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Uniform partitions of 3-space, their relatives and embedding

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Abstract

We review 28 uniform partitions of 3-space in order to find out which of them have graphs (skeletons) embeddable isometrically (or with scale 2) into some cubic lattice $\mathbf{Z}_n$. We also consider some relatives of those 28 partitions, including Achimedean 4-polytopes of Conway-Guy, non-compact uniform partitions, Kelvin partitions and those with unique vertex figure (i.e. Delone star). Among last ones we indicate two continuums of aperiodic tilings by semi-regular 3-prisms with cubes or with regular tetrahedra and regular octahedra. On the way many new partitions are added to incomplete cases considered here.

1 Introduction

A polyhedron is called *uniform* if all its faces are regular polygons and its group of symmetry is vertex-transitive. A normal partition of 3-space is called *uniform* if all is facets (cells) are uniform polyhedra and group of symmetry is vertex-transitive. There are exactly 28 uniform partitions of 3-space. A short history of this result follows. Andreini in 1905 proposed, as the complete list, 25 such partitions. But one of them ($13'$, in his notation) turns out to be not uniform; it seems, that Coxeter [Cox35], page 334 was the first to realize it. Also Andreini missed partitions 25-28 (in our numeration given below). Till recent years, mathematical literature was abundant with incomplete lists of those partitions. See, for example, [Cri70], [Wil72] and [Pea78] (all of them does not contain 24-28) and [Gal89]. The first to publish the complete list was Grünbaum in [Grü94]. But he wrote there that, after obtaining the list, he realized that the manuscript [Joh91] already contained all 28 partitions. We also obtained all 28 partitions independently but, perhaps, the full classification should be attributed to Andreini-Johnson-Grünbaum.

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We say that given partition P has a l_1-*graph* and embeds up to *scale* λ into the cubic lattice $\mathbf{Z}_m$, if there exists a mapping f of the vertex-set of the skeleton of P into $\mathbf{Z}_m$ such that

$$\lambda d_P(v_i, v_j) = ||f(v_i), f(v_j)||_{l_1} = \sum_{1 \le k \le m} |f_k(v_i) - f_k(v_j)| \text{ for all vertices } v_i, v_j.$$

We take, of course, the smallest such number λ.

Call an l_1-partition l_1-*rigid*, if all its embeddings (as above) into cubic lattices are pairwise equivalent. All embeddable partitions in this paper turn out to be l_1-rigid and so having scale 1 or 2. Those embeddings were obtained by constructing a complete system of alternated zones; see [CDG97], [DSt96], [DSt97], [DSt98].

The following 5-gonal inequality [Dez60] is an important necessary condition for embedding of graphs:

$$d_{xy} + (d_{ab} + d_{ac} + d_{bc}) \le (d_{xa} + d_{xb} + d_{xc}) + (d_{ya} + d_{yb} + d_{yc})$$

for distances between any five its vertices a, b, c, x, y. It turns out that all non-embeddable partitions considered in this paper are, moreover, not 5-gonal.

Denote by $De(T)$ and $Vo(T)$ the Delone and Voronoi partitions of 3-space associated with given set of points T. By an abuse of language, we will use same notation for the graph, i.e. the skeleton of a partition. The Voronoi and Delone partitions are dual one to each other (not only combinatorially, but metrically). Denote by P^* the partition dual to partition P; it should not be confounded with the same notation for dual *lattice*.

2 28 uniform partitions

In the Table 1 of 28 partitions, the meaning of the column is:

1. the number which we give to the partition;
2. its number in [And05] if any;
3. its number in [Grü94];
4. a characterization (if any) of the partition;
5. tiles of partition and their numbers in Delone star;

5*. tiles of its dual;

6. embeddability (if any) of partition;

6*. embeddability (if any) of its dual.

Notation $\frac{1}{2}\mathbf{Z}_m$ in columns 6, 6* means that the embedding is isometric up tp scale 2.

