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Embedding of all regular tilings and star-honeycombs

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Abstract

We review the regular tilings of d-sphere, Euclidean d-space, hyperbolic d-space and Coxeter's regular hyperbolic honeycombs (with infinite or star-shaped cells or vertex figures) with respect of possible embedding, isometric up to a scale, of their skeletons into a m-cube or m-dimensional cubic lattice. In section 2 the last remaining 2-dimensional case is decided: for any odd $m \geq 7$, star-honeycombs $m\frac{m}{2}$ are embeddable while $\frac{m}{2}m$ are not (unique case of non-embedding for dimension 2). As a spherical analogue of those honeycombs, we enumerate, in section 3, 36 Riemann surfaces representing all nine regular polyhedra on the sphere. In section 4, non-embeddability of all remaining star-honeycombs (on 3-sphere and hyperbolic 4-space) is proved. In the last section 5, all cases of embedding for dimension $d > 2$ are identified. Besides hyper-simplices and hyper-octahedra, they are exactly those with bipartite skeleton: hyper-cubes, cubic lattices and 8, 2, 1 tilings of hyperbolic 3-, 4-, 5-space (only two, 435 and 4335, of those 11 are compact).

1 Introduction

We say that given tiling (or honeycomb) T has a l_1-*graph* and embeds up to *scale* λ into m-cube H_m (or, if the graph is infinite, into cubic lattice $\mathbf{Z}_m$), if there exists a mapping f of the vertex-set of the skeleton graph of T into the vertex-set of H_m (or $\mathbf{Z}_m$) such that

$$\lambda d_P(v_i, v_j) = ||f(v_i), f(v_j)||_{l_1} = \sum_{1 \leq k \leq m} |f_k(v_i) - f_k(v_j)| \text{ for all vertices } v_i, v_j.$$

We take, of course, the smallest such number λ.

Denote by $T \to H_m$ (by $T \to \mathbf{Z_m}$) isometric embedding of the skeleton graph of T into m-cube (respectively, into m-dimensional cubic lattice); denote by $T \to \frac{1}{2}H_m$ and by $T \to \frac{1}{2}\mathbf{Z_m}$ isometric up to scale 2 embedding.

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Call an embeddable tiling l_1-*rigid*, if all its embeddings as above are pairwise equivalent. All, except 3- and 4-simplex, embeddable tilings in this paper turn out to be l_1-rigid and so having scale 1 or 2. Those embeddings were obtained by constructing a complete system of *alternated zones* ; see [CDG97], [DSt96],[DSt97].

Actually, a tiling is a special case of a honeycomb, but we reserve the last term for the case when the cell or the vertex figure is a star-polytope and so the honeycomb covers the space several times; the multiplicity of the covering is called its *density*.

Embedding of Platonic solids was remarked in [Kel75] and precised, for the dodecahedron, in [ADe80]. Then [Ass81] showed that 36, 63, and mk (for even $m \geq 8$ and $m = \infty$) are embeddable. The remaining case of odd m and limit cases of $m = 2, \infty$ was decided in [DSt96]; all those results are put together in the Theorem 1 below.

All four star-polyhedra are embeddable. The great icosahedron $3\frac{5}{2}$ of Poinsot and the great stellated dodecahedron $\frac{5}{2}3$ of Kepler have the skeleton (and, moreover, the surface) of, respectively, icosahedron and dodecahedron; each of them has density 7. All ten star-4-polytopes are *not* embeddable; see Theorem 3 below. [DSt97] considered also embedding of all 6 Coxeter-Petri infinite regular polyhedra; they are also not tilings of the space but of *sponge* surface. (They are 36, 63, 44 and 46, 64, 66, each taking a region of 3-space, which has identical shape with its complement.)

The case of Archimedean tilings of 2-sphere and of Euclidean plane was decided in [DSt96]; it turns out that for any such tilings (except embeddable $Prism_3$) exactly one of two (a tiling and its dual) is embeddable. For 3-sphere and 3-space it was done in [DSt98b]. All 92 *regular-faced* 3-polytopes were considered in [DGr97b] and, for all higher dimensions, in [DSt96]. The *truncations* of regular polytopes were considered in [DSt97]. Another large generalization of Platonic solids - *bifaced* polyhedra - were considered in [DGr97b]. (Some generalizations of Archimedean plane tilings, *2-uniform* ones and *equitransitive* ones, were treated in [DSt96], [DSt97], respectively.) Finally, skeletons of (Delaunay and Voronoi tilings of) lattices were dealt with in [DSt98a].

Embeddable ones, among all compact regular tilings of all dimensions, were identified in [DSt96], [DSt97].

Coxeter (see [Cox54]) extended the concept of regular tiling, permitting infinite cells and vertex figures, but with the fundamental region of the symmetry group of a finite content. His second extension was to permit *honeycombs*, i.e. star-polytopes can be cells or vertex figures. For the 2-dimensional case, on which we are focusing here,above extensions produced only following new honeycombs - $\frac{m}{2}m$ and $m\frac{m}{2}$ for any odd $m \geq 7$ - which are hyperbolic analogue of spherical star-polyhedra $\frac{5}{2}5$ (the small stellated dodecahedron of Kepler) and $5\frac{5}{2}$ (the great dodecahedron of Poinsot). Both $\frac{5}{2}5$ and $5\frac{5}{2}$ have the skeleton of the icosahedron. For any odd m above honeycombs cover the space (2-sphere for $m = 5$) 3 times. The skeleton of $m\frac{m}{2}$ is, evidently, the same as of 3m, because it can be seen as 3m with the same vertices and edges forming m-gons instead of triangles. The faces of $\frac{m}{2}m$ are stellated faces of m3 and it have the same vertices as 3m.

