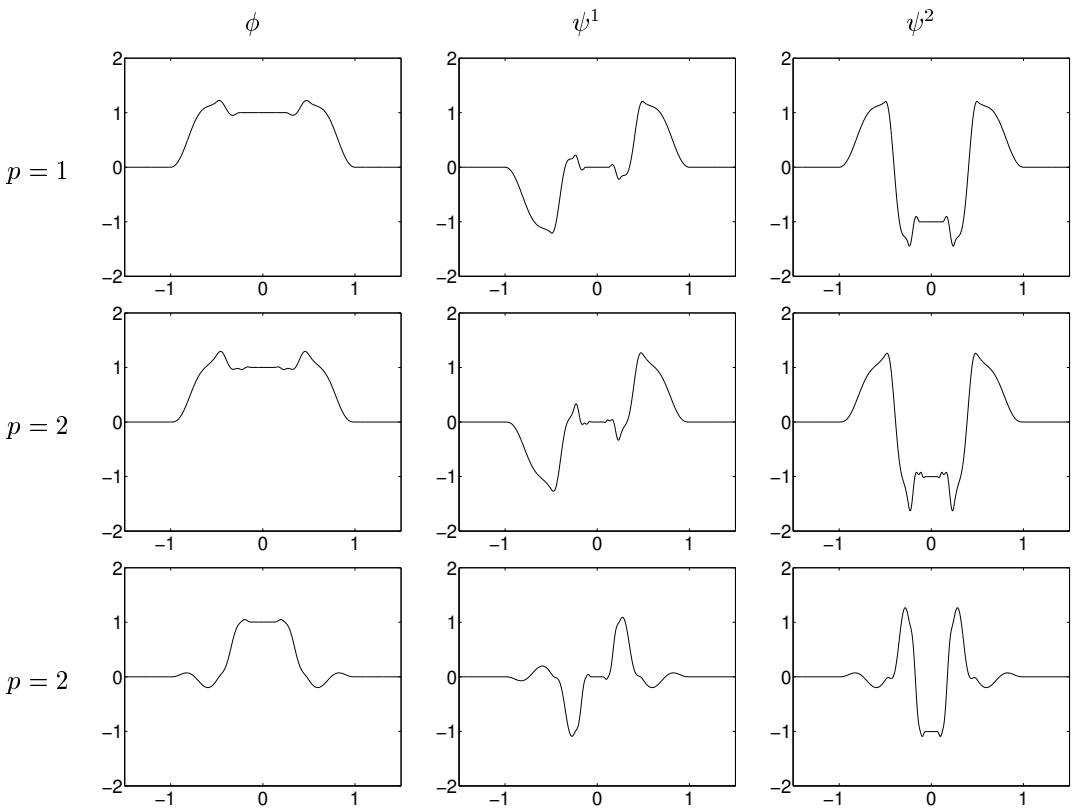


Figure 7: Foveal wavelets that yield an orthogonal basis and reproduce polynomials of degree p, calculated by minimizing the Sobolev semi-norm $S_{p+1}(\psi^1)$.

6.1 Detection and Characterization of Hölder Exponents

A function f is pointwise Hölder $\alpha \geq 0$ at u, if there exist $K > 0$ and a polynomial $q_u(t)$ of degree $m = \lfloor \alpha \rfloor$ such that

$$\forall t \in \mathbb{R} \quad , \quad |f(t) - q_u(t)| \leq K\,|t-u|^{\alpha} \ . \tag{101}$$

If $0 \leq \alpha \leq 1$ then $q_u(t) = f(u)$ so (101) becomes

$$\forall t \in \mathbb{R} \quad , \quad |f(t) - f(u)| \leq K\,|t-u|^{\alpha} \ .$$

We say that f is singular at u if it is not Hölder 1 at u.

It is well known that wavelet coefficients can characterize the pointwise Hölder regularity of a function f. Let $\psi(t)$ be a wavelet with one vanishing moment and $\psi_{j,u}(t) = 2^{-j/2}\psi(2^{-j}(t-u))$. If f is Hölder $\alpha \leq 1$ at u then [11]

$$|\langle f, \psi_{j,u}\rangle| = O(2^{j(\alpha+1/2)}) \ . \tag{102}$$

This necessary condition is however not sufficient. In general, to control the Hölder regularity of f at u, one must control the amplitude of wavelet coefficients $|\langle f, \psi_{j,v}\rangle|$ for v in a whole neighborhood of u. However, for a pair of foveal wavelets constructed with a positive window, if f is not oscillatory in the neighborhood of u, the following theorem proves that it is sufficient to control the amplitude of wavelet coefficients at $v = u$, which simplifies considerably the detection of singularities.

Theorem 6 *Let ψ^1 and ψ^2 be foveal wavelets constructed from a continuous and compactly supported foveal window $\phi \geq 0$ with $\phi(0) > 0$. Suppose there exists $\epsilon > 0$ such that f is monotonous on $[u-\epsilon, u)$ and $(u, u+\epsilon]$, and that f is bounded. If there exists $0 < \alpha \leq 1$ such that*

$$|\langle f, \psi^1_{j,u}\rangle| = O(2^{j(\alpha+1/2)}) \quad \textit{and} \quad |\langle f, \psi^2_{j,u}\rangle| = O(2^{j(\alpha+1/2)}) \tag{103}$$

then f is Hölder α at u.