Notation for the tiles given in the Table 1 is: tr$P$ for truncated polyhedron P; $Prism_n$ for semi-regular n-prism; α_3, β_3 and γ_3 for the Platonic tetrahedron, octahedron and cube; Cbt and $Rcbt$ for Archimedean Cuboctahedron and Rhombicuboctahedron; $RoDo$, tw$RoDo$ and $RoDo - v$ for Catalan Rhombic Dodecahedron, for its twist and for $RoDo$ with deleted vertex of valency 3; Pyr_4 and $BPyr_3$ for corresponding pyramid and bipyramid; BDS^* for dual bidisphenoid.

Remark that [Cox35] considered 12 of all 28 partitions; namely, No's 8, 7, 18, 2, 16, 23, 9 denoted there as $t_A\delta_4$ for $A = \{1\}$, $\{0,1\}$, $\{0,2\}$, $\{1,2\}$, $\{0,1,2\}$, $\{0,1,3\}$, $\{0,1,2,3\}$, respectively, and No's 6, 5, 20, 19, 17 denoted as $q\delta_4$, $h\delta_4$, $h_2\delta_4$, $h_3\delta_4$, $h_{2,3}\delta_4$.

Table 1. Embedding of uniform partitions and their duals.

1	2	3	4	5		5*	6	6*
No				tiles		of dual	emb.	dual
1	1	22	$De(\mathbf{Z}_3) = Vo(\mathbf{Z}_3)$	γ_3	8	γ_3	$\mathbf{Z}_3$	$\mathbf{Z}_3$
2	3	28	$Vo(A_3^* = \mathrm{bcc})$	$tr\beta_3$	4	$\sim \alpha_3$	$\mathbf{Z}_6$	–
3	4	11	$De(A_2 \times \mathbf{Z}_1)$	$Prism_3$	12	$Prism_6$	$\frac{1}{2}\mathbf{Z}_4$	$\mathbf{Z}_4$
4	5	26	$Vo(A_2 \times \mathbf{Z}_1)$	$Prism_6$	6	$Prism_3$	$\mathbf{Z}_4$	$\frac{1}{2}\mathbf{Z}_4$
5	2	1	$De(A_3 = \mathrm{fcc})$	α_3, β_3	$8, 6$	$RoDo$	$\frac{1}{2}\mathbf{Z}_4$	$\mathbf{Z}_4$
6	13	6	Föppl partition	$\alpha_3, tr\alpha_3$	$2, 6$	$\sim \gamma_3$	–	$\mathbf{Z}_4$
7	14	8	boron CaB_6	$\beta_3, tr\gamma_3$	$1, 4$	Pyr_4	–	–
8	15	7	$De(J\text{-complex})$	β_3, Cbt	$2, 4$	$\sim \beta_3$	–	–
9	22	27		$Prism_8,$ $trCbt$	$2, 2$	$\sim \alpha_3$	$\mathbf{Z}_9$	–
10	6	24	$De(x(4.8^2 \times \mathbf{Z}_1)$	$Prism_8, \gamma_3$	$4, 2$	$\sim Prism_3$	$\mathbf{Z}_5$	–
11	8	18	$De(3.6.3.6 \times \mathbf{Z}_1)$	$Prism_3,$ $Prism_6$	$4, 4$	$\sim \gamma_3$	–	$\mathbf{Z}_4$
12	12	17	$De(3^4.6 \times \mathbf{Z}_1)$	$Prism_3,$ $Prism_6$	$8, 2$	$\sim Prism_5$	$\frac{1}{2}\mathbf{Z}_7$	–
13	11	13	$De(3^3.4^2 \times \mathbf{Z}_1)$	$Prism_3, \gamma_3$	$6, 4$	$\sim Prism_5$	$\frac{1}{2}\mathbf{Z}_4$	–
14	11′	14	$De(3^2.4.3.4 \times \mathbf{Z}_1)$	$Prism_3, \gamma_3$	$6, 4$	$\sim Prism_5$	$\frac{1}{2}\mathbf{Z}_5$	–
15	7	19	$De(3.12^2 \times \mathbf{Z}_1)$	$Prism_3,$ $Prism_{12}$	$2, 4$	$\sim Prism_3$	–	$\frac{1}{2}\mathbf{Z}_\infty$
16	18	25	a zeolit	$\gamma_3, tr\beta_3, trCbt$	$1, 1, 2$	$\sim \alpha_3$	$\mathbf{Z}_9$	–
17	20	21		$tr\alpha_3, tr\gamma_3,$ $trCbt$	$1, 1, 2$	$\sim \alpha_3$	–	–
18	17	9		$\gamma_3, Cbt, Rcbt$	$2, 12$	$\sim BPyr_3$	–	–
19	16	5		$\alpha_3, \gamma_3, Rcbt$	$1, 3, 1$	$\sim BPyr_3$	$\frac{1}{2}\mathbf{Z}_7$	–
20	21	10	boron UB_{20}	$tr\alpha_3, tr\beta_3, Cbt$	$2, 1, 2$	$\sim Pyr_4$	–	–
21	9	16	$De(3.4.6.4 \times \mathbf{Z}_1)$	$\gamma_3, Prism_3,$ $Prism_6$	$4, 2, 2$	$\sim \gamma_3$	$\frac{1}{2}\mathbf{Z}_4$	–
22	10	23	$De(4.6.12 \times \mathbf{Z}_1)$	$\gamma_3, Prism_3,$ $Prism_{12}$	$2, 2, 2$	$\sim Prism_3$	$\mathbf{Z}_7$	–
23	19	20		$\gamma_3, Prism_8,$ $tr\gamma_3, Rcbt$	$1, 2, 1, 1$	$\sim Pyr_4$	–	–
24	2′	2	$De(\mathrm{hcp})$	α_3, β_3	$8, 6$	$twRoDo$	–	–
25	–	3	$De(\text{elong. } A_3)$	$Prism_3, \beta_3, \alpha_3$	$6, 4, 3$	$RoDo - v$	$\frac{1}{2}\mathbf{Z}_4$	$\mathbf{Z}_4$
26	–	4	$De(\text{elong. hcp})$	$Prism_3, \beta_3, \alpha_3$	$6, 4, 3$	$RoDo - v$	–	–
27	–	12		$Prism_3$	12	BDS^*	$\frac{1}{2}\mathbf{Z}_5$	–
28	–	15	$De(\text{elong. } 27)$	$Prism_3, \gamma_3$	$6, 4$	$\sim Prism_5$	$\frac{1}{2}\mathbf{Z}_5$	–