2 Planar tilings and hyperbolic honeycombs

They are presented in the Table 1 below; notation there are as follows:

1. The row indicates the facet (cell) of the tiling (or honeycomb), the column indicates its vertex figure. The tilings and honeycombs are denoted usually by shortened (i.e. without parentheses and commas) their Schläfli notation.

2. By m ($m \geq 2$) we denote m-gon, by $\frac{m}{2}$ star m-gon (if m is odd); the pentagram $\frac{5}{2}$ is considered separately. By α_3, β_3, γ_3, Ico, Do and δ_2 we denote regular ones tetrahedron, octahedron, cube, icosahedron, dodecahedron and the square lattice Z_2. In the Table 1, the numbers are: any $m \geq 7$ in 8th column, row and any *odd* $m \geq 7$ in 9th column, row.

3. We consider that: $2m$ is a 2-vertex multi-graph with m edges; $m2$ can be seen as a m-gon; all vertices of $m\infty$ are on the absolute conic at infinity (it has an infinite degree); the faces ∞ of ∞m are inscribed in horocycles.

Table 1. 2-dimensional regular tilings and honeycombs.

	2	3	4	5	6	7	m	∞	$\frac{m}{2}$	$\frac{5}{2}$
2	22	23	24	25	26	27	$2m$	2∞		
3	32	α_3	β_3	Ico	**36**	37	$3m$	3∞		$3\frac{5}{2}$
4	42	γ_3	$\boldsymbol{\delta_2}$	45	46	47	$4m$	4∞		
5	52	Do	54	55	56	57	$5m$	5∞		$5\frac{5}{2}$
6	62	**63**	64	65	66	67	$6m$	6∞		
7	72	73	74	75	76	77	$7m$	7∞		
m	$m2$	$m3$	$m4$	$m5$	$m6$	$m7$	mm	$m\infty$	$m\frac{m}{2}$	
∞	$\infty 2$	$\infty 3$	$\infty 4$	$\infty 5$	$\infty 6$	$\infty 7$	∞m	$\infty\infty$		
$\frac{m}{2}$							$\frac{m}{2}m$			
$\frac{5}{2}$		$\frac{5}{2}3$		$\frac{5}{2}5$						

Theorem 1 *All 2-dimensional tilings mk are embeddable, namely:*

(i) if $\frac{1}{m} + \frac{1}{k} > \frac{1}{2}$ (2-sphere), then
$2m \to H_1$ *for any* m, $m2 \to \frac{1}{2}H_m$ *for odd* m *and* $m2 \to H_m$ *for even* m*;*
$33 = \alpha_3 \to \frac{1}{2}H_3, \frac{1}{2}H_4$; $43 = \gamma_3 \to H_3$; $34 = \beta_3 \to \frac{1}{2}H_4$;
$35 = Ico(\sim 3\frac{5}{2} \sim 5\frac{5}{2} \sim \frac{5}{2}5) \to H_6$ *and* $53 = Do(\sim \frac{5}{2}3) \to \frac{1}{2}H_{10}$;

(ii) if $\frac{1}{m} + \frac{1}{k} = \frac{1}{2}$ (Euclidean plane), then
$2\infty \to H_1$, $\infty 2 \to Z_1$; $44 = \delta_2 \to \mathbf{Z}_2$, $36 \to \frac{1}{2}\mathbf{Z}_3$, $63 \to \mathbf{Z}_3$;

(iii) if $\frac{1}{2} > \frac{1}{m} + \frac{1}{k}$ (hyperbolic plane), then
$mk \to \frac{1}{2}\mathbf{Z}_\infty$ *if* m *is odd,* $k \leq \infty$ *and* $mk \to \mathbf{Z}_\infty$ *is* m *is even or* ∞, $k \leq \infty$.

The following theorem, the main result of this note, gives the family, containing all non-embeddable regular planar cases.

Theorem 2 *For any odd $m \geq 7$ we have*

(i) $\frac{m}{2}m$ is not embeddable;

(ii) $m\frac{m}{2}(\sim 3m) \to \frac{1}{2}\mathbf{Z}_\infty$.

The assertion (ii) is trivial. The proof of (i) will be preceded by 3 lemmas and first two of them are easy but of independent interest for embedding of (not necessary planar) graphs.

Let G be a graph, scale λ embeddable into $\mathbf{Z_m}$, let C be an oriented circuit of length t in G and let e be an arc in C. Then there are λ *elementary vectors* , i.e. steps in the cubic lattice $\mathbf{Z_m}$, corresponding to the arc e; denote them by $x_1(e), ..., x_{\lambda(e)}$. Clearly, the sum of all vectors $x_i(e)$ by all i and all arcs e of the circuit, is the zero-vector.

Now, if t is even, denote by e^* the arc opposite to e in the circuit C; if t is odd, denote by e',e'' two arcs of C opposite to e. For even t, call the arc e *balanced* if the set of all its vectors $x_i(e)$ coincides with the set of all $x_i(e^*)$, but the vectors of arc e^* go in opposite direction *on the circuit C* to the vectors of e. For odd t, call the arc e *balanced* if a half of vectors of e' together with a half of vectors of the second opposite arc e'' form a partition of the set of vectors of e and those vectors go in opposite direction (on C) to those of arc e.

Remind, that the *girth* of the graph is the length of its minimal circuit.

Lemma 1. *Let G be an embeddable graph of girth t. Then*

(i) any arc of a circuit of length t is balanced;

(ii) if t is even, then any arc of a circuit of length $t+1$ is also balanced.