Proof Since $f(t)$ is monotonous in a left and right neighborhood of u, the following limits exist:

$$\lim_{\substack{t \to u \\ t<u}} f(t) = f(u^-) \quad \text{and} \quad \lim_{\substack{t \to u \\ t>u}} f(t) = f(u^+) \ .$$

Let us first prove that $f(t)$ is continuous at $t = u$ by verifying that $f(u^-) = f(u^+) < +\infty$. We know that

$$\int_0^{+\infty} \psi^1(t)dt = \frac{1}{2}\int_{-\infty}^{+\infty} \phi(t)\,dt \ .$$

Since ϕ has a compact support, $\psi^1(t) = \bar{\phi}(t) - \bar{\phi}(2t)$ also has a compact support and using the monotonicity of f on each side of u we get

$$\lim_{j \to -\infty} 2^{-j/2}|\langle f, \psi^1_{j,u}\rangle| = \frac{f(u^+) - f(u^-)}{2} \int_{-\infty}^{+\infty} \phi(t)\,dt \ .$$

Since $|\langle f, \psi^1_{j,u}\rangle| = O(2^{j(\alpha+1/2)})$ it results that $f(u^+) = f(u^-)$. We now verify that $f(u^+) = f(u^-) < +\infty$. Since $\psi^2(t) = \phi(t) - 2\phi(2t)$, for any $l < J$

$$\sum_{j=l}^{J} 2^{-j/2}\psi^2_{j,u}(t) = 2^{-J}\phi(2^{-J}(t-u)) - 2^{-l+1}\,\phi(2^{-l+1}(t-u)) \ .$$

and hence

$$\sum_{j=l}^{J} 2^{-j/2} \langle f\ ,\ \psi^2_{j,u} \rangle = \int_{-\infty}^{+\infty} 2^{-J} \phi(2^{-J}(t-u))\, f(t)\, dt - \int_{-\infty}^{+\infty} 2^{-l+1} \phi(2^{-l+1}(t-u))\, f(t)\, dt\ . \tag{104}$$

Since $|\langle f, \psi^2_{j,u} \rangle| = O(2^{j(\alpha+1/2)})$ with $\alpha > 0$, the left hand-side sum converges when l goes to $-\infty$ and hence

$$\lim_{l \to -\infty} \int_{-\infty}^{+\infty} 2^{-l}\, \phi(2^{-l}(t-u))\, f(t)\, dt = \frac{f(u^+) + f(u^-)}{2} \int_{-\infty}^{+\infty} \phi(t)\, dt < +\infty\ ,$$

so $f(u^+) = f(u^-) = f(u) < +\infty$.

We now prove that f is Hölder α at u. Letting l go to $-\infty$ in (104) together with $|\langle f, \psi^2_{j,u} \rangle| = O(2^{j(\alpha+1/2)})$ proves that

$$\left| \int_{-\infty}^{+\infty} 2^{-J} \phi(2^{-J}(t-u)) f(t)\, dt - f(u) \int_{-\infty}^{+\infty} \phi(t) dt \right| = O(2^{\alpha J})$$

and hence

$$\left| \int_{-\infty}^{+\infty} \phi(2^{-J}(t-u))(f(t) - f(u))\, dt \right| = O(2^{(\alpha+1)J})\ . \tag{105}$$

For any $l < J$

$$\sum_{j=l}^{J} 2^{j/2} \psi^1_{j,u}(t) = \bar{\phi}(2^{-J}(t-u)) - \bar{\phi}(2^{-l+1}(t-u))$$

so

$$\sum_{j=l}^{J} 2^{j/2}\, \langle f, \psi^1_{j,u} \rangle = \int_{-\infty}^{+\infty} f(t)\, \bar{\phi}(2^{-J}(t-u))\, dt - \int_{-\infty}^{+\infty} f(t)\, \bar{\phi}(2^{-l+1}(t-u))\, dt\ .$$

Since ϕ has a compact support and $f(t)$ is continuous at u we get

$$\sum_{j=-\infty}^{J} 2^{j/2}\, \langle f, \psi^1_{j,u} \rangle = \int_{-\infty}^{+\infty} f(t)\, \bar{\phi}(2^{-J}(t-u))\, dt\ .$$

Since $\int_{-\infty}^{+\infty} \bar{\phi}(2^{-j}(t-u))\, dt = 0$ and $|\langle f, \psi^1_{j,u} \rangle| = O(2^{j(\alpha+1/2)})$ we derive that

$$\left| \int_{-\infty}^{+\infty} (f(t) - f(u))\, \bar{\phi}(2^{-J}(t-u))\, dt \right| = O(2^{(\alpha+1)J})\ . \tag{106}$$

Putting together (105) and (106), and the fact that $\phi \geq 0$ and that f is monotonous in the left and right neighborhoods of u, we derive that for 2^J sufficiently small

$$\int_{-\infty}^{u} \phi(2^{-J}(t-u))\, |f(t) - f(u)|\, dt = O(2^{(\alpha+1)J}) \tag{107}$$

and

$$\int_{u}^{+\infty} \phi(2^{-J}(t-u))\, |f(t) - f(u)|\, dt = O(2^{(\alpha+1)J})\ . \tag{108}$$