Remarks on the Table 1:

1. The partition 15^* is only one embeddable into $\mathbf{Z}_\infty$ (in fact, with scale 2).

2. All partitions embeddable with scale 1 are, except 25^*, zonohedral. The Voronoi tile of 25^* is not centrally-symmetric. It will be interesting to find a normal tiling of 3-space

embeddable with scale 1 such that the tile is centrally-symmetric; such non-normal tiling is given in [Sht80]: see item 35 in Table 2 below.

3. Embedding of its tiles is necessary but not sufficient for embedding of whole tiling; for example, 26^* and 27^* are not embeddable while their tiles embeddable into H_4 and $\frac{1}{2}H_8$, respectively.

4. Among all 28 partitions only No's 1, 2, 5, 6, 8 have same surrounding of edges: polygons (4.4.4.4), (4.6.6), (3.3.3.3), (3.3.6.6), (3.3.4).

5. Partitions 8 and 24 are Delone partitions of lattice complexes: 3-lattice called J-complex and bi-lattice hcp; the tile of Vo(J-complex) has form of jackstone (it explains the term "J-complex")) and it is combinatorially equivalent to β_3.

6. Partitions 1, 3, 5 are Delone partitions of lattices $\mathbf{Z}_3$, $A_2 \times \mathbf{Z}_1$, A_3=fcc. Partitions 2 and 4 are Voronoi partitions of lattices A_3^* =bcc and $A_2 \times \mathbf{Z}_1$. No's 10, 11, 12, 13, 14, 15, 21, 22 are Delone prismatic partitions over 8 Archimedean partitions of the plane.

