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A Zoo of ℓ_1-embeddable Polyhedra II

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A zoo of l_1-embeddable polyhedra II

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Abstract

We complete here the study of l_1-polyhedra started in our previous paper on this subject, [DeGr97]. New classes are considered, especially small polyhedra, some operations on Platonic solids and k-valent polyhedra with only two types of faces.

1 Introduction

We use definitions and notation from [DeGr97]. Call a polyhedron P l_1-*polyhedron* (or l_1-*embeddable* if its skeleton is embedded isometrically (or with doubled distance) into an m-cube, and use the notation $P \to H_m$ (or $P \to \frac{1}{2}H_m$, respectively).

In this paper we group, in compact form, many results specifying l_1-polyhedra in the classes defined in the titles of sections. All proofs are obtained by the techniques of [CDGr97], [DeGr97] and by direct check.

We remind only the following necessary condition (called *5-gonal inequality*) for embedding of a graph which is sufficient for any bipartite and "many" planar graphs:

$$F(x,y;a,b,c) := d_{xy} + (d_{ab} + d_{bc} + d_{ac}) - \sum_{p=x,y;q=a,b,c} d_{pq} \leq 0$$

for any distinct vertices x, y, a, b, c of the graph.

Moreover: (i)$F(x,y;a,b,c) \geq 4 - 6\max_{p=x,y;q=a,b,c} d_{pq}$ with equality, for example, for a graph such that x, y belong to a triangle $K_{x,y,z}$ and a, b, c belong to $K_{a,b,c,d}$, and the triangle and K_4 are joined by a path from z to d.

(ii) $F(x,y;a,b,c) \leq d_{xy}$ with equality iff both x and y lie on a geodesic between each pair (a,b), (a,c), (b,c) (for example, as in $K_{\{x,y\},\{a,b,c\}}$).

(iii) $F(x,y;a,b,c) \leq (d_{ab} + d_{ac} + d_{bc}) - 2d_{xy}$ with equality iff each of a, b, c lies on a geodesic between x and y (for example, as in (snub $APrism_n$)* for any $n \geq 5$, see §5).

2 Small polyhedra

l_1-status of all 10 polyhedra with at most 6 faces and their duals is such that they are either l_1-embeddable or non 5-gonal.

The l_1-embeddable are

$\to \frac{1}{2}H_4 : \alpha_3 \simeq \alpha_3^*$, $Pyr_4 \simeq Pyr_4^*$, $BPyr_3 = Prism_3^*$, $\gamma_3^* = \beta_3$,

$\to \frac{1}{2}H_5$: $Pyr_5 \simeq Pyr_5^*$, $Prism_3 \simeq BPyr_3^*$, (2-truncated $\alpha_3)^*$, a dual with skeleton $K_6 - P_5$ (# 43 in Proposition 2.1);

$\to \frac{1}{2}H_6 : \gamma_3$, 2-truncated α_3.

Remaining are: self-dual one with the skeleton $K_6 - P_6$, one with dual skeleton $K_{3\times 2} - e$, its dual (# 44 in Proposition 2.1 below) and one with dual skeleton $K_6 - P_5$.

Proposition 2.1. *Between all 34 polyhedra (# # 11–44 on Fig.2) with 7 faces and their duals, we have*

$\to \frac{1}{2}H_5$*:* 20^*, 21^*, 29^*, 35, 36^*, 38^*, $39 \simeq 39^*$, 43*;*

$\to \frac{1}{2}H_6$*:* 12^*, 14^*, 16^*, 17^*, 18, $19 \simeq 19^*$*;*

$\to \frac{1}{2}H_7$*:* 12, 20, 23*;*

extreme hypermetric: $13^* = G_4$*;*

non 7-gonal: 32^*, 37^*, *and*

non 5-gonal: all others (incl. self-dual 34, 41 *and* $40^* = 42$*).*

Remark. (i) Between above l_1-polyhedra only γ_3 is bipartite and only 3 are not l_1-rigid: $\alpha_3 \simeq \alpha_3^* \to \frac{1}{2}H_3, \frac{1}{2}H_4$, $BPyr_3$ and $39 \simeq 39^* \to \frac{1}{2}H_5$, $\frac{1}{2}H_6$. All simple ones between above 10+34 polyhedra with at most 7 faces are: α_3, $Prism_3$ =1-truncated α_3, 2-truncated α_3, γ_3 and 11, 12 (3-truncated α_3), 13, 20 (1-truncated γ_3), 23=$Prism_5$.

(ii) 20 between all 44 polyhedra with at most 7 faces are combinatorially equivalent to a space-filler: all 10, except Pyr_5, with at most 6 faces (including all 3 non 5-gonal) and # # 12, 15, 18, 20, 22, 23, 28, 30, 36, 43, 44 (between them # # 12, 18, 20, 23, 43 are l_1-graphs).

Proposition 2.2. *Between all 27 cubic graphs on up to 10 vertices, 19 are non 5-gonal while remaining 8 are l_1-graphs: non-polytopal Petersen graph $\to \frac{1}{2}H_6$ and 7 simple polyhedra: i-truncated $\alpha_3 \to \frac{1}{2}H_{4+i}$ for $0 \le i \le 3$, i-truncated $\gamma_3 \to \frac{1}{2}H_{6+i}$ for $i = 0, 1$ and $Prism_5 \to \frac{1}{2}H_7$.*

Proposition 2.3. *Between all 14 simple polyhedra with 8 faces (i.e. with 12 vertices) 4 are l_1-graphs (all are embedded into $\frac{1}{2}H_8$): $Prism_6$, dual bisdisphenoid, Dürer's octahedron and γ_3 truncated on 2 adjacent vertices. Duals of first two are not 5-gonal and of last two $\to \frac{1}{2}H_6$. 4-truncated α_3 and γ_3 truncated on 2 vertices at distance 2 are not 5-gonal; their duals $\to \frac{1}{2}H_7$. 5 polyhedra from Fig.3 are resp. non 5-, 5-, 7-, 7-, 9-gonal; their duals are resp. embeddable into $\frac{1}{2}H_8$, $\frac{1}{2}H_7$, $\frac{1}{2}H_7$, $\frac{1}{2}H_6$, non 5-gonal. Remaining 3 polyhedra and their duals are non 5-gonal.*

The Dürer's octahedron above is γ_3 truncated on 2 opposite vertices; it is called so, because it appears in Dürer's "Melancolia", 1514, staying on a triangular face.