Since $\phi(t)$ is continuous and $\phi(0) > 0$, there exists $\gamma > 0$ and $\beta > 0$ such that $\phi(t) > \beta$ for $|t| \leq \gamma$. Hence

$$\int_{u-\gamma 2^J}^{u} |f(t) - f(u)|\, dt = O(2^{(\alpha+1)J}) \tag{109}$$

and

$$\int_{u}^{u+\gamma 2^J} |f(t) - f(u)|\, dt = O(2^{(\alpha+1)J})\ . \tag{110}$$

The function f is Hölder α at u if there exists $K > 0$ such that $|f(t) - f(u)| \leq K|t-u|^\alpha$. Suppose that it is not the case, say in a right neighborhood of u. Since f is monotonous in a right neighborhood of u, then for any $K > 0$ there exists a sequence of integers $\{j_n\}_{n\in\mathbb{N}}$ with $\lim_{n\to+\infty} j_n = -\infty$ such that

$$|f(u+\gamma 2^{j_n}) - f(u)| \geq K\,(\gamma\, 2^{j_n})^\alpha \ .$$

For $t \in [u+\gamma 2^{j_n}\ ,\ u+\gamma 2^{j_n+1}]$, the monotonicity implies that

$$|f(t) - f(u)| \geq |f(u+\gamma 2^{j_n}) - f(u)|$$

hence

$$\int_u^{u+\gamma 2^{j_n+1}} |f(t)-f(u)|\,dt \geq \int_{u+\gamma 2^{j_n}}^{u+\gamma 2^{j_n+1}} |f(t)-f(u)|\,dt \geq K\,(\gamma\, 2^{j_n})^{\alpha+1} \ .$$

But since this can be obtained for any $K > 0$, it contradicts (110). The same applies in the left neighborhood of u. So there exists a neighborhood of u for which $|f(t) - f(u)| \leq K|t-u|^\alpha$. Since f is bounded this can be extended for any $t \in \mathbb{R}$ for K sufficiently large, so f is Hölder α at u. ■

We say that f is non-oscillating if it is monotonous in a left and right neighborhood of any $u \in \mathbb{R}$. Together with the necessary condition (102), this theorem proves that the Hölder regularity of a non-oscillating f at u is characterized by the decay of its two foveal wavelet coefficients at u. The condition (103) is necessary and sufficient to prove that f is Hölder $\alpha < 1$ at u. Singularities can thus be detected by computing $|\langle f, \psi^1_{j,u}\rangle|$ together with $|\langle f, \psi^2_{j,u}\rangle|$, and measuring their decay across scales for all $u \in \mathbb{R}$.

Non-oscillating singularities can be detected by computing at each location u a *foveal energy* defined by:

$$e(u) = \lim_{l\to-\infty} \frac{1}{J-l} \sum_{j=l+1}^{J} 2^{-3j}\left(|\langle f, \psi^1_{j,u}\rangle|^2 + |\langle f, \psi^2_{j,u}\rangle|^2\right) \ ,$$

where 2^J is a fixed maximum scale. The following proposition gives a necessary and a sufficient condition on this energy to detect a singularity at u

Proposition 11 *Let ψ^1 and ψ^2 be foveal wavelets constructed from a continuous and compactly supported foveal window $\phi \geq 0$ with $\phi(0) > 0$.*

- *If f is Hölder 1 at u then $e(u) < +\infty$.*
- *Suppose that f is non-oscillating. If f is not Hölder alpha < 1 at u then $e(u) = +\infty$.*

Proof: The necessary condition (102) proves that if f is Hölder 1 at u then $|\langle f, \psi^k_{j,u}\rangle| = O(2^{3j/2})$ for $k = 1, 2$ so $e(u) < \infty$. Conversely if f is non-oscillating and is not Hölder $\alpha < 1$ at u then (103) proves that we do not have $|\langle f, \psi^k_{j,u}\rangle| = O(2^{(\alpha+1/2)j})$ for $k = 1, 2$. It results that $e(u) = +\infty$. ■

For discrete signals measured at a finite resolution 2^{-l}, foveal wavelet coefficient can only be calculated up to the finest scale 2^{l+1} and the resulting energy is thus defined by:

$$e(u) = \frac{1}{J-l} \sum_{j=l+1}^{J} 2^{-3j}\left(|\langle f, \psi^1_{j,u}\rangle|^2 + |\langle f, \psi^2_{j,u}\rangle|^2\right) \ . \tag{111}$$

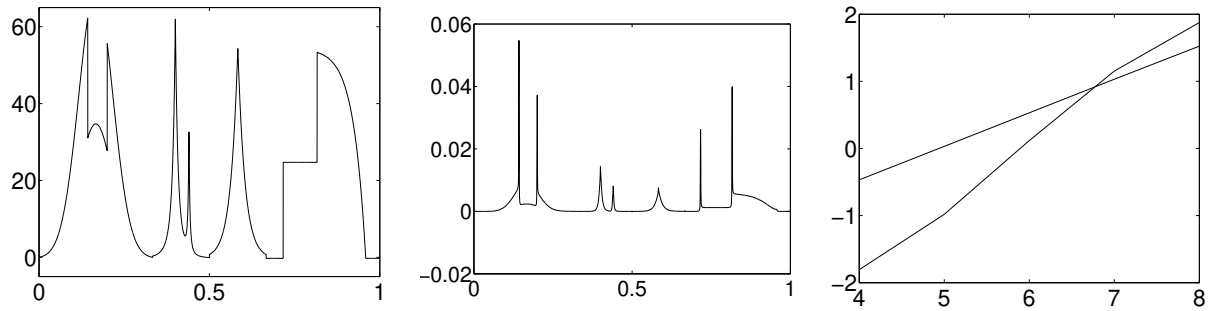
Since this sum is finite, the amplitude of $e(u)$ remains finite at singularity locations. Similarly to the wavelet maxima approach introduced in [9], potential singularities are detected by finding the local maxima of $e(u)$. As

opposed to the wavelet maxima algorithm described in [9], we do not need to detect and follow local maxima across scales because the singularity behavior at u is characterized directly by the overall foveal energy $e(u)$.