7. Partitions 7 and 20 occur in Chemistry as borons CaB_6 and UB_{20}, respectively. Partition 20 occurs in zeolites.

8. The ratio of tiles in partition is 1:1 for 6, 7, 8, 10; 2:1 for 5, 11, 13, 14, 24, 28; 3:1 for 9; 8:1 for 12; 2:1:1 for 17, 19, 20; 3:1:1 for 16, 18; 3:2:1 for 21, 22, 25, 26; 3:3:1:1 for 23.

3 The table of other partitions

Table 2. Embedding of some other partitions.

29	$De(L_5)$	α_3, Pyr_4	$ElDo$	$\frac{1}{2}\mathbf{Z}_4$	$\mathbf{Z}_5$
30	$De(D$-complex$)$	$\alpha_3, \sim\beta_3$	triakis trα_3	$\frac{1}{2}\mathbf{Z}_5$	$-$
31	De(Kelvin)	α_3, β_3	$RoDo, twRoDo$	$-$	$-$
32	De(Grünbaum)	$Prism_3$	$Prism_6, BDS^*$	$\frac{1}{2}\mathbf{Z}_5$	$-$
33	De(elong. Kelvin)	$\alpha_3, \beta_3, Prism_3$	$RoDo - v$	$-$	$-$
34	De(elong. Grünbaum)	$Prism_3, \gamma_3$	$\sim Prism_5$	$\frac{1}{2}\mathbf{Z}_5$	$-$
35	$P(S_1)$	S_1		$\mathbf{Z}_3$	$-$
36	$P(S_2)$	S_2		$\mathbf{Z}_4$	$-$
37	$P(S_3)$	S_3		$\mathbf{Z}_5$	$-$
38	A-19	$Prism_\infty$		$\mathbf{Z}_2$	
39		$Prism_\infty$		$\mathbf{Z}_2$	
40	A-20, n even	$C_n \times P_\mathbf{Z}$		$\mathbf{Z}_\infty$	
	A-20, n odd	$C_n \times P_\mathbf{Z}$		$\frac{1}{2}\mathbf{Z}_\infty$	
41	A-22	$Aprism_\infty$		$\frac{1}{2}\mathbf{Z}_3$	
42	A-23	$Prism_\infty, Aprism_\infty$		$\frac{1}{2}\mathbf{Z}_3$	
43	$\| -type$	$\gamma_3, C_4 \times P_\mathbf{Z}$		$\mathbf{Z}_3$	
44	$\perp -type$	$\gamma_3, C_4 \times P_\mathbf{Z}$		$\mathbf{Z}_3$	
45	$chess - type$	$\gamma_3, C_4 \times P_\mathbf{Z}$		$\mathbf{Z}_3$	
46	A-13$'$	α_3, trα_3	R, twisted R	$-$	$-$

In the Table 2 we group some other relevant partitions. Here L_5 denotes a representative of 5-th Fedorov's type (i.e. by the Voronoi polyhedron) of lattice in 3-space and $ElDo$

denotes its Voronoi polyhedron, called elongated dodecahedron. Remaining four lattices appeared in the Table 1 as No 1 $=De(\mathbf{Z}_3) = Vo(\mathbf{Z}_3)$, No 5 $=De(A_3)$, No 2 $=Vo(A_3^*)$, No 3 $=De(A_2 \times \mathbf{Z}_1)$, No 4 $=Vo(A_2 \times \mathbf{Z}_1)$. Remark that $De(L_5)$ and $De(A_2 \times \mathbf{Z}_1)$ coincide as graphs, but differ as partitions.

In notation De(Kelvin) below we consider any Kelvin packing by α_3 and β_3 (in proportion 2:1) which is *proper*, i.e. different from the lattice A_3=fcc (face-centered lattice) and the bi-lattice hcp (hexagonal closed packing). Any proper Kelvin partition, as well as partition $13'$ in [And05] (given as 46 in Table 2 and which Andreini wrongly gave as uniform one), have exactly two vertex figures. (The Voronoi tiles of tiling 46 are a rhombohedron, say, R, i.e. the cube contracted along a diagonal, and twisted R; both are equivalent to γ_3.) The same is for Grünbaum partitions; see Section 5 below for those notions and items 32–34 of Table 2.