Remark. (i) All simple l_1-polyhedra with f, $f \le 8$, faces (there are 11 of them) embed into $\frac{1}{2}H_f$; all of them, except α_3, are l_1-rigid and $\gamma_3 \to H_3$, $Prism_6 \to H_4$.

(ii) the polyhedron in centre of Fig.3 and one of 3 non 5-gonal simple octahedra, having non 5-gonal duals, are two smallest 3-regular graphs with trivial group of automorphisms.

(iii) Between all 11 simple polyhedra with only k-gons, $k \le 5$, as faces (duals of all 8 convex deltahedra, 1-truncated α_3, 1-truncated γ_3 and Dürer's octahedron) only 2 (duals of 3-augmented $Prism_3$ and of 2-capped $APrism_4$) are not l_1-graphs.

3 Truncations and cappings of Platonic solids

Call by *i-truncation* and *i-capping* the (short) truncation on i vertices of a polytope P and, respectively, adding a pyramid on i its faces. A *triakis* (*tetrakis, pentakis, hexakis*) i-capping is an i-capping only on 3-faces (4-,5-,6-faces, respectively).

Proposition 3.1.

(i) i-truncated $\alpha_3 \to \frac{1}{2}H_{4+i}$, $0 \le i \le 3$; *4-truncated* α_3 *is not 5-gonal;*

dual i-truncated $\alpha_3 =$ *i-capped* $\alpha_3 \to \frac{1}{2}H_{3+i}$, $0 \le i \le 4$;

(ii) i-truncated $\beta_3 \to \frac{1}{2}H_4, \frac{1}{2}H_6, H_4, H_6$ *for* $i = 0,1,2$ *(on opposite vertices), 6(=all), non 7-gonal if only 2 vertices of an edge are not truncated; non 5-gonal otherwise.*

dual i-truncated $\beta_3 =$ *i-capped* $\gamma_3 \to \frac{1}{2}H_6$ *if* $i \le 2$ *or* $i = 3$ *and any 2 capped faces of* γ_3 *are not opposite; otherwise non 5-gonal.*

(iii) i-truncated $\gamma_3 \to \frac{1}{2}H_{6+i}$ *if* $i = 0,1,2$ *(truncated vertices are at distance 1 or 3), 3 (they form* P_3*), 4 (they form* C_4*), otherwise non 5-gonal;*

dual i-truncated $\gamma_3 =$ *i-capped* $\beta_3 \to \frac{1}{2}H_{4+i}$ *for* $0 \le i \le 8$.

(iv) i-capped Icosahedron $\to \frac{1}{2}H_{6+i}$ *for* $0 \le i \le 20$;

i-capped Dodecahedron $\to \frac{1}{2}H_{10}$ *for* $0 \le i \le 12$.

Remark. (i) Between above l_1-graphs only non l_1-rigid one is $\alpha_3 \to \frac{1}{2}H_3, \frac{1}{2}H_4$.

(ii) 4-truncated γ_3, on 4 non-adjacent vertices and on 4 vertices forming 2 opposite edges, are $Cham(\alpha_3)$ and twisted $Cham(\alpha_3)$, respectively. Remaining 4 ways to 4-truncate γ_3 are when these 4 vertices induce one of the graphs C_4, P_4, $P_3 + K_1$ and $K_{1,3}$. All 3 ways to 3-truncate γ_3 are on P_3, $P_2 + K_1$ and $3K_1$.

Consider now capping of some almost regular l_1-polyhedra on 3- and 4-gonal faces.

Clearly, any triakis i-capping of an l_1-polyhedron P embeds into $\frac{1}{2}H_{m+i}$ if $P \to \frac{1}{2}H_m$.

1) i-capped $Pyr_4 \to \begin{cases} \frac{1}{2}H_{4+i} & \text{if the 4-face is truncated,} \\ \frac{1}{2}H_{3+i} & \text{otherwise;} \end{cases}$

1-truncated Pyr_4 (=dual 1-capped Pyr_4)$= \gamma_3$ if the apex is truncated, or # 43^* otherwise; (so, non 5-gonal).

2) *i-augmented* $Prism_m$ is $Prism_m$ capped on i (4-gonal) faces, which are not 2 base faces. i-augmented $Prism_m (m = 3,4) \to \frac{1}{2}H_{m+2}$ for $i = 0,1,2$, non l_1 for $i \ge 3$.

3) Snub cube $\to \frac{1}{2}H_9$ but any tetrakis i-capping of it is non 5-gonal.

4) Rhombicuboctahedron $\to \frac{1}{2}H_{10}$. Only 5-gonal tetrakis i-capping of it (which, moreover, embeds into $\frac{1}{2}H_{10}$) is capping on at most 2 or on 3 non-opposite 4-faces, chosen between 6 4-faces adjacent only to 4-faces. Remark similarity with cappings of γ_3 (Proposition 3.1 (ii)).

5) Tetrakis omni-capping of truncated $\beta_3 \to \frac{1}{2}H_{12}$.

6) Omni-cappings of α_3 (the $3^*_{12}(T_d)$), of truncated α_3 (the $3^*_{36}(T_d)$) $\to \frac{1}{2}H_7, \frac{1}{2}H_{11}$, respectively, but omni-cappings of $3_{36}(T_d)$ (the $3^*_{108}(T_d)$) and of truncated cube are not l_1.

4 Chamfering of Platonic solids

Cham P denotes the *chamfering* of the polyhedron P. It is obtained by putting prisms on all faces of P and deleting original edges; see also §6.

Proposition 4.1 *Let P be one of five Platonic solids and $t \geq 1$. Then $Cham_t P$ or its dual is l_1-graph only in the following cases:*

(i) $(Cham\alpha_3)^* \rightarrow \frac{1}{2}H_8$,

(ii) $Cham\gamma_3 \rightarrow H_7$ (zonohedron, generated by $e_1, e_2, e_3, e_1 \pm e_2, e_1 \pm e_3$),

(iii) Cham(Dodecahedron)$\rightarrow \frac{1}{2}H_{22}$

Remark.

(i) $ChamP$ is partially truncated zonohedron in the following cases:

$Cham\alpha_3 = \gamma_3$, truncated on 4 non-adjacent vertices;

$Cham\beta_3$, $Cham\gamma_3$ are rhombic dodecahedra truncated on all 3-valent (resp., all 4-valent) vertices,

$Cham(Icosahedron)$, $Cham(Dodecahedron)$ are triacontrahedra truncated on all 3-valent (resp., all 5-valent) vertices.