At any local maxima location, the Hölder regularity of f is estimated by computing

$$a_j(u) = \frac{1}{2} \log_2\Big(|\langle f, \psi^1_{j,u}\rangle|^2 + |\langle f, \psi^2_{j,u}\rangle|^2 \Big) \ . \tag{112}$$

If f is non-oscillating, Theorem 6 proves that f is Hölder α at u if and only if $a_j(u) = O((\alpha + 1/2)j)$. The Hölder regularity at u is thus calculated with a linear regression of $a_j(u)$ as a function of j. Figure 8 gives an example of signal and the corresponding foveal energy $e(u)$. There are 8 local maxima located at $0.14, 0.2, 0.4, 0.44, 0.58, 0.72, 0.82$ and 0.96. Figure 8c shows that at $u = 0.58$ and $u = 0.72$ we have $a_j(u) \approx (\alpha + 1/2)\, j + \beta$ with $\alpha = 0$ at $u = 0.72$ which indicates that the signal is discontinuous at the location, and $\alpha = 1/2$ at $u = 0.58$.



(a) (b) (c)

Figure 8: (a): Original signal. (b): Foveal energy $e(u)$ calculated with the foveal wavelets derived from $\phi \geq 0$ shown in Figure 7 ($p = 1$). (c): Variations of $a_j(u)$ defined in (112) as a function of j, at $u = 0.58$ and $u = 0.72$.

When f has oscillations that have an accumulation point then (103) is not a sufficient condition. For example, $f(t) = \sin(1/t)$ has such an accumulation point at $u = 0$. One can then verify that although f is discontinuous at $u = 0$, if ψ^1 and ψ^2 are differentiable then

$$|\langle f, \psi^1_{j,u}\rangle| = O(2^{j(2+1/2)}) \quad \text{and} \quad |\langle f, \psi^2_{j,u}\rangle| = O(2^{j(2+1/2)}) \ .$$

The analysis of the Hölder regularity of oscillating functions with wavelets can be found in [10], but let us mention that our definition of non-oscillation is stronger that the one of Jaffard and Meyer [10].

6.2 Reconstruction of Singularities

Transition singularities occur at points u between regions where f is regular. This section studies the precision of a foveal approximation and gives conditions on the foveal wavelets to obtain a uniformly regular approximation error. Uniformly regular functions are well approximated with linear projectors in standard bases such as Fourier or wavelet bases. One can thus define a double layer approximation of f which first recovers the singular parts with foveal wavelets and then represents efficiently the foveal approximation error with a standard linear approximation scheme.

Suppose that f is regular on the left and right of u but not at u where it is can be discontinuous. To characterize the singularity at u, we introduce a definition of left and right Hölder regularity at a point. A function f is said to be Hölder α on the left and on the right of u if there exists two polynomials $q_u^-(t)$ and $q_u^+(t)$ of degree $m \leq \lfloor \alpha \rfloor$ and a constant K such that

$$\forall t < u \quad , \quad |f(t) - q_u^-(t)| \leq K\,|t-u|^\alpha \tag{113}$$

and

$$\forall t > u \quad , \quad |f(t) - q_u^+(t)| \leq K\,|t-u|^\alpha \ . \tag{114}$$

The following theorem proves that if polynomials of sufficiently high degree are reproduced by foveal wavelets then the singularity is reconstructed by projecting f in the foveal space at u.

Theorem 7 *Let $\{\psi_j^{1-}\,,\,\psi^{1+}\}_{j\in\mathbb{Z}}$ be an orthogonal basis of a foveal space $\mathbf{V}_0$ with*

$$\forall t > 0 \quad \sum_{j=-\infty}^{+\infty} 2^{kj}\,\psi^1(2^{-j}t) = a_k\,t^k \quad \textit{with } a_k \neq 0 \textit{ for } 0 \leq k \leq p\ . \tag{115}$$

Suppose that $|\psi^1(t)| = O(|t|^\beta)$ for $|t| \leq 1$ and $|\psi^1(t)| = O(|t|^{-\beta-1})$ for $|t| \geq 1$. If f is Hölder α on the left and on the right of u with $\alpha \leq p+1$ and $\alpha < \beta$ then

$$\forall t \in \mathbb{R} \quad , \quad |f(t) - P_{\mathbf{V}_u} f(t)| = O(|t-u|^\alpha)\ . \tag{116}$$

Proof: The translated family $\{\psi_{j,u}^{1-}\ \psi_{j,u}^{1+}\}_{j\in\mathbb{Z}}$ is an orthonormal basis of $\mathbf{V}_u$ so

$$P_{\mathbf{V}_u} f = \sum_{j=-\infty}^{+\infty} \Big(\langle f\,,\,\psi_{j,u}^{1-}\rangle\,\psi_{j,u}^{1-} + \langle f\,,\,\psi_{j,u}^{1+}\rangle\,\psi_{j,u}^{1+} \Big)\ .$$

Since f is left and right Hölder α at u, there exists two polynomials $q_u^-(t)$ and $q_u^+(t)$ of degree $m < \lfloor \alpha \rfloor$ such that

$$\forall t < u \quad , \quad f(t) = q_u^-(t) + \epsilon_u^-(t) \quad \text{and} \quad \forall t > u \quad , \quad f(t) = q_u^+(t) + \epsilon_u^+(t)$$

with

$$\forall t < u \quad |\epsilon_u^-(t)| \leq K\,|t-u|^\alpha \ \text{ and } \ \forall t > u \quad |\epsilon_u^+(t)| \leq K\,|t-u|^\alpha\ . \tag{117}$$