Lemma 2. *Let G be an embeddable graph of girth t and let P be an isometric oriented path of length at most $\lfloor \frac{t}{2} \rfloor$ in G. Then there are no two arcs on this path having vectors, which are equal, but have opposite directions on the path.*

Lemma 3 *The girth of the skeleton of $\frac{m}{2}m$ is 3 for $m = 5$ and $m - 1$ for any odd $m \geq 7$.*

Proof of Lemma 3

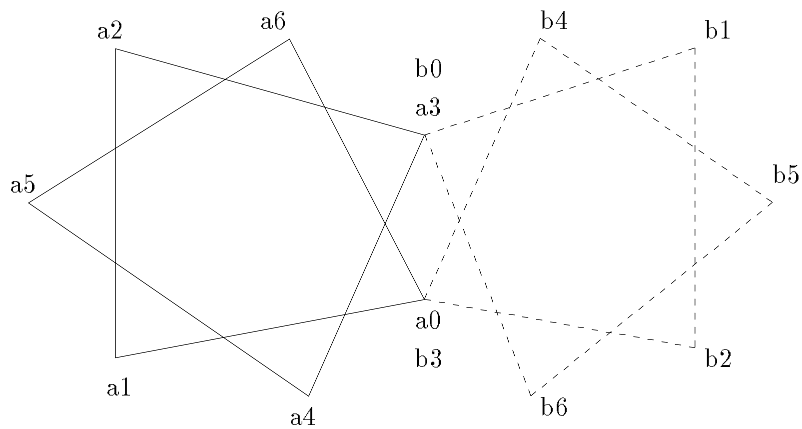

Fig . 1a. A fragment of 7/2 7

Consider Fig.1a.Take a cell $A = (a_0, ..., a_m = a_0)$ of the $\frac{m}{2}m$, i.e. a star m-gon, seen as an oriented cycle of length $m = 2k + 1$. Consider following automorphism of the honeycomb: a turn by 180 degrees around the midpoint of the segment $[a_0, a_k]$. The image of A is the oriented star m-gon $B = (b_0, ..., b_m = b_0)$ with $b_0 = a_k$,$b_k = a_0$. Consider now oriented cycle $C = (a_0, a_1, ..., a_k = b_0, ..., b_k = a_0)$ of even length $m-1 = 2k$. In order to prove the Lemma 3, we will show that C is a cycle of minimal length.

First we show that the graph distance $d(a_0, a_k) = k$ (i.e. the path $P := (a_0, a_1, ..., a_k)$ is a shortest path from a_0 to a_k). It will imply that $d(a_0, c(A)) = d(a_k, c(A)) = k$, where $c(A)$ is the center of the cell A, because all vertices of $\frac{m}{2}m$ are vertices of regular triangles of $3m$.

Let Q be a shortest path from a_0 to a_k. Then it goes around the vertex $c(A)$ or the center $c(B)$ of the cell B, because otherwise Q goes through at least one of the vertices a_{k+1}, a_{2k}, b_{k+1}, b_{2k} and so Q contains at least one of the pairs of vertices (a_0, a_{k+1}), $(a_0 = b_k, b_{2k})$, $(b_k = a_0, a_{2k})$, $(a_k = b_0, b_{k+1})$. But each of those pairs has, by the symmetry of our honeycomb $\frac{m}{2}m$, same distance between them as (a_0, a_k); it contradicts to the supposition that Q is a shortest path. So, we can suppose that Q goes around $c(A)$ (the argument is the same if it goes around $c(B)$). Now, to each edge (s, t), corresponds, from the center $c(A)$ of A, the angle $(s, c(A), t)$. The $2k+1$ edges of A are only edges, for which this angle is $\frac{4k\pi}{2k+1}$; for any other edge, the angle is smaller, since it is more far from $c(A)$. So, if Q contains an edge, other than one from A, then, in order to reach a_k from a_0, it should be of length more than k. Therefore, any shortest path from a_0 to a_k, should consist only of edges of A and then it is of length k. So, $d(a_0, c(A)) = k$ also, as well as for any edge of $3m$. Same holds for $m = 5$.

We will show now that: (i)any path R of length $2k-2$ is not closed and (ii)R cannot be closed by only one edge. But C is a closed path of length $2k$; so (i), (ii) will imply that $2k$ (respectively, $2k+1$) is the minimal length of any (respectively, any odd) simple isometric cycle in the graph. For $m = 5$ (ii) does not holds.

Suppose that R is closed; let as see it as a $2k-2$-gon on hyperbolic plane. Any angle of R is a multiple $i\frac{2\pi}{m}$, but $i > 1$ for at least one angle, because $(2k-2)\frac{2\pi}{m} < 2\pi$. Suppose that a angle has $1 < i \leq k$; the argument will be the same, if $k+1 \leq i < m-1$, but for the complementary angle $(m-i)\frac{2\pi}{m}$ with $1 < m-i \leq k$.

See Fig.1b for the following argument. Fix an angle r, s, t between two adjacent edges (r, s) and (s, t) of R, has i with $1 < i \leq k$. Let $s*$ be the opposite vertex to s on R, let (s, r'), (s, t') be the edges such that the angles r, s, r', t, s, t' are $\frac{2\pi}{m}$. Let A, B be the cells $\frac{m}{2}$, defined by pairs $(r, s), (s, r')$ and $(t, s), (s, t')$ of their adjacent edges and $c(A)$, $c(B)$ are their centers. The vertex $c(A)$ not belongs to the path from s to $s*$ of length $k-1$, since we proved above that $d(s, c(A) = k)$; so this path should go around $c(A)$. Let p be the vertex of A, reachable from s by $k-1$ steps on A, starting by r, let q be the vertex of B, reachable from s by $k-1$ steps on B, starting by t. By mirror on (r, s) (respectively, (s, t)) we obtain the cells A', B', their centers $c(A')$, $c(B')$ and vertices p', q', which are reflections of p, q. Call *A-domain*, the part of the hyperbolic plane, bounded by half-lines $c(A), p, \infty$, $c(A'), p'\infty$ and the angle $c(A), s, c(A')$; call *B-domain*, the part, bounded similarly for B. Actually, B-domain is the reflection of A-domain by the bisectrisse of the angle (r, s, t).