See Section 4 below for items 38–45 of Table 2. D-complex is the diamond bi-lattice; triakis trα_3 denotes truncated α_3 with Pyr_3 on each its triangular faces. Partitions 29 and 30 from Table 2 are both vertex-transitive, but they have some non-Archimedean tiles: Pyr_4 for 29 and non-regular octahedron for 30.

The partitions 35, 36, 37 of Table 2 are all 3 non-normalizable tilings of 3-space by convex parallelohedron, which where found in [Sht80]. The polyhedra denoted by S_1, S_2 and S_3 are centrally symmetric 10-hedra obtained by a decoration of the paralelipiped. S_1 is equivalent to β_3 truncated on two opposite vertices. $P^*(S_i)$ for $i = 1, 2, 3$ are different partitions of 3-space by non-convex bodies, but they have the same skeleton, which is not 5-gonal.

4 Non-compact uniform partitions

[And05] introduced in subsections 19, 20, 22, 23 non-compact uniform partitions, which we denote A-19, A-20, A-22, A-23 and put in Table 2 as No's 38, 40, 41, 42. Denote by $Prism_\infty$ ($Aprism_\infty$) and $C_n \times P_{\mathbf{Z}}$ the ∞-sided prisms (anti-prisms, respectively) and the cylinder on C_n.

A-19 is obtained by putting $Prism_\infty$ on (4^4) and so its skeleton is $\mathbf{Z}_2$. We add, as item 39, the partition which differs from 38 only by other disposition of infinite prisms *under* net (4^4), i.e. perpendicular to those above it.

A-20 is obtained by putting the cylinders on (4^4), A-22 by putting $Aprism_\infty$ on (3^6) and A-23 by putting $Prism_\infty$ and $Aprism_\infty$ on $(3^3.4^2)$.

In subsection $20'$ [And05] mentions also the partition into two half-spaces separated by some of 10 Archimedean (and one degenerated) nets, i.e. uniform plane partitions. We can also take two parallel nets (say, T) and fill the space between them by usual prisms (so, the skeleton will be direct product of the graph of T and K_2) or, for $T = (4^4)$ or $(3^3.4^2)$, by a combination of usual and infinite prisms. Similar uniform partitions are obtained if we will take an infinity of parallel nets T.

The partitions 43, 44, 45 of Table 2 differ only by the disposition of cubes and cylinders. In 43 the layers of cylinders stay parallel ($\|$-type); in 44 they are perpendicular to the cylinders of each previous layer. In 45 we see (4^4) as infinite chess-board; cylinders stay on "white" squares while piles of cubes stay on the "black" ones.

By a decoration of $Prism_\infty$ in 38, 39, one can get other non-compact uniform partitions.

5 Almost-uniform partitions

Call a normal partition of the 3-space into Platonic and Archimedean polyhedra, *almost-uniform* if the group of symmetry is not vertex-transitive but all vertex figures are congruent. Grünbaum [Grü94] gave two infinite classes of such partitions and indicated that he do not know other examples. In our terms, given below, they called *elongated proper Kelvin* and *elongated proper Grünbaum* partitions. A Kelvin and Grünbaum partition is defined uniquely by infinite binary sequence characterizing the way how layers follow each other. In Kelvin partition, the layers of α_3 and β_3 follow each other in two different ways (say, a and b) while in Grünbaum partition the layers of $Prism_3$ follow each other in parallel or perpendicular mutual disposition of heights. Unproper Kelvin partitions give uniform partitions 5 and 24 for sequences *...aaa...* (or *...bbb...*) and *...ababab...*, respectively. Proper Kelvin and Grünbaum partitions are not almost-uniform; there are even ∞-uniform ones (take a non-periodic sequence).