If $q(t)$ is a polynomial of degree $m < \beta$ then we derive from (115) that it can be decomposed into

$$q(t) = \sum_{j=-\infty}^{+\infty} \Big(a^-[j]\,\psi_{j,u}^{1-}(t) + a^+[j]\,\psi_{j,u}^{1+}(t) \Big)$$

and from the orthogonality of $\{\psi_j^{1-}\,,\,\bar{\psi}^{1+}\}_{j\in\mathbb{Z}}$ we get

$$q = \sum_{j=-\infty}^{+\infty} \Big(\langle q\,,\,\psi_{j,u}^{1-}\rangle\,\psi_{j,u}^{1-} + \langle q\,,\,\psi_{j,u}^{1+}\rangle\,\psi_{j,u}^{1+} \Big)\ . \tag{118}$$

For $t < u$

$$P_{\mathbf{V}_u} f(t) = \sum_{j=-\infty}^{+\infty} \langle f\,,\,\psi_{j,u}^{1-}\rangle\,\psi_{j,u}^{1-}(t)\ .$$

Inserting $\langle f\,,\,\psi^{1-}_{j,u}\rangle = \langle q^-_u\,,\,\psi^{1-}_{j,u}\rangle + \langle \epsilon^-_u\,,\,\psi^{1-}_{j,u}\rangle$ in this equation together with (118) yields

$$\forall t < u \quad , \quad P_{\mathbf{V}_u} f(t) = q^-_u(t) + \sum_{j=-\infty}^{+\infty} \langle \epsilon^-_u\,,\,\psi^{1-}_{j,u}\rangle\, \psi^{1-}_{j,u}(t) \tag{119}$$

so

$$\forall t < u \quad , \quad |P_{\mathbf{V}_u} f(t) - f(t)| = |\epsilon^-_u(t) - \sum_{j=-\infty}^{+\infty} \langle \epsilon^-_u\,,\,\psi^{1-}_{j,u}\rangle\, \psi^{1-}_{j,u}(t)| \;.$$

Since $|\epsilon^-_u(t)| \leq K\,|t-u|^\alpha$, with a change of variable we get

$$\begin{aligned} |\langle \epsilon^-_u\,,\,\psi^{1-}_{j,u}\rangle| &\leq K \int |t-u|^\alpha\, 2^{-j/2}\, |\psi^{1-}(2^{-j}(t-u))|\, dt \\ &\leq K'\, 2^{(\alpha+1/2)j} \;. \end{aligned}$$

We know that there exist $C_1, C_2 > 0$ such that $|\psi^1(t)| \leq C_1$ and $|\psi^1(t)| \leq C_2\,|t|^\beta$, and since $\alpha < \beta$

$$\begin{aligned} \left| \sum_{j=-\infty}^{+\infty} \langle \epsilon^-_u\,,\,\psi^{1-}_{j,u}\rangle\, \psi^{1-}_{j,u}(t) \right| &\leq \sum_{j=-\infty}^{\lfloor \log_2 |t-u| \rfloor} K' 2^{(\alpha+1/2)j}\, C_1\, 2^{-j/2} \\ &\quad + \sum_{j=\lceil \log_2 |t-u| \rceil}^{+\infty} K' 2^{(\alpha+1/2)j}\, C_2\, 2^{-j/2} |2^{-j}(t-u)|^\beta \\ &\leq K''\, |t-u|^\alpha \;. \end{aligned} \tag{120}$$

So $|P_{\mathbf{V}_u} f(t) - f(t)| = O(|t-u|^\alpha)$ for $t < u$. Similarly we prove that $|P_{\mathbf{V}_u} f(t) - f(t)| = O(|t-u|^\alpha)$ for $t \geq u$.

■

This theorem gives conditions so that the projection of f over a foveal space $\mathbf{V}_u$ reconstructs the singularity at u, up to a small residual error that depends upon the regularity of f on a left and a right neighborhood of u. The error bound (116) is the same as the one obtained in Proposition 2 with foveal spaces constructed with translated orthogonal wavelets, but Theorem 7 does not require any uniform regularity condition as opposed to Proposition 2 and the approximation is performed with only two generating foveal wavelets.

Figure 9 shows the residue obtained with different foveal wavelets centered at $u = 0$, when approximating a signal that is $\mathbf{C}^\infty$ for $t < 0$ and for $t > 0$ but whose derivatives are discontinuous at $t = 0$. For spline foveal wavelets of degree $p \geq 1$, Table 1 shows that $\beta < p + 1$. To guarantee that $|f(t) - P_{\mathbf{V}_u} f(t)| = O(|t-u|^\alpha)$, Theorem 7 requires that $\alpha < \beta$. If $p = 1$ then Table 1 shows that $\beta = 1.497$ and if $p = 2$ then $\beta = .8645$. In the example of Figure 9, the residue in the neighborhood of $u = 0$ is indeed smaller for $p = 1$ than for $p = 2$.