We will show now that the vertex $s*$ should belong to both A- and B-domains, but they do not have common points, besides s. This contradiction will show, that our R, a closed path of length $2k-2$, do not exists. Any edge of the path $(s, t, ..., s*)$ of length $k-1$ is seen from $c(A)$ under angle at most $\frac{4\pi}{m}$ with equality if and only if the edge belongs to

A (as, for example, the edge (r, s)). Summing up those angles along the path $(st, ..., s*)$, we get less than $(k-1)\frac{4\pi}{m}$, obtained for the path of length $k-1$ from s to p, going along A. It implies that $s*$ belongs to A-domain and also, by reflection, to B-domain.

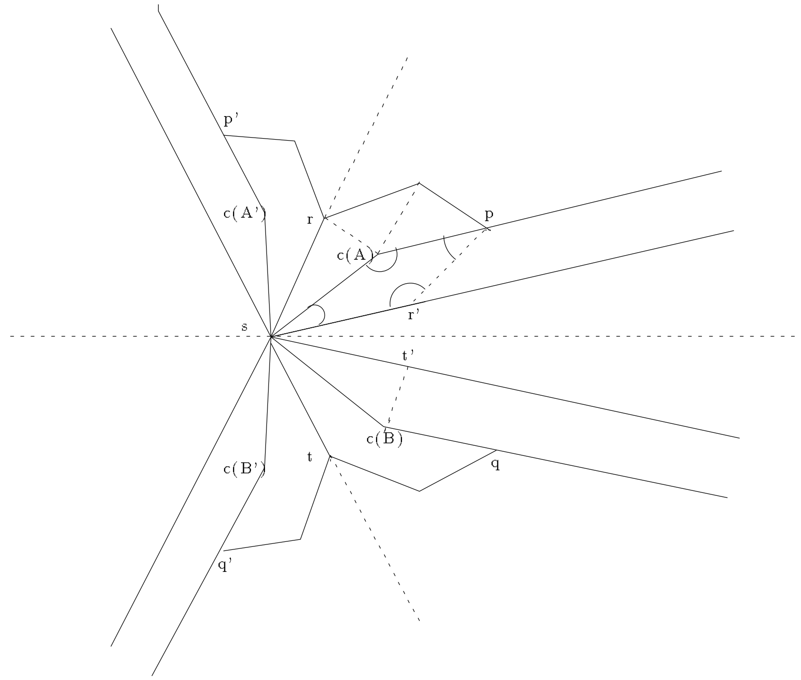

Fig. 1b. A fragment of 9/2 9

But A- and B-domains intersect only in point s, because the lines through $(c(A), p)$ and (s, r') diverge on the hyperbolic plane. In fact, denote by α_1, α_2, β_1, β_2 the angles (p, c(A), s), (c(A), s, r'), (c(A), p, r'), (p, r', s), respectively. They are equal to $\frac{4\pi}{m} + \frac{2\pi}{m}$, $\frac{\pi}{m}$, $\frac{\pi}{m} + \frac{\pi}{m}$, $\frac{2\pi}{m} + \frac{\pi}{m}$, respectively. So $\alpha_1 + \alpha_2 = \frac{7\pi}{m} \leq \pi$, since $m \geq 7$ and the lines, if they converge or parallel, do it on the right side of Fig. 1b. Now, $\beta_1 + \beta_2 = \frac{5\pi}{m} < \pi$ and the lines, if they converge or parallel, do it on the left. So, they diverge.

We demonstrated ad absurdum, the non-existence of the vertex $s*$ and so, of the *closed* path R. So, a path R of length $2k-2$ is not closed. But p, q is never an edge; so we need at least two edge in order to close R. If two edges are enough, then points r', t' coincide, i.e. $i = 2$; actually, two edges will be enough if, moreover, $m = 7$. The proof of Lemma is completed.

Proof of Theorem 2

Consider star-m-gons A, B and the circuit C as in beginning of the proof of Lemma 3 above. Take the arc $e = (a_0, a_1)$ on the circuit C; by Lemma 1 (i), e is balanced, i.e. the vectors $x_i(e^*)$ of the opposite arc $e^* = (b_0, b_1)$ are same, as of the arc e, but they have opposite directions with respect of the circuit C. The same arc e, seen as a arc of the circuit B of length m, is opposite to two arcs in this odd circuit and, in particular, to the arc (a_k, a_{k+1}). The last arc has, by Lemma 1 (ii), $\frac{\lambda}{2}$ vectors, coinciding with vectors of e, but with opposite direction on the circuit B. Finally, consider the oriented path

$(a_{k+1}, a_k = b_0, b_1)$ of length 2 in our $\frac{m}{2}m$. Its two arcs have vectors, coinciding, but going in opposite direction on this path. But it contradicts to Lemma 2, because $2 < k$.

3 Spherical analogue of Coxeter's honeycombs

In this Section we consider, for any pair (i, m) of integers, such that $1 \leq i < \frac{m}{2}$ and $g.c.d.(i, m) = 1$, star-polygons $\frac{m}{i}$. Clearly, $\frac{m}{1}$ denotes now a convex m-gon; so we see star-polygons as a generalization of convex ones. We will allow further extension: star-polygons $\frac{m}{i}$ with $\frac{m}{2} < i < m$, let us call them *large* star-polygons. They cannot be represented on Euclidean or hyperbolic plane, because they have there the same representation as $\frac{m}{m-i}$. But they can be represented on the sphere by the following way; see Fig.2 for the simplest $\frac{3}{1}$ and $\frac{3}{2}$. Let $a_0, ..., a_{m-1}$ be m points, placed in this order, on a great circle of the sphere, in order to form a regular m-gon. Then the spherical (great circle) distance $d(a_0, a_i)$ is $\frac{2\pi i}{m}$, but on $\frac{m}{i}$, the length of the way is $d(a_0, a_i)$ for $i < \frac{m}{2}$ and $2\pi - d(a_0, a_i)$ otherwise. Using this larger set of polygons, we will look for spherical representations of regular (i.e. with a group of symmetry acting transitively on all j-faces, $0 \leq j \leq 2$) polyhedra.