(ii) Moreover, we have $Cham_t\alpha_3$ $(t = 1, 2)$, $Cham_t\beta_3$ $(t = 1, 2)$, $Cham_t\gamma_3$ $(t \geq 2)$, $Cham_2(Icosahedron)$, $(Cham_t\beta_3)^*$ $(t \geq 1)$, $(Cham\gamma_3)^*$, $(Cham(Icosahedron))^*$, $(Cham(Dodecahedron))^*$ are not 5-gonal. Example of proving it: $ChamP$ is not 5-gonal if the polyhedron P contains induced $K_4 - e$, because then $ChamP$ contains isometric non 5-gonal graph G_{11} consisting of the cycle C_{10} with a new point adjacent only to the points 1,5,6 of the cycle C_{10}.

(iii) $Cham(Prism_6)$, $Cham(Rhombic\ Dodecahedron)$ are not 5-gonal.

Iterated chamferings and their duals are used in the image compressing; for example, $(Cham_t\gamma_3)^*$ appears there as $(t+1)$-th approximation of some fractal distribution on β_3.

The notion of chamfering can be extended on non-polyhedral graphs. For example, $Prism_m$ can be seen as the chamfering of the cycle C_m. On the other hand, this notion can be extended naturally for a simple n-polytope P. Denote by $Cham\ P$ the dual of the convex hull of all verices of P^* and of the mid-points of all edges of P^*. As a generalization of Proposition 4.1(i), we get $(Cham\ \alpha_n)^* \rightarrow \frac{1}{2}H_{2n+2}$. Remark that $\alpha_n \rightarrow \frac{1}{2}H_{n+1}$, $ambo\ \alpha_n \rightarrow \frac{1}{2}H_{n+1}$ (see §6) and $(Cham\ \alpha_n)^* = \mathrm{conv}(V(\alpha_n) \cup V(ambo\ \alpha_n))$.

5 Bifaced polyhedra

Denote by $(k; a, b; p_a, p_b)$ and call *bifaced* any k-valent polyhedron whose faces are only p_a a-gons and p_b b-gons, where $3 \leq a < b$ and $p_a > 0 \leq p_b$. Any bifaced polyhedron $(k; a, b; p_a, p_b)$ with n vertices has $kv/2 = (ap_a + bp_b)/2$ edges and satisfies the Euler relation $n - k\frac{n}{2} + (p_a + p_b) = 2$, i.e. $n = 2(p_a + p_b - 2)/(k-2)$ and

(*) $p_a(2k - a(k-2)) + p_b(2k - b(k-2)) = 4k$.

So, $a < 2k/(k-2)$ and only possible (k, a) are (3,5), (3,4), (3,3), (4,3), (5,3).

The case $p_b = 0$ of above 5 classes is (combinatorially) Dodecahedron, γ_3, α_3, β_3, Icosahedron, respectively. For $p_b = 1$ and for $p_b = 2$, $(k, a) = (3, 3)$, such polyhedra do not exist. For $p_b = 2$ classes $(k, a) =$ (3,4), (4,3), (3,5), (5,3) consist of a unique polyhedron each: $Prism_b$, $APrism_b$, $Barrel_b :=$ (2-capped $APrism_b)^*$ and $snub\ APrism_b$ consisting of the following 3 circuits: outer $(x_1, ..., x_b)$, middle $(y_1, y_1', ..., y_b, y_b')$, and inner $(z_1, ..., z_b)$, where $x_i \sim y_i, y_i', y_{i+1}$ and $z_i \sim y_{i-1}', y_i, y_i'$ for all $i = 1, ..., b$ ordered cyclically. The last polyhedron is called *snub* $APrism_b$, since for $b = 4$ it is (combinatorially) well known regular-faced *snub* $APrism_4$ which is 5-gonal, but not 7-gonal. $Barrel_b$ is an analog of

$Prism_b$, where the layer of 4-gons is replaced by two layers of 5-gons; it has $4b$ vertices. We have

Proposition 5.1 *(i)* $Prism_b \to \frac{1}{2}H_{b+2}$ *(moreover,* $\to \frac{1}{2}H_{(b+2)/2}$ *for even* b*),*
$Prism_b^* \to \frac{1}{2}H_4$, $\frac{1}{2}H_4$, *non 5-gonal for* $b = 3, 4 \geq 5$, *respectively;*
(ii) $APrism_b \to \frac{1}{2}H_{b+1}$,
$APrism_b^* \to H_3$, *non 5-gonal for* $b = 3, \geq 4$*;*
(iii) $Barrel_3$ *=Dürer's octahedron* $\to \frac{1}{2}H_8$, $Barrel_3^* \to \frac{1}{2}H_6$,
$Barrel_5$ *=Dodecahedron* $\to \frac{1}{2}H_{10}$, $Barrel_5^* \to \frac{1}{2}H_6$,
$Barrel_4$ *is non 5-gonal,* $Barrel_4^*$ *(a convex deltahedron) is an extereme hypermetric. For* $b > 5$ *both* $Barrel_b$ *and its dual are non 5-gonal;*
(iv) snub $APrism_3 = Icosahedron \to \frac{1}{2}H_6$*, its dual* $\to \frac{1}{2}H_{10}$*;*
snub $APrism_4$ *is not 7-gonal, its dual is not 5-gonal.*
For $b \geq 5$ *both snub* $APrism_b$ *and its dual are not 5-gonal.*

From now on we consider mainly bifaced polyhedra with $p_b \geq 3$.

Consider first *simple* bifaced polyhedra, i.e. the case of $k = 3$. For $b < 6$ there are only 6 such polyhedra: the Dürer's octahedron (γ_3 truncated on two opposite vertices) and 5 dual deltahedra ($Prism_3$ and duals of $BPyr_5$, of bisdisphenoid, of 3-augmented $Prism_3$, of 2-capped $APrism_4$). Moreover, there are only 11 simple polyhedra with $\sum_{i\geq 3} p_i = p_3 + p_4 + p_5$: 6 above polyhedra, remaining 3 dual deltahedra (α_3, γ_3, Dodecahedron) and 1-truncated γ_3, 2-truncated α_3, having $(p_3, p_4, p_5) = (1, 3, 3)$, (2,2,2), respectively. The l_1-status of all these polyhedra and their duals is known (see §2 above).

Simple bifaced n-vertex polyhedra with $b = 6$ will be denoted as 3_n, 4_n, 5_n if $a = 3, 4, 5$, respectively. They will be considered in the next sections as well as interesting case of $(4; 3, 4; p_3, p_4)$, having constant $p_3 = 8$; denote such a polyhedron by $(3, 4)_n$.