Section 5.1 constructs foveal wavelets ψ^1 which generate orthogonal bases, reproduce high order polynomials and have a compact support included in $[-C_2, -C_1] \cup [C_1, C_2]$ with $C_2 > C_1 > 0$. These foveal wavelets thus satisfy the decay conditions of Theorem 7 for any $\beta > 0$. Their approximation capabilities is then only limited by the degree p of the polynomials they reproduce. If f is Hölder $\alpha = p+1$ on the left and on the right of u then $|f(t) - P_{\mathbf{V}_u} f(t)| = O(|t-u|^{p+1})$. For the optimized foveal wavelets shown in Figure 7 with $p = 1$, the residue obtained in Figure 9 is indeed smaller than for linear and quadratic polynomial spline foveal wavelets. As expected, the amplitude of this residue in the neighborhood of $u = 0$ is further reduced for the two optimized foveal wavelets that reproduce polynomials of degree $p = 2$, shown in Figure 7.

Figure 10 displays the reconstruction of a signal from multiple foveal points, using the optimized foveal wavelets in Figure 7 and maximum foveal scales given by (7). Since $\phi \geq 0$, singularities are detected from the

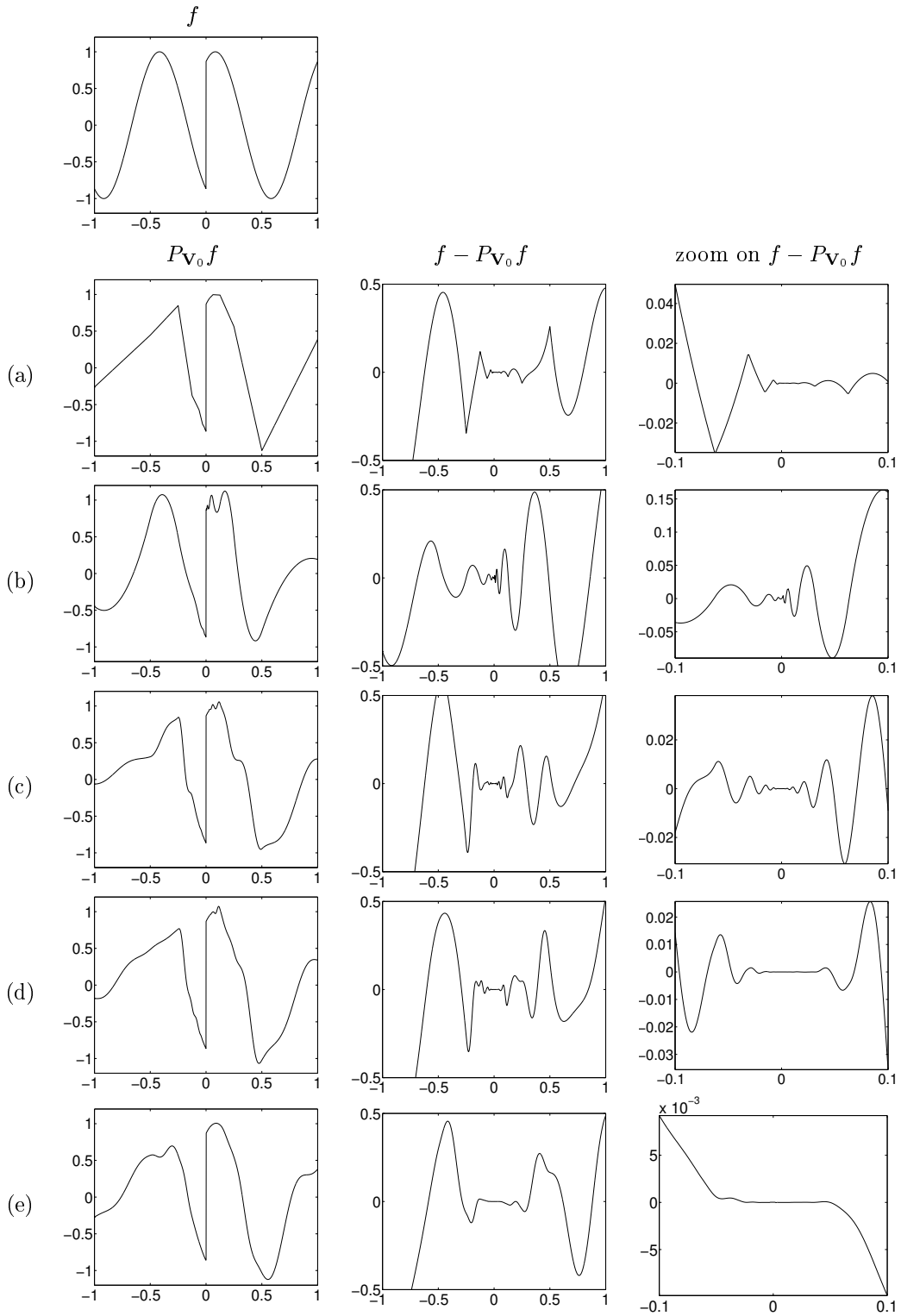


Figure 9: Foveal approximations $P_{\mathbf{V}_0} f$ and the corresponding residue $f - P_{\mathbf{V}_0} f$ computed with different foveal wavelets. (a,b): with foveal polynomial splines of order p, respectively equal to 1 and 2. (c,d): with the optimized foveal wavelets in Figure 7 having a positive ϕ, for p respectively equal to 1 and 2. (e): with the optimized foveal wavelets for $p = 2$, shown at the bottom of Figure 7.