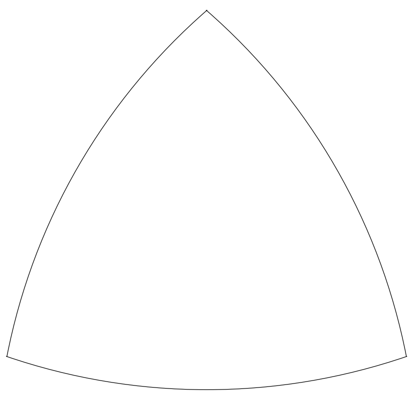

Fig. 2a. 3/1

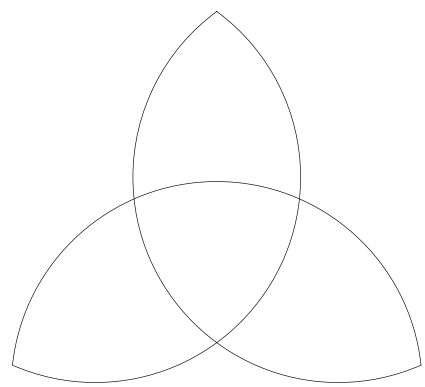

Fig. 2b. 3/2

In the Table 2 below, the rows (columns) denote a cell (respectively, a vertex figure) of would-be representations. If the representation, corresponding to a given pair of $(\frac{m}{i}, \frac{n}{j})$ of polygons, exists, we denote it by this pair and write its density in corresponding cell of the Table 2. The densities were counted directly, by superposing the representation on corresponding regular polyhedron. But the expression of the density, given in the formula 6.41 of [Cox73] for multiply-covered sphere is valid for our representations, i.e. the density of $(\frac{m}{i}, \frac{n}{j})$ is $N_1(\frac{i}{m} + \frac{j}{n} - \frac{1}{2})$, where N_1 is the number of edges. (Above expression is equivalent to Cayley's generalization of Euler's Formula, given as the formula 6.42 in [Cox73].) Our representations are *Riemann surfaces*, i.e. d-sheeted spheres (or d almost coincident, almost spherical surfaces) with the sheets connected in certain branch-points.

We see a $\frac{m}{i}$ as a representation of the m-cycle on the sphere, together with a bi-partition of i-covering of the sphere. Call *interior* the part with angle, which is less than π. For representations below, the vertex figure selects uniquely the part of the cell;

namely, the vertex figure $\frac{n}{j}$ gives the value $\frac{2\pi j}{n}$ for the angle of the cell. It takes interior of the cell if $j < \frac{n}{2}$ and exterior otherwise.

The Table 2 shows that each of all nine regular polyhedra (seen as abstract surfaces) admits four such Riemann surfaces and we checked, case by case, that all 36 are different and that remaining 28 possible representations do not exist. Each of four representations for each regular polyhedron has same genus as corresponding abstract surface; so the genus is four for 8 representations of the form $(\frac{5}{i}, \frac{5}{j})$ and zero for all others.

In the Table 2, the column with $\frac{2}{1}$ corresponds to doubling of regular polygons. Alexandrov ([Ale58]) considered, for other purpose, the doubling of any convex polygon as an abstract sphere, realized as a degenerated (i.e. with volume 0) convex polyhedron. $m2$ and $2n$ on the plane and the sphere appeared also in Section 7 of [FTo64]. By analogy, we will do such doubling for star-polygons $\frac{m}{i}$ with $i < \frac{m}{2}$. But for large star-polygons we should do doubling on the sphere. The row and the column with $\frac{m}{i}$ correspond to any pair of mutually prime integers (i, m), $1 \leq i < m$. As Table 2 shows, there exist all representations $(\frac{2}{1}, \frac{m}{i})$ and $(\frac{m}{i}, \frac{2}{1})$ and each of them has density i (and the genus 0).

An infinity of other representations can be obtained by permitting polygons $\frac{m}{i+tm}$ for any integer $t \geq 0$; the way on the edge (a_0, a_{i+tm}) will be $2\pi t - d(a_0, a_{i+tm})$.

Table 2. 36 representations of regular polyhedra on the sphere.

	$\frac{2}{1}$	$\frac{3}{1}$	$\frac{3}{2}$	$\frac{4}{1}$	$\frac{4}{3}$	$\frac{5}{1}$	$\frac{5}{4}$	$\frac{5}{2}$	$\frac{5}{3}$	$\frac{m}{i}$	$\frac{m}{m-i}$
$\frac{2}{1}$	1	1	2	1	3	1	4	2	3	i	$m-i$
$\frac{3}{1}$	1	1	3	1	7	1	19	7	13		
$\frac{3}{2}$	2	3	5	5	11	11	29	17	23		
$\frac{4}{1}$	1	1	5								
$\frac{4}{3}$	3	7	11								
$\frac{5}{1}$	1	1	11					3	9		
$\frac{5}{4}$	4	19	29					21	27		
$\frac{5}{2}$	2	7	17			3	21				
$\frac{5}{3}$	3	13	23			9	27				
$\frac{m}{i}$	i										
$\frac{m}{m-i}$	$m-i$										

4 Star-honeycombs

Besides star-polygons and four regular star-polyhedra on 2-sphere, which are all embeddable (last four are isomorphic to Ico or Do), there are ([Cox54]) only following regular star-honeycombs: ten regular star-polytopes on 3-sphere and four star-honeycombs in hyperbolic 4-space; see the Tables 1, 3-5. In this section we show that none of last 14 is embeddable. Consider first the case of 3-sphere.