Consider now elongation of those partitions, i.e. we add alternatively the layers of $Prism_3$ for Kelvin and of cubes for Grünbaum partitions.

Remark that $RoDo - v$ (the Voronoi tile of partitions 25, 26, 33) can be seen as a half of $RoDo$ cut in two, and that tw$RoDo$ is obtained from $RoDo$ by a twist (a turn by 90^o) of two halves.The Voronoi tiles for proper Kelvin partition 31 are both $RoDo$ and tw$RoDo$ while only one of them remains for two unproper cases 5, 24. Similarly, the Voronoi tile of 34 (a special 5-prism) can be seen as a half of $Prism_6$ cut in two, and BDS^* can be seen as twisted $Prism_6$ in similar way. The Voronoi tiles for proper Grünbaum partition 32 are both $Prism_6$ and BDS^* while only one of them remains for two unproper cases 3, 27.

Besides of two unproper cases 25, 26 (elongation of uniform 5, 24) which are uniform, we have a continuum of proper elongated Kelvin partitions (denoted 33 in Table 2) which are almost-uniform. Among them there is a countable number of periodic partitions corresponding to periodic (a, b)-sequences. Remaining continuum consists of aperiodic filings of 3-space by α_3, β_3, $Prism_3$ with very simple rule: each has unique Delone star consisting of 6 $Prism_3$ (put together in order to form a 6-prism), 3 α_3 and 3 β_3 (put alternatively on 6 triangles subdividing the hexagon) and one α_3 filling remaining space in the star. Each b in the (a, b)-sequence, defining such tiling, corresponds to the twist interchanging 3 α_3 and 3 β_3 above (i.e. to the turn of the configuration of 4 α_3, 3 β_3 by 60^o).

Similar situation occurs for elongated Grünbaum partitions. Besides of two uniform unproper cases 13, 28 (elongation of uniform 3, 27), we have a continuum of proper elongated Grünbaum partitions (denoted 34 in Table 2) which are almost-uniform. The aperiodic (a, b)-sequences give a continuum of aperiodic tilings by $Prism_3$, γ_3 with similar simple rule: unique Delone star consisting of 4 γ_3 (put together in order to form a 4-prism), 4 $Prism_3$ put on them and 2 $Prism_3$ filling remaining space in the star. Each b in (a, b)-sequence, defining the tiling, corresponds to a turn of all configuration of 6 $Prism_3$ by 90^o.

6 Archimedean 4-polytopes

Finite relatives of uniform partitions of 3-space are 4-dimensional Archimedean polytopes, i.e. those having vertex-transitive group of symmetry and whose cells are Platonic or Archimedean polyhedra and prisms or anti-prisms with regular faces. [Con65] enumerated all of them:

1) 44 polytopes (others than prism on γ_3) obtained by Wythoff's kaleidoscope construction from 4-dimensional irreducible reflexion (point) groups;

2) 17 prisms on Platonic (other than γ_3) and Archimedean solids;

3) Prisms on n-anti-prisms with $n > 3$;

4) A doubly infinity of polytopes which are direct products of two regular polygons (if one of polygons is a square, then we get a prisms on 3-dimensional prisms);

5) Gosset's semi-regular polytope called *snub 24-cell*;

6) A new polytope, called *Grand Anti-prism*, having 100 vertices (all from 600-cell), 300 cells α_3 and 20 cells 5-anti-prisms (those anti-prisms form two interlocking tubes).

Using the fact that the direct product of two graphs is l_1-embeddable if and only if each of them is, and the characterization of embeddable Archimedean polyhedra in [DSt96], we can decide about embeddability in cases 2)–4). In fact, the answer is "yes" always in cases 2)–4), except prisms on trα_3, trγ_3, Cuboctahedron, truncated Icosahedron, truncated Dodecahedron and Icosidodecahedron, which are all not 5-gonal.

Now, the snub 24-cell embeds into $\frac{1}{2}H_{12}$ and the Grand Anti-prism (as well as 600-cell itself) violates 7-gonal inequality, which is necessary for embedding.

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