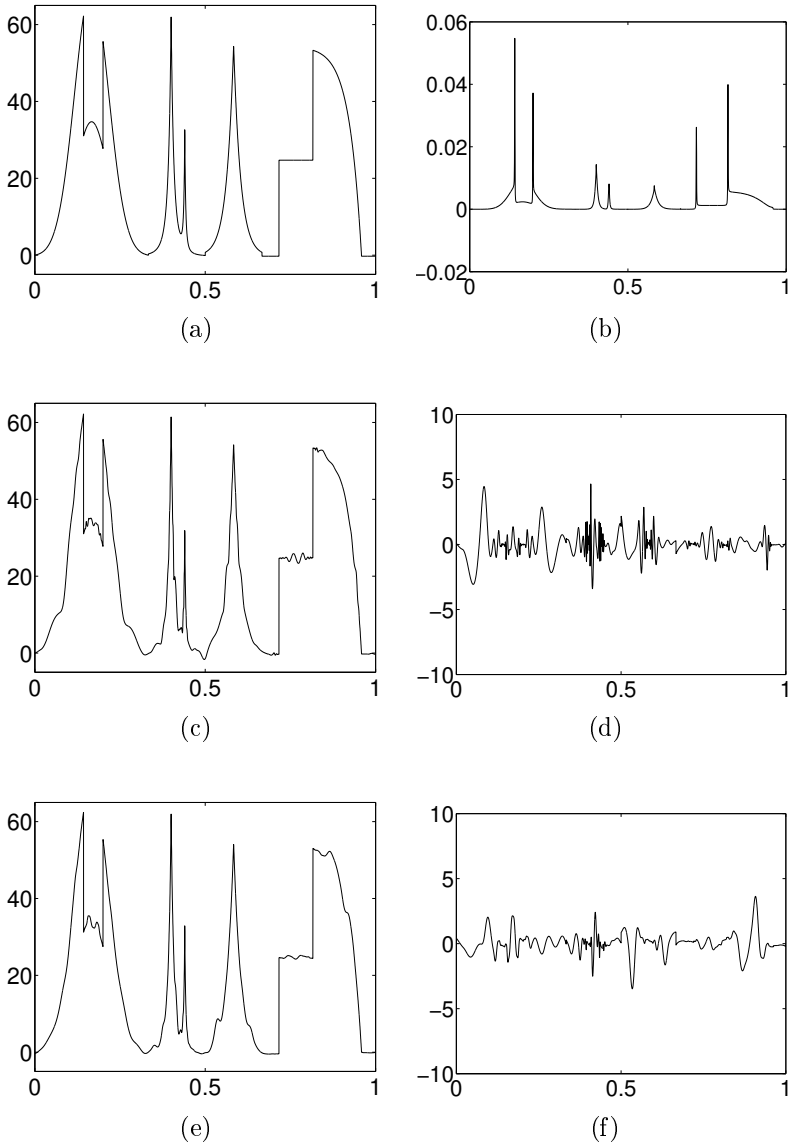


Figure 10: (a): Original signal. (b): Foveal energy $e(u)$ calculated with the optimized foveal wavelets in Figure 7 for $p = 1$. (c): Foveal approximation $P_{\mathbf{V}}f$ computed from the detected foveal points at $0.14, 0.2, 0.4, 0.44, 0.58, 0.72, 0.82, 0.96$. (d): Residue $f - P_{\mathbf{V}}f$. (e) Foveal approximation $P_{\mathbf{V}}f$ computed from the same detected foveal points with the optimized foveal wavelets at the bottom of Figure 7, for $p = 2$. (f) Residue $f - P_{\mathbf{V}}f$.

local maxima of the foveal energy $e(u)$ defined in (111) and displayed in Figure 8b. The resulting approximation reconstructs precisely all singularities. One can observe that the residue has a smaller amplitude and is more regular for $p = 2$ than for $p = 1$.

If ψ^1 is a regular function then the following theorem proves that the residue $f - P_{\mathbf{V}_u} f$ is uniformly regular. In Figure 9, one can indeed verify that the regularity of the residues depend upon the regularity of the foveal wavelets used to compute them.

Theorem 8 *Let $\{\psi_j^{1-}, \psi^{1+}\}_{j\in\mathbb{Z}}$ be an orthogonal basis of a foveal space $\mathbf{V}_0$ with*

$$\forall t > 0 \quad \sum_{j=-\infty}^{+\infty} 2^{kj}\, \psi^1(2^{-j}t) = a_k\, t^k \quad \textit{with } a_k \neq 0 \textit{ for } 0 \leq k \leq p\ . \tag{121}$$

Suppose that ψ^1 has a support included in $[-C_2, -C_1] \cup [C_1, C_2]$ for $C_2 > C_1 > 0$ and that it is uniformly Hölder $p+1$. If the restrictions of f to $(-\infty, u]$ and $[u, +\infty)$ are uniformly Hölder $\alpha \leq p+1$ then $r = f - P_{\mathbf{V}_u} f$ is uniformly Hölder α on $\mathbb{R}$.