There are six regular 4-polytopes (4-simplex α_4, 4-cross-polytope β_4, 4-cube γ_4, self-dual 24-cell and the pair of duals 600-cell and 120-cell) and ten star-4-polytopes; see the Chapter 14 in [Cox73]. [Ass81] showed non-embeddability of 24- and 600-cell; [DGr97c] did it for 120-cell. Clearly, γ_4 and β_4 are H_4 and $\frac{1}{2}H_4$ themselves and they are l_1-rigid.

But α_4 embeds into $\frac{1}{2}H_5$ (i.e. an embedding of scale 2 into 5-cube) and it it embeds also, for example, with scale 6 into 10-cube.

Embeddable ones among Archimedean tilings of 3-sphere and 3-space, were identified in [DSt98b]; for example, *snub* 24-cell (semi-regular Gosset's 4-polytope $s(3,4,3)$) embeds into $\frac{1}{2}H_{12}$ while the Grand Antiprism of [Con67] is not embeddable.

The isomorphisms among ten star-4-polytopes, see [vOs15] and pp. 266-267 of [Cox73], imply, of course, the isomorphisms of the skeletons of those polytopes. Using Schläfli notation, those isomorphisms *of graphs* are:

(i) $\frac{5}{2}53 \sim 5\frac{5}{2}3$;

(ii) $\frac{5}{2}33$ ~120-cell (remind the isomorphism of $\frac{5}{2}3$ and 53);

(iii) all remaining seven skeletons are isomorphic with the skeleton of 600-cell (moreover, $35\frac{5}{2}$ has same faces; remind the isomorphism of $3\frac{5}{2}$ and 35).

So eight star-polytopes from (ii) and (iii) above are not embeddable. Remaining case (i) is decided by the Theorem 3 below, using following *5-gonal* inequality, which is necessary condition (see [Dez60]) for embedding of graphs:

$d_{ab} + (d_{xy} + d_{xz} + d_{yz}) \leq (d_{ax} + d_{ay} + d_{az}) + (d_{bx} + d_{by} + d_{bz})$

for distances between any five its vertices a, b, c, x, y.

Theorem 3 *None of ten star-4-polytopes is embeddable.*

Proof of Theorem 3

In view of above isomorphisms, it will be enough to show that (the skeleton of) 4-polytope $P := \frac{5}{2}53$ is not 5-gonal. P is the stellated 120-cell and $\frac{5}{2}$ is the stellated dodecahedron, i.e. all face-planes are extended until their intersections form a pyramid on each face. P has 120 vertices, as 600-cell; namely, the centers of all 120 (dodecahedral) cells of 120cell. For any vertex s of P, denote by Do(s) the corresponding dodecahedron. P has (as 120-cell) 1200 edges, 720 faces and 120 cells; its density is 4. Any edge (s,t) of P goes through interiors of $Do(s)$, $Do(t)$ and the edge of 120-cell, linking those dodecahedra; (s,t) is a continuation of this edge in both directions till the centers of dodecahedra $Do(s)$,$Do(t)$.

Consider now Fig. 3. Take as vertices a and b (for future contre-example for 5-gonal inequality,given before of Theorem 3) some two vertices of $\frac{5}{2}5$ (a cell of P), which are centers of two face-adjacent dodecahedral cells of 120-cell. Let $Q = (q_1, q_2, q_3, q_4, q_5)$ be this face of adjacency, presented by the 5-cycle of its vertices. For any q_i there is unique star-5-gon (a, d_i, b, d'_i, d''_i), such that sides (b, d'_i) and (d''_i, a) intersect in the point q_i. Now, $D := (d_1, d_2, d_3, d_4, d_5)$ is a 5-cycle in P, because each (d_{i-1}, d_i) is an edge in one of five cells $\frac{5}{2}5$ of P, containing vertices a and b. Put $x := d_1$, $y := d_2$, $z := d_4$ and check 5-gonal inequality for five vertices a, b, x, y, z of P.

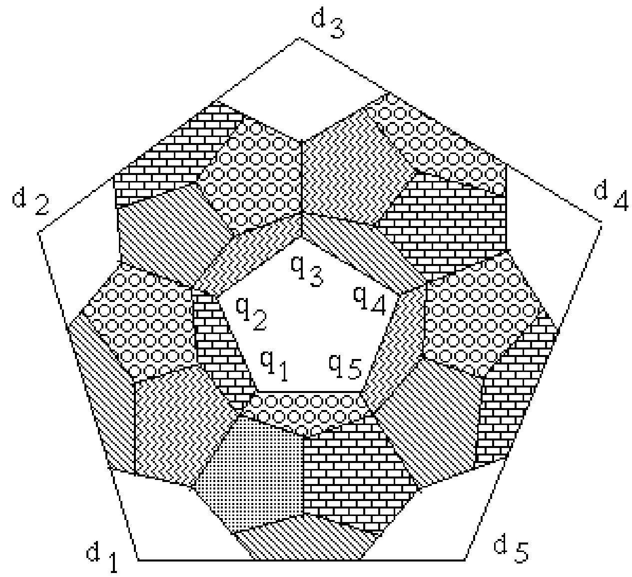

Fig. 3. A fragment of 5/2 5 3

In fact, $d_{xy} = 1 = d_{ax} = d_{ay} = d_{az} = d_{bx} = d_{by} = d_{bz}$, because of the presence of corresponding edges in P. Therefore, d_{xz}, d_{yz} and d_{ab} are at most 2. So, the absence of edges (x,z), (y,z) and (a,b) will complete the proof of the Theorem. The edge (a, b) does not exist, because $Do(a)$ is face-adjacent to $Do(b)$. The edge (x, z) does not exists, because the line, linking vertices x and z, goes, besides $Do(x)$ and $Do(z)$, through two other dodecahedra (such that their stellations are $\frac{5}{2}5$, containing vertices a, b, d_2, d_3 or a, b, d_3, d_4. By symmetry, the edge (y, z) does not exist also. We are done.