Simple bifaced polyhedra with $b > 6$ were considered in [Malk70]. Besides Euler's relation $(6 - a)p_a = 12 + (b - 6)p_b$, we have

if $a = 3$, then $b \in \{7, 8, 9, 10\}$;

if $a = 4$, $b \equiv 0 \pmod 8$, then p_b is even;

if $a = 5$, $b \equiv 0 \pmod{10}$, then p_b is even.

[Malk70] asserts that above necessary conditions (for existence of such a polyhedron) are sufficient, with only a finite number of exceptions. Polyhedra $(3; 3, b; p_3, p_b)$ with $b < 6$ are only $Prism_3$ and Dürer's octahedron; both are l_1-embeddable into $\frac{1}{2}H_5$, $\frac{1}{2}H_8$, respectively. The polyhedra $(3; 3, b; p_3, p_b)$ with $b = 6, 8, 10$ are non 5-gonal , since they contain (triangles are isolated) isometric non 5-gonal subgraph consisting of a vertex surrounded by a triangle and two even-gons. Examples of l_1-graphs between their duals are 3 omni-capped Platonic solids α_3, β_3, Icosahedron.

Simple bifaced polyhedra were studied in order to find non-Hamiltonian memebers between them; for example, they exist (J.Zaks) for $(a, b) = (5, 8)$, but do not exist ([Good75], [Good77]) for $(a, b) = (3, 6), (4, 6)$.

Consider now *non-simple* bifaced polyhedra with $b \leq 6$. All possible $(k; a, b)$ are (4;3,6), (4;3,5), (4;3,4), (5;3,6), (5;3,5), (5;3,4). In each of those cases there is an infinity of such polyhedra. Namely, we have

[Fisch75]: $(5; 3, 4; p_3, p_4)$ exists for any $p_4 > 1$,

[Grün67], p.282: $(4; 3, 4; p_3, p_4)$exists for any $p_4 > 1$,

[Grün96]: an infinity of examples is constructed for each of 4 remaining cases.

Remark that cases (ii), (iii), (i_3) of *ambo P* construction of §6, produce also an infinity of bifaced polyhedra for $(k=4; a=3, b)$ with $b=4,5,6$.

There are only two classes of non-simple bifaced polyhedra, namely, $(4;3,b;p_3,p_b)$ and $(5;3,b;p_3,p_b)$. Examples of l_1-polyhedra between them are Rhombicuboctahedron $\to \frac{1}{2}H_{10}$ ((4;3,4;8,18)), tetrakis truncated $\beta_3 \to \frac{1}{2}H_{12}$ ((4;3,6;24,8)), *snub* $\gamma_3 \to \frac{1}{2}H_9$ ((5;3,4;32,6)), snub dodecahedron $\to \frac{1}{2}H_{15}$ ((5;3,5;80,12)), and, between their duals, both Catalan zonohedra. Duals of above six l_1-polyhedra are non 5-gonal. Only 3 Archimedian polyhedra are not bifaced: Rhombicosidodecahedron $\to \frac{1}{2}H_9$ and two large zonohedra.

The list of known l_1-polyhedra between bifaced polyhedra and their duals is given in Tables 2 and 3.

Remark (i) For all l_1-polyhedra of the tables below (except $Prism_3$, Dürer's octahedron and 2-capped $C_4 \times P_{m+1}$) the dual polyhedra are not 5-gonal.

(ii) For majority of l_1-polyhedra P from both Tables 2 and 3, $P \to \frac{1}{2}H_{2d}$, where d is the diameter of P. But, for example, the diameter of dual truncated α_3 is 2 and of dual truncated dodecahedron is 4.

(iii) Between triangulations ## 1–12 of Table 3 (and denoting by "<" isometric subgraph) we have: Icosahedron $<$ #4 $<$ #12; $\beta_3 <$ # 2, # 3 $<$ # 11; # 6 $<$ # 5.

Self-dual polyhedron with $p=(p_a,p_b)$, $3 \le a < b$, exists, applying [Juco70] iff $a=3$, all $p_i = v_i$ and $p_a = p_3 = n - p_b$, $p_b = \frac{n-4}{b-3}$. So, it is α_3, Pyr_b for $p_b = 0,1$. For $b=4$, the subcase $n \equiv 1 \pmod 4$ is realized by k-elongated Pyr_4 having $p=(p_3=4, p_4=4k+1)$. All above polyhedra are embedded into a half-cube. But the polyhedra with $b=4$ and $n=6,7,8$ are not 5-gonal. Also the gyrobifastigum (a regular-faced polyhedron) has $p=(p_3,p_4)=v=(v_3,v_4)=(4,4)$, but it is not self-dual; it and its dual are non 5-gonal.

Table 1. All k-valent polyhedra with only a-gonal and b-gonal faces,
$3 \le a < b \le 6$, $p_a > 0 < p_b$.

$(a,b) \setminus k$	3	4	5
$(5,6)$	5_n(fullerene) exists iff $p_6 \ge 2$	$--$	$--$
$(4,6)$	4_nexists iff $p_6 \ge 2$	$--$	$--$
$(3,6)$	3_nexists iff $p_6 \ge 4$, even	$p_3 = 8+2p_6$, $n=6+3p_6$	$p_3 = 20+8p_6$, $n=12+6p_6$
$(4,5)$	only 4 dual deltahedra	$--$	$--$
$(3,5)$	only Dürer's octahedron	$p_3=8+p_5$, $n=6+2p_5$	$p_3=20+5p_5$, $n=12+4p_5$
$(3,4)$	only $Prism_3$	$p_3=8$, $n=6+p_4$ exists iff $p_4 \ge 2$	$p_3=20+2p_4$, $n=12+2p_4$ exists iff $p_4 \ge 2$