Proof: Theorem 7 proves that $|r(t)| = O(|t-u|^\alpha)$. To prove that r is uniformly Hölder α on $\mathbb{R}$, one can verify that it is therefore sufficient to prove that its restrictions to $(-\infty, u]$ and $[u, +\infty)$ are uniformly Hölder α. For $t < u$, as in the proof of Theorem 7 we write $f(t) = q^-(t) + \epsilon_u^-(t)$, and $r = \epsilon_u^- - P_{\mathbf{V}_u}\epsilon_u^-$ with

$$P_{\mathbf{V}_u}\epsilon_u^-(t) = \sum_{j=-\infty}^{+\infty} \langle \epsilon_u^-\ ,\ \psi_{j,u}^{1-}\rangle\, 2^{-j/2}\, \psi^{1-}(2^{-j}(t-u))\ ,$$

and there exists $A > 0$ such that

$$|\langle \epsilon_u^-\ ,\ \psi_{j,u}^{1-}\rangle| \leq A\, 2^{(\alpha+1/2)j}\ . \tag{122}$$

Since f is uniformly Hölder α on $(-\infty, u]$ and $q^-(t)$ is a polynomial necessarily $\epsilon_u^- = f - q^-$ is also uniformly Hölder α on $(-\infty, u]$. We prove that r is uniformly Hölder α on $(-\infty, u]$ by verifying that $P_{\mathbf{V}_u}\epsilon_u^-$ is uniformly Hölder α on $(-\infty, u]$.

For any $v \leq u$ we want to approximate $P_{\mathbf{V}_u}\epsilon_u^-$ by a polynomial $q_v(t)$ for t in the neighborhood of v. Since ψ^1 is uniformly Hölder α, at any $v_j = 2^{-j}(v-u)$ there exists a polynomial q_{v_j} of degree $m = \lfloor \alpha \rfloor$ such that

$$|\psi^1(t) - q_{v_j}(t)| \leq K\, |t - v_j|^\alpha\ . \tag{123}$$

Since the support of ψ^1 is $[-C_2, -C_1] \cup [C_1, C_2]$ clearly $q_{v_j}(t) \neq 0$ only if $C_1 \leq |v_j| \leq C_2$ and hence if $C_2^{-1}|u-v| \leq 2^j \leq C_1^{-1}|u-v|$. Let us define the polynomial

$$\begin{aligned} q_v(t) &= \sum_{j=-\infty}^{+\infty} \langle \epsilon_u^-\ ,\ \psi_{j,u}^{1-}\rangle\, 2^{-j/2}\, q_{v_j}\left(2^{-j}(t-u)\right) \\ &= \sum_{j=\log_2(C_2^{-1}|u-v|)}^{\log_2(C_1^{-1}|u-v|)} \langle \epsilon_u^-\ ,\ \psi_{j,u}^{1-}\rangle\, 2^{-j/2}\, q_{v_j}\left(2^{-j}(t-u)\right)\ . \end{aligned} \tag{124}$$

Let us evaluate $I(t) = |P_{\mathbf{V}_u}\epsilon_u^-(t) - q_v(t)|$. If $|t-v| > |u-v|/2$ then $|t-v| > |t-u|/3$. We saw in (120) that

$$|P_{\mathbf{V}_u}\epsilon_u^-(t)| \leq K''|t-u|^\alpha\ . \tag{125}$$

Since $|\psi^1(t)|$ is bounded by a constant C, we derive from (123) that $|q_{v_j}(t)| \leq C + K|t - v_j|^\alpha$. Inserting this in (124) proves that there exists C' and K' such that

$$|q_v(t)| \leq C' \, |u - v|^\alpha + K' \, |t - v|^\alpha \ . \tag{126}$$

If $|t - v| > |u - v|/2$ then $|t - v| > |t - u|/3$ so (125) and (126) imply that there exists K_2 such that $I(t) \leq K_2 |t - v|^\alpha$. Suppose now that $|t - v| \geq |u - v|/2$ and let us evaluate

$$I(t) = \left| \sum_{j=-\infty}^{+\infty} \langle \epsilon_u^-, \psi^{1-} \rangle \, 2^{-j/2} \left(\psi^{1-}(2^{-j}(t-u)) - q_{v_j}(2^{-j}(t-u)) \right) \right| \ .$$

The terms of the series are zero for $j < l_1 = C_2^{-1} \min(|u - v|, |t - u|)$ and $j > l_2 = C_1^{-1} \max(|u - v|, |t - u|)$. Applying (123) together with (122) gives

$$|I(t)| \leq \sum_{j=\log_2(l_1)}^{\log_2(l_2)} A \, 2^{\alpha j} \, K \, |2^{-j}(t-u) - v_j|^\alpha \ .$$

Since $|t - v| \leq |u - v|/2$ we have $|u - v|/2 < |t - u| < 3|u - v|/2$ so $l_2/l_1 \leq 3C_2/C_1$ and hence

$$|I(t)| \leq A \, K \, |t - v|^\alpha \, \log_2(3C_2/C_1) \ .$$

It results that $P_{\mathbf{V}_u} \epsilon_u^-$ is uniformly Hölder α on $(-\infty, u]$, and hence $r(t)$ is uniformly Hölder α on $(-\infty, u]$. We prove similarly that $r(t)$ is uniformly Hölder α on $[u, +\infty)$ and therefore that it is uniformly Hölder α on $\mathbb{R}$.

■

This theorem proves that the residual error of a foveal approximation is a uniformly regular function. If f includes several singularities, a foveal approximation is computed as a projection in a space that is a sum of foveal spaces corresponding the singularity locations, as explained in Section 2.1. If f is uniformly Hölder α between the singularities then we derive from Theorem 8 that the residue is uniformly Hölder α. This residue may have a large amplitude far away from the signal singularities but since it is uniformly regular, it can be efficiently approximated with a linear scheme in a Fourier or a wavelet basis [4]. This suggests using a double layer approximation scheme, with foveal wavelets to reconstruct singularities, and any standard linear approximation procedure to approximate the regular residue. Orthogonal foveal wavelets can therefore be used both to detect singularities and to reconstruct a precise signal approximation from these singularities, where as the residue is represented by a standard linear approximation procedure.

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