Corollary *None of four star honeycombs in hyperbolic 4-space is embeddable*

Proof of Corollary

In fact, $\frac{5}{2}533$ has cell which contains (because of Theorem 3), as an induced subgraph, non-5-gonal graph $K_5 - K_3$. But any induced graph of diameter 2 is isometric; so $\frac{5}{2}533$ is not 5-gonal. $335\frac{5}{2}$ has cell 335 = 600-cell. Two other have cells which are isomorphic to 600-cell. But 600-cell (seen by Gosset's construction as capping of all 24 icosahedral cells of snub 24-cell) contains also a forbidden induced graph of diameter 2: pyramid on icosahedron (it violates *7-gonal* inequality, which is also necessary for embedding; see [Dez60], [DSt96]). So, three other star-4-polytopes are also non-7-gonal and non-embeddable.

5 Regular tilings of dimension $d \geq 3$

The Tables 3-5 below present all of them and also all regular honeycombs in the dimensions 3, 4, 5; for higher dimensions, $(d+1)$-simplices α_{d+1}, $(d+1)$-cross-polytopes β_{d+1}, $(d+1)$-cubes γ_{d+1} and cubic lattices δ_d are only regular ones.

In those Tables, 24−, 600−, 120− are regular spherical 4-polytopes 343, 335, 533 with indicated number of cells and $De(D_4)$, $Vo(D_4)$ are regular partitions 3343, 3433 of Euclidean 4-space, which are also Delaunay (Voronoi, respectively) partitions associated with point lattice D_4.

All cases of embeddability are marked be the star * in the Tables.

Table 3. 3-dimensional regular tilings and honeycombs.

	α_3	γ_3	β_3	Do	Ico	δ_2	63	36	$3\frac{5}{2}$	$\frac{5}{2}3$	$5\frac{5}{2}$	$\frac{5}{2}5$
α_3	α_4*		β_4*		600−			336	$33\frac{5}{2}$			
β_3		24−				344						
γ_3	γ_4*		δ_3*		435*			436*				
Ico				353							$35\frac{5}{2}$	
Do	120−		534		535			536	$53\frac{5}{2}$			
δ_2		443*				444*						
36							363					
63	633*		634*		635*			636*				
$\frac{5}{2}3$	$\frac{5}{2}33$				$\frac{5}{2}35$							
$3\frac{5}{2}$												$3\frac{5}{2}5$
$\frac{5}{2}5$				$\frac{5}{2}53$							$\frac{5}{2}5\frac{5}{2}$	
$5\frac{5}{2}$										$5\frac{5}{2}3$		$5\frac{5}{2}5$

Table 4. 4-dimensional regular tilings and honeycombs.

	α_4	γ_4	β_4	24−	120−	600−	δ_3	$35\frac{5}{2}$	$\frac{5}{2}53$	$5\frac{5}{2}5$
α_4	α_5*		β_5*			3335				
β_4				$\mathbf{De(D_4)}$						
γ_4	γ_5*		δ_4*			4335*				
24−		$\mathbf{Vo(D_4)}$					3434			
600−								$335\frac{5}{2}$		
120−	5333		5334			5335				
δ_3				4343*						
$\frac{5}{2}53$					$\frac{5}{2}533$					
$35\frac{5}{2}$										$35\frac{5}{2}5$
$5\frac{5}{2}5$									$5\frac{5}{2}53$	

Table 5. 5-dimensional regular tilings and honeycombs.

	α_5	γ_5	β_5	$Vo(D_4)$	$De(D_4)$	δ_4
α_5	α_6*		β_6*			
β_5					33343	
γ_5	γ_6*		δ_5*			
$De(D_4)$				33433		
$Vo(D_4)$		34333				34334
δ_4					43343$*$	

Theorems 1, 2 above show that all regular 2-dimensional tilings and star-honeycombs are embeddable except $\frac{m}{2}m$ for all odd $m \geq 7$. The following Theorem decides all remaining regular cases.

Theorem 4 *All embeddable regular tilings and honeycombs of dimension $d \geq 3$ are tilings:*

(i) either α_{d+1} and β_{d+1}, or

(ii) all with bipartite skeleton:

(ii-1) all with cell γ_d: γ_{d+1}, δ_d and 3 hyperbolic ones: 435, 4335, *non-compact* 436*;*

(ii-2) all 4 with cell δ_{d-1}: hyperbolic non-compact 443, 444, 4343, 43343*;*

(ii-3) all 4 with cell 63*: hyperbolic non-compact* 633, 634, 635, 636.

All embeddable regular tilings or honeycombs, except any α_n and (for $n \geq 5$) any β_n (see Remark 2 below) are l_1-rigid. All bipartite ones are embeddable (with scale 1) tilings; in particular, all 11 above hyperbolic tilings embed into $\mathbf{Z}_\infty$.

Proof of Theorem 4

In fact, we review all cases of Tables 3-5. All compact cases (on first 5 rows,columns of Table 3 and first 6 rows,columns of Table 4) were decided in [DSt97]. Non-embeddability for all 14 star-polytopes and star-honeycombs (in Tables 3, 4) was established in section 3. It remains 11, 2, 5 non-compact tilings of hyperbolic 3-, 4-, 5-space; we will show that 7, 1, 1, respectively, of them are embeddable into $\mathbf{Z}_\infty$, while 8 others are not 5-gonal.