Table 2. Known l_1-polyhedra between bifaced polyhedra

k	a, b	p_a, p_b	polyhedron	embeds into
3	$4, 6$	$6, 8$	truncated β_3	H_6
3	$4, 6$	$6, 12$	chamfered γ_3	H_7
3	$4, 6$	$6, 12$	twisted chamfered γ_3	H_7
3	$5, 6$	$12, 3$	fullerene $5_{26}(D_{3h})$	$\frac{1}{2}H_{12}$
3	$5, 6$	$12, 12$	fullerene $5_{44}(T_d)$	$\frac{1}{2}H_{16}$
3	$5, 6$	$12, 30$	chamfered Dodecahedron	$\frac{1}{2}H_{22}$
3	$3, 5$	$2, 6$	Dürer's octahedron	$\frac{1}{2}H_8$
3	$4, 5$	$4, 4$	dual bisdisphenoid	$\frac{1}{2}H_8$
3	$4, b$	$b, 2$	Prism$_b$ (incl. $b = 3$)	$\frac{1}{2}H_{b+2}$
4	$3, b$	$2b, 2$	APrism$_b$	$\frac{1}{2}H_{b+1}$
4	$3, 4$	$8, 4m$	2-capped $(C_4 \times P_{m+1})$	$\frac{1}{2}H_{2m+4}$
4	$3, 4$	$8, 3$	ambo Prism$_3$	$\frac{1}{2}H_6$
4	$3, 4$	$8, 18$	Rhombicuboctahedron	$\frac{1}{2}H_{10}$
4	$3, 6$	$24, 8$	tetrakis truncated β_3	$\frac{1}{2}H_{12}$
5	$3, 4$	$32, 6$	snub γ_3	$\frac{1}{2}H_9$
5	3.5	$80, 12$	snub dodecahedron	$\frac{1}{2}H_{15}$

Table 3. Known l_1-polyhedra between dual bifaced polyhedra

k	a, b	p_a, p_b	polyhedron	embeds into
3	$3, 6$	$4, 4$	truncated α_3	$\frac{1}{2}H_7$
3	$3, 6$	$4, 6$	chamfered α_3	$\frac{1}{2}H_8$
3	$3, 6$	$4, 6$	twisted chamfered α_3	$\frac{1}{2}H_8$
3	$3, 6$	4.12	4-truncated Dodecahedron	$\frac{1}{2}H_{10}$
3	$3, 6$	4.18	truncated omnicapped α_3	$\frac{1}{2}H_{11}$
3	$5, 6$	$12, 4$	fullerene $5_{28}(T_d)$	$\frac{1}{2}H_7$
3	$5, 6$	$12, 8$	fullerene $5_{36}(D_{6h})$	$\frac{1}{2}H_8$
3	$5, 6$	$12, 20$	5_{60} = truncated Icosahedron	$\frac{1}{2}H_{10}$
3	$3, 4$	$2, 4$	Prism$_3$	$\frac{1}{2}H_4$
3	$3, 5$	$2, 6$	Dürer's octahedron	$\frac{1}{2}H_4$
3	$3, 8$	$8, 6$	truncated γ_3	$\frac{1}{2}H_{12}$
3	$3, 10$	$20, 12$	truncated Dodecahedron	$\frac{1}{2}H_{26}$
4	3.4	$8, 6$	Cuboctahedron	H_4
4	3.5	$20, 12$	Icosidodecahedron	H_6
4	$3, 4$	$8, 4m$	2-$(C_4 \times P_{m+1}) = (C_4 \times P_{m+2})^*$	H_{m+3}

6 Constructions of bifaced polyhedra

Consider some operations on a bifaced polyhedron P with parameters $(k; a, b; p_a, p_b)$. If $b = 2k$, then the dual of omnicapped P, called *leapfrog* of P in [Fowl93], has parameters $(3; a, b; p_a, p_b + |V(P)|)$ and $k|V(P)|$ vertices. If $b = 2k = 6$, then the *chamfering* of P (replace all edges by hexagons) has parameters $(3; a, b; p_a, p_b + |E(P)|)$ and $4|V(P)|$ vertices.

All 5_n, 4_n, 3_n with icosahedral, octahedral, tetrahedral symmetry, respectively, are characterized in [Gold37]. They have $n = 20(a^2+ab+b^2)$, $8(a^2+ab+b^2)$, $4(a^2+ab+b^2)$, resp. for $a \geq b \geq 0$ (in Coxeter notation they are $\{m+, 3\}_{a,b}$ for $m = 5, 4, 3$) and cases $b = 0$ and $a = b$ correspond to full such symmetry I_h, O_h, T_d. The leapfrog (and chamfering) of Dodecahedron, γ_3, α_3 correspond to the case $a = b = 1$ ($a = 2$, $b = 0$, resp.); above leapfrogs are truncated Icosahedron, β_3, α_3, respectively. Clearly, each of classes of all 5_n, 4_n, 3_n is closed under operations of leapfrog and chamfering.

Let P, P' be two Platonic polyhedra having (k, l, n, f) and $(k' = k, l', n', f')$ resp. as size of faces, valency, the number of vertices and the number of faces. Consider the convex polyhedron $P + fP'$ obtained by adjoining a copy of P' on each face of P. Then $(P + fP')^*$ is a bifaced polyhedron with parameters $(k; l', l(l'-1); f(n'-k), n)$. The case $P' = \alpha_3$ corresponds to omnicapping of simplicial P. While, clearly, $\alpha_3 + i\alpha_3$, $\beta_3 + i\alpha_3$ are l_1-embeddable for $0 \leq i \leq f$, $\alpha_3 + i\beta_3$ for $2 \leq i \leq 4$ and $\beta_3 + i\beta_3$ for $1 \leq i \leq 8$ are not l_1.

Given a polyhedron P, let $P + Pyr_3$ ($P + Prism_q$, respectively) be the polyhedron defined by join of new vertex to all vertices of a triangular face (respectively, join of $Prism_q$ to a q-face). It is easy to check the following

Proposition 6.1 *(i) $P + Pyr_3$ is l_1-embeddable iff P is l_1-embeddable;*

(ii) $P + Prism_q$ is l_1-embeddable iff P is l_1-embeddable.

In particular, $P \to \frac{1}{2}H_m$ implies $P + Pyr_3 \to \frac{1}{2}H_{m+1}$, but, for example, for $P = Prism_3 \to \frac{1}{2}H_5$, we have $P + Pyr_3 \to \frac{1}{2}H_6$ and $\frac{1}{2}H_5$, also. While $\alpha_3 + iPyr_3 \to \frac{1}{2}H_{3+i}$ ($1 \leq i \leq 4$) and $Prism_3$ are l_1-rigid, we have $\alpha_3 + iPrism_3 \to \frac{1}{2}H_{3+2i}, \frac{1}{2}H_{4+2i}$ ($0 \leq i \leq 4$). Now, $P \to \frac{1}{2}H_m$ implies $P + Prism_m \to \frac{1}{2}H_{m+2}$. The case (i) above implies that for a planar triangulation T, the subgraph T_1 induced by all vertices of degree ≥ 4 is an isometric triangulation and T is embeddable iff T_1 is embeddable. Remark that $APrism_n \to \frac{1}{2}H_{n+1}$ but $APrism_n + APrism_n$ is not 5-gonal. The *golden dodecahedron* of [HiPe89], i.e. Dodecahedron+12$APrism_5$ is not 5-gonal. The property (ii) above implies the following embeddings. Let P be a polyhedron with a vertex v such that $P - v \to \frac{1}{2}H_m$. Then P, truncated on the vertex v, is embedded into $\frac{1}{2}H_{m+2}$. For example, this truncation of P on two opposite caps is embedded into: $\frac{1}{2}H_{10}$ if P is the icosahedron (i.e. 2-capped $APrism_5$); $\frac{1}{2}H_9$ if P is 2-capped $APrism_4$ (which is an extreme hypermetric); H_4 if P is $\beta_3 = APrism_3$; $\frac{1}{2}H_{11}$ if P is 5^*_{24} (i.e. 2-capped $APrism_6$, which is not 5-gonal).

Among 92 regular-faced polyhedra, 19 are *elongated* P (i.e. obtained from P by adjoining or inserting a prism), 27 are i-*augmented* P i.e. i-cappings of P. For example, pentakis of $APrism_5$ and of the Dodecahedron are embeddable, pentakis of 5-Pyramid and of $Prism_5$ are not 5-gonal, tetrakis of $Prism_3$ and of $APrism_4$ are extreme hypermetrics.

The following constructions produce bifaced polyhedra from columns of prisms and antiprisms. Denote by $Prism_n^t$, $APrism_n^t$ the column of t n-prisms (resp. of t n-antiprisms) pilled up on their n-gonal faces. Denote by 2-$Prism_n^t$, 2-$APrism_n^t$ their cappings on both external n-gonal faces. Remark that 2-$Prism_n^t = (Prism_n^{t+1})^*$, that $APrism_n^t$ can be seen as a helix and that identifying of both external faces in $Prism_n^t$, $APrism_n^t$ gives a realization of maps (4^4), (3^6) on the torus. Now $(APrism_3^t)^*$, $(2-APrism_5^t)^*$, $(2\text{-}APrism_6^t)^*$ are 4_{6t+2}, 5_{10t+10}, 5_{12t+12} (all are non 5-gonal for $t > 1$) and $(Prism_4^t)^*$ is a $(3,4)_{4t+2} \to \frac{1}{2}H_{2t+2}$ (its dual $\to H_{t+2}$). (An example of use of such columns of prisms and antiprisms in Chemistry is that $APrism_3^t$ ($t = 1, 2, 3, \infty$) and $Prism_3^t$ (t=1,2), 2-$Prism_5^2$ are metal cluster polyhedra for molybdenum sulfides $Mo_6S_8^{4-}$, $Mo_9S_{11}^{4-}$, $Mo_{12}S_{14}^{6-}$, $(Mo_6S_6)_\infty^{2-}$ and for platinum carbonils $Pt_6(CO)_{12}^{2-}$, $Pt_9(CO)_{18}^{2-}$, $Pt_{19}(CO)_{22}^{4-}$, respectively; see [King87)].)

The next construction of bifaced polyhedra is based on the notion of an ambo-polytope. For given polytope P, denote by ambo P the convex hull of the midpoints of all edges of P; i.e. ambo P is the mid-edge truncation of P. Sometimes, moreover, the skeleton of ambo P is the line graph of the skeleton of P; in such a case we write ambo $P = L(P)$. For example, ambo $\alpha_n = L(\alpha_n) \to \frac{1}{2}H_{n+1}$, ambo $\alpha_3 = \beta_3$, ambo β_4 =24-cell, ambo $\gamma_3 = L(\gamma_3)$ =Cuboctahedron=ambo β_3, ambo $Dodecahedron = L(Dodecahedron) = Icosidodecahedron =$ ambo $Icosahedron$, $L(Cuboctahedron) = Rhombicuboctahedron.$

Clearly, ambo P, if P is a polyhedron $(k, a, b; p_a, p_b)$ with n vertices, is a 4-valent polyhedron with p_a a-faces, p_b b-faces and, in addition, n k-faces. So, ambo P is bifaced iff $k \in \{a, b\}$; all possible cases for its parameters are:

(i) $k = a = 3, 4, 5$,

(ii) $(3, 4)_{2n}$ if $P = (3, 4)_n$,

(iii) $(4; 3, 5; 20 + 5p_5, 12 + 5p_5)$ if P has parameters $(5; 3, 5; p_3, p_5)$.

If we consider only P with $b \leq 6$, then (see Table 1) we have, in addition to (ii), (iii):

(i_1) $(3, 4)_9$ if $P = Prism_3$,

(i_2) (4;3,5;14,6) if P is the Dürer's octahedron,

(i_3) $(4; 3, 6; n + 4, \frac{n}{2} - 2)$ if $P = 3_n$.

Finally, we give two constructions (by O. Delgado and M. Deza) generalizing two smallest *fulleroids* (it will mean here simple polyhedra with $p = (p_5, p_7)$ and of the icosahedral symmetry) the existence and unicity of which is proved in [DrBr96]. Clearly, $p_7 = p_5$-12 and divisible by 60; a fulleroid has $v = 4(p_7 + 5)$ vertices.For any positive integer m, we get a fulleroid with $v = 120m(m + 1) + 20$ from the $5_{20(2m+1)^2}(I_h)$ by the *triacon* decoration of some $\frac{p_6}{4} = 10m(m + 1)$ of its hexagonal faces with no pentagonal neighbors. (Th triacon means adding a vertex connected to 3 alternated mid-edges of the hexagon.) Also we get simple polyhedra of the icosahedral symmetry and with $p = (p_5 = 72, p_7 = 60)$, $(p_5 = 72, p_8 = 30)$, $(p_5 = 72, p_9 = 20)$, from the icosahedral fullerenes $5_{140}, 5_{80}, 5_{60}$, respectively, by following decoration of each of 12 pentagonal faces.Add the 5-cycle of new vertices $u_1, \ldots, u_5$ to the 5-cycle of vertices $v_1, \ldots, v_5$ of above face; then connect each u_i to the mid-point of corresponding edge (v_i, v_{i+1}).Those polyhedra have, respectively, 260,200,180 vertices.

7 Polyhedra 3_n

Theorem 2 of [GrMo63] gives that 3_n exists for any $n \equiv 0 \pmod 4$, except $n = 8$, and provides complete description of their skeletons. [Good77] showed that all 3_n are Hamiltonian.

$3_4 = \alpha_3 \to \frac{1}{2}H_3$, $\frac{1}{2}H_4$ is not l_1-rigid. The "would-be" 3_8 is not 3-connected and non 5-gonal.

There are exactly $N_3(n)$ polyhedra 3_n ([Dill96]) for $1 \leq \frac{n}{4} \leq 7$, where $N_3(n)$ is given below

$\frac{n}{4}$	1	2	3	4	5	6	7
$N_3(n)$	1	0	1	2	1	2	2

Clearly, α_3 is unique 3_n with abutting triangles. So, any 3_n, $n>4$, is not 5-gonal since it contains an isometric subgraph G_{11} given in §4.

We know six l_1-polyhedra 3_n^* (no other exists for $n \le 28$): $3_4^* \to \frac{1}{2}H_3$ (and $\to \frac{1}{2}H_4$), $3_{12}^* \to \frac{1}{2}H_7$, both $3_{16}^* \to \frac{1}{2}H_8$, a $3_{28}^* \to \frac{1}{2}H_{10}$, a $3_{36}^* \to \frac{1}{2}H_{11}$; last 5 are 4-capped α_3, β_3, β_3, 5_{20}^* (on disjoint faces), $5_{28}^*(T_d)$= hexakis (truncated α_3), respectively.

Except α_3 and unique 3_{12} =truncated α_3, any 3_n is a 4-vertex truncation of a simple poyhedron with $\sum_{i\ge 3} p_i = p_4+p_5+p_6$ (so $p_5 = 12-p_4$ from the well-known equality $\sum_{i\ge 3}(6-i)p_i = 12$) and $n-8$ vertices (so $n = 28+2p_6-2p_4$). In particular, 3_n is 4-vertex truncation of a fullerene 5_{n-8} iff this 5_{n-8} has 4 vertices of type (5,5,5) at pairwise distance at least 3; so any pair of triangles is separated by more than one hexagon. Such 5_{n-8} has either 4 isolated triples of pentagons, or two isolated clusters of 6 pentagons. $3_n^* \to \frac{1}{2}H_m$ iff $5_{n-8}^* \to \frac{1}{2}H_{m-4}$.

For small values of n we have (see Fig. 4; marked 4 vertices indicate truncation):

1) exactly seven of 3_n come as 4-vertex truncation of dual deltahedra: unique 3_{12} from α_3, both 3_{16} (chamfered and twisted chamfered α_3) from γ_3, unique 3_{20} from dual bisdisphenoid, both 3_{24} from dual 2-capped $APrism_4$ and one (of two) 3_{28} from the (pentagonal) dodecahedron. The second one 3_{28} comes from a simple dodecahedron with $p = (p_4, p_5, p_6) = (4, 4, 4)$.

2) exactly six 3_n come as 4-vertex truncation of fullerenes 5_m, $20 \le m \le 36$: one 3_{28} from 5_{20}, a 3_{32} from 5_{24}, a 3_{36} from $5_{28}(D_2)$, another 3_{36} from $5_{28}(T_d)$, a 3_{40} from $5_{32}(D_2)$, a 3_{44} from $5_{36}(D_2)$; each of these 6 fullerenes has a unique, up to a symmetry, set of 4 vertices at pairwise distance ≥ 3.

The class of 5_{n-8} with 4 isolated triples of pentagons contains all tetrahedral 5_{n-8}; those have (using [Gold37]) $n = 4(a^2+ab+b^2)$ with $a \ge b \ge 0$, $a \ge 2$. In particular, $Cham_t\alpha_3$, $t \ge 2$, comes from $5_{4^{t+1}-8}(T_d)$. (But already $5_{56}^*(T_d)$ is not l_1; so this 3_{64}^* is not l_1 implying that $(Cham_t\alpha_3)^*$ is l_1 iff $t=1$.) But such fullerenes 5_{n-8} exist also for $n-8 = 40, 44, 56$ (unique for each), 68. On the other hand, for $n = 40, 44, 48, 52, 68$ there are 5_{n-8} with two isolated groups of 6 pentagons (3 such fullerenes for $n=48$, unique for others) such that a 4-vertex truncation of them is a 3_n (see, for example, last 3 on Fig. 4).

8 Polyhedra 4_n

Theorem 1 of [GrMo63] gives that 4_n exists for any even $n \ge 8$ except $n=10$. [Good75] showed that all 4_n are Hamiltonian. Clearly, 4_n is bipartite, and there is an infinity of centrally symmetric 4_n. Hence it either $\to H_m$ or is non 5-gonal. For $4 \le \frac{n}{2} \le 22$, there are $N_4(n)$ polyhedra 4_n ([Dill96]), where $N_4(n)$ is given in the table below.

$\frac{n}{2}$	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22
$N_4(n)$	1	0	1	1	1	1	3	1	3	3	3	2	8	3	7	7	7	5	14

Unique l_1-polyhedron between 4_n^* is $4_8^* = \beta_3 \to \frac{1}{2}H_4$, since other 4_n^* contain the following non 5-gonal isometric subgraph G_7: $Cyc_{1,\dots,6}$ with chordes (1,3), (4,6) plus a new vertex connected with vertices 1,3,4,6. Clearly $4_8 = \gamma_3$, $4_{12} = Prism_6$, truncated β_3 is 4_{24} (one of 3), two 4_{32} are $Cham\gamma_3$ and twisted $Cham\gamma_3$ (see Fig. 5). Those 5 polyhedra are all known l_1-4_n. They are embedded into H_3, H_4, H_6, H_7, H_7, respectively. The first 3 are Voronoi polyhedra, $Cham\gamma_3$ is non space-filling zonohedron, twisted one is not centrally symmetric.

$Cham_t\gamma_3$ is l_1 iff $t = 1$. For $n \equiv 2$ ((mod 6) the dual of column of $\frac{n-2}{6}$ octahedra $\beta_3 = APrism_3$ gives a 4_n; for $n > 8$, it and its dual are not 5-gonal. Examples of 4_n without abutting pairs of 4-gons are $Cham_t\gamma_3$, $t \geq 1$) and duals of tetrakis cube, Cuboctahedron, triangular ortobicupola ("anticuboctahedron"), giroelongated triangular bicupola and snub cube, having, respectively $8a^2$ $(a \geq 2)$, 24, 32, 32, 44, 60 vertices. Last two are non 5-gonal. "Dual tetrakis" above, means a truncation on six 4-valent vertices of the dual polyhedra.

On the other hand, any 4_n with each pair of 4-gons separated by at least 3 edges (for example, $Cham_t\gamma_3$, $t \geq 2$) comes as *6-edge* truncation (put 4-gons instead of edges) of a fullerene 5_{n-12}. Hence such a 4_n has no abutting pair of 4-gons and, moreover, the corresponding 5_{n-12} has 6 pairs of pentagons separated by only one edge. Many 4_n come as 6-edge truncation of 5_n under weaker conditions: $Cham\ \gamma_3$ from 5_{20}, truncated β_3 from α_3. Also truncated β_3 is 6-(disjoint) edges truncation of $4_{12} = Prism_6$, $Prism_6$ is 2-(disjoint) edges truncation of $4_8 = \gamma_3$, γ_3 is 2-(disjoint) edges truncation of α_3. Suitable 6-(disjoint) edges truncation of 5_{24} (dual 2-capped $APrism_6$) gives a 4_{36} and so on (see Fig. 5).

9 Polyhedra 5_n

Theorem 1 of [GrMo63] gives that 5_n exists for any even $n \geq 20$ except $n = 22$. The polyhedra 5_n, i.e. the bifaced polyhedra $(3; 5, 6; 12, p_6)$, are called *fullerenes* in Chemistry; see, for example, [Fowl93], [DDGr96] for a sample of vast literature on them. In fact, all 5_n are the cases $f = 2 + \frac{n}{2} \geq 12$ of *medial* polyhedra, introduced in [Gold35] as putative best (isoperimetrically) approximation of a sphere within the class of polyhedra having given number f of faces. A medial polyhedron with f faces is a bifaced polyhedron $(3; a = \lfloor 6 - \frac{12}{f} \rfloor, b = a + 1; p_a, p_b = f - p_a)$; it exists for any $f \geq 4$, except $f = 11, 13$. For $f = 4, ..., 10, 12$ they are exactly 8 dual convex deltahedra: α_3, $Prism_3$, γ_3, $Prism_5$, dual bisdisphenoid, dual 3-augmented $Prism_3$, dual 2-capped $APrism_4$ and the Dodecahedron (see the case $k = 3$ of Table 1 above).

All known l_1-5_n are the Dodecahedron $5_{20} \rightarrow \frac{1}{2}H_{10}$, $5_{26} \rightarrow \frac{1}{2}H_{12}$, $5_{44}(T) \rightarrow \frac{1}{2}H_{16}$ and the chamfered Dodecahedron $5_{80}(I_h) \rightarrow \frac{1}{2}H_{22}$. The last one is dual pentakis Icosidodecahedron; its twisted version is not l_1. All known l_1-5_n^* are the Icosahedron $5_{20}^* \rightarrow \frac{1}{2}H_6$, hexakis(truncated α_3) $= 5_{28}(T_d) \rightarrow \frac{1}{2}H_7$, hexakis$(APrism_6^2) = 5_{36}^*(D_{6h}) \rightarrow \frac{1}{2}H_8$ and pentakis Dodecahedron $5_{60}^*(I_h) \rightarrow \frac{1}{2}H_{10}$. In fact, no other l_1-5_n, l_1-5_n^* exist for $n < 60$ (see [DDGr96]) and they are not expected for other n.

Some interesting classes of 5_n were mentioned above: $5_{10(t+1)} = (2\text{-}APrism_5^t)^*$, $5_{12(t+1)} = (2\text{-}APrism_6^t)^*$ and those (with 4 isolated triples of pentagons) coming from collapsing of 4 triangles in some 3_n. Let us consider a generalization of the last 2 classes.

We say that a fullerene 5_n is PR_i (*pentagon-regular of degree i*) if each of 12 pentagons is adjacent to exactly i other pentagons. Clearly, the dodecahedron 5_{20} and the hexagonal barrel $Barrel_6 = 5_{24}$ are unique PR_5 and PR_4, respectively. RP_0's are fullerenes with isolated pentagons; chemists call them *preferable* or IP fullerenes. The smallest ones among them are the $5_{60}(I_h)$ and a $5_{70}(D_{5h})$. PR_1's are those having 6 isolated pairs of abutting pentagons; all 130 of such 5_n with $n \leq 84$ are listed in [Fowl93] the 4 smallest ones are a $5_{50}(D_3)$, a $5_{52}(T)$, the $5_{52}(C_2)$, a $5_{54}(D_3)$. The only PR_3 are the $5_{28}(T_d)$ and the $5_{32}(D_{3h})$. It will be interesting to characterize all fullerenes PR_2. Their 12 pentagons form isolated k-cycles ($k = 3, 6, 9, 12$): four 3-cycles (including a $5_{48}(D_2)$ and at least one tetrahedral 5_n for any $n = 4(a^2 + ab + b^2) - 8$, starting with the $5_{40}(T_d)$, the $5_{44}(T)$, the $5_{56}(T_d)$, a $5_{58}(T)$ and so on); two 3-cycles and one 6-cycle; one 3-cycle and one 9-cycle (including the $5_{38}(C_{3v})$); two 6-cycles (including any $5_{12t} = (2\text{-}APrism_6^{t-1})^*$ with $t \geq 3$, it has symmetry D_{6h} for odd t and D_{6d} for even t); one 12-cycle. The number of vertices of type (5^3), $(5^2.6)$, (5.6^2) for above 5 classes (by faces surrounding a vertex) are (4,12,24), (2,18,18), (1,21,15), (0,24,12), (0,24,12), respectively. We not found PR_2 fullerenes of second class; they are also absent among all 1812 fullerenes 5_{60}. All PR_i with $n \leq 50$ are 15 mentioned above: one for $i = 5, 4, 1$, two for $i = 3$ and ten for $i = 2$. We find that in the third and in the fifth class there are exactly 5 fullerenes of Fig. 6.

Proposition 9.1 *All fullerenes with 12-cycle of pentagons are those four given on Fig.6.*

In fact, let C_{12} and C'_{12} be the inner and the outer cycles of a belt of 12 adjacent squares. We have to set 12 new vertices on edges of C_{12} and C'_{12} such that the 12 squares of the ring are transformed into 12 pentagons. Let a and $a' = 12 - a$ be the numbers of the new vertices on C_{12} and C'_{12}, respectively. Now we connect the a vertices of C_{12} by lines such that the inner domain of the ring is partitioned into hexagons. Similarly we partition the outer domain. The Euler relation implies that $a = a' = 6$. If k is the number of new vertices in the inner part (their degree is 3), then the number of hexagons