The tilings 3434, 34333, 33433, 34334 have non-5-gonal graph $K_5 - K_3$ as induced subgraph of the cell. 363 (respectively,344) contain induced $K_5 - K_3$, because each its edge is common to 3 (respectively, to 4) triangles. 336 is a simplicial manifold with 6 triangles on an edge; taking 1-st, 3-rd and 5-th of them, we get again induced $K_5 - K_3$. A particularity of $T := 33343$ is that the cell β_4 of its vertex figure $De(D_4)$ is also the equatorial section of the cell β_5 of T. All neighbors of a vertex s of T form $De(D_4)$. Take an isometric subgraph $K_5 - K_3$ in $De(D_4)$, given in [DSt98a]. The vertex s is a neighbor of each of its five vertices; obtained 6-vertex graph is non-5-gonal graph of diameter 2, which is, using above particularity of T, is an induced subgraph of T. (Compare with embeddable tiling 43343 by γ_5, having the same vertex figure.) All seven above tilings are not 5-gonal, because any induced graph of diameter 2 is isometric. Finally, each edge of 536 is common to 6 disjoint pentagons; taking 1-st, 3-rd and 5-th of them we obtain non-5-gonal 11-vertex induced subgraph of diameter 4 of 536; a routine check shows that it is isometric.

Other hyperbolic tilings embed into $\mathbf{Z}_\infty$, because of Lemma 5 below; it is easy to find reflections, required by Lemma 5 in each case. It is easy to check l_1-rigidity (for cases of embedding) for dimension 2; any bipartite embeddable graph is l_1-rigid, because it has scale 1. The proof is complete.

Let T be any (not necessary regular) convex d-polytope or tiling of Euclidean or hyperbolic d-space by convex polytopes, such that the skeleton is a bipartite graph. (We admit infinite cells and, if regular, infinite vertex figures.) Then the set of its edges can be partitioned into *zones*, i.e. sequences of edges, such that any edge of a sequence is the opposite to the previous one on an (even necessary) 2-face.

Lemma5. *Let T is as above; suppose that the mid-points of edges of each zone lie on hyperplanes, different for each zone, which are (some of) reflection hyperplanes of T and perpendicular to edges of their zones. Then T embeds into $\mathbf{Z}_m$ with m no more than the number of zones.*

Proof of Lemma 5

It follows directly from the fact that each geodesic path (in the skeleton of T) intersects any zone in at most one edge.

Remark 1 Embedding of any bipartite regular tiling can be obtained, using Lemma 5. The reflections, required by Lemma 5 (let us call them *zonal* reflections) generate (because of simple connectedness of T) a vertex-transitive group of automorphisms of T (call it *zonal* group; so T is uniform and the zonal group is generated by the zonal reflections of all edges incident to a fixed vertex of T. For any fixed $2k$-gonal 2-face of T, let $m_1,...,m_k$ be the zonal reflections of its edges, considered in the cyclic order. Then the product $m_1...m_k m_1...m_k =< 1 >$ (i.e. $m_1...m_k$ is an involution) and those relations, for all 2-faces around a fixed vertex of T, are all defining relations for the zonal group of T. So, the zonal group is not 2-transitive on vertices. For example, the zonal group of Archimedean truncated β_3 is an 1-transitive subgroup of index 2 of the octahedral group $Aut(T) = O_h$, which is 2-transitive. Also, a polytope in the conditions of Lemma 5 is not necessary zonotope. For example, any centrally-symmetric non-Archimedean (by choice of the length of truncation) truncated β_3 fits in it; it is a zonohedron in original sense of Fedorov, but not in usual sense of Minkowski (with all edges of each zone having same length).

Remark 2

All infinite families of regular tilings are embeddable. In fact, m-gons, $\delta_{n-1} = \mathbf{Z}_n$, $\gamma_n = H_n$, α_n, β_n are embeddable and, moreover, first three are l_1-rigid. But embeddings of skeletons of α_n and, for $n \geq 4$, β_n, is more complicate. It is considered in detail (in terms of corresponding complete graph K_{n+1} and Cocktail-Party graph $K_{n\times 2}$ in Chapter 23 [DLa97] and Section 4 of Chapter 7 [DLa97], respectively. Any $\alpha_n, n \geq 3$ is not l_1-rigid, i.e. it admits at least two different embeddings. We give now two embeddings of α_n into m-cubes with scale λ , realizing, respectively, maximum and minimum of $\frac{m}{\lambda}$. The first one is $\alpha_n \rightarrow \frac{1}{2}H_{n+1}$. Now define $m_n = \frac{2n}{n+1}$ for odd n and $= \frac{2n+2}{n+2}$ for even n; define λ_n be the minimal even positive number t such that tm_n is an integer. Then α_n embeds into tm_n-cube with scale λ_n. Any β_n, $n > 4$, is not l_1-rigid. All embeddings of β_n are into 2λ-cube with any such even scale λ that α_{n-1} embeds into m-cube, $m \geq 2\lambda$ with scale λ. For minimal such scale, denote it μ_n, the following is known: $n > \mu_n \geq 2\lceil \frac{n}{4} \rceil$ with equality in the lower bound for any $n \leq 80$ and, in the case of n divisible by 4, if and only if there exists an Hadamard matrix of order n. In particular, $\beta_3 \to \frac{1}{2}H_4$, $\beta_4 \to \frac{1}{2}H_4$ (in fact, they coincide), but β_5 (the smallest non-l_1-rigid one) embeds only with scale 4 (into H_8).

Remark 3

This note finalizes the study of embeddability for regular tilings done in [DSt96], [DSt97]; we correct now following misprints there: a) in the sentence "Any l_1-graph, not containing K_n, is l_1-rigid" on p.199 [DSt96], should be K_4 instead of K_n; b) in the sentence, on p.200 [DSt96], about partitions of Euclidean plane, embeddable into $\mathbf{Z}_m$, $m < \infty$, should be $\leq$ instead of $<$; c) in the sentence about Föppl partition on p.155 [DSt97], should be α_3 and truncated α_3 instead of α_3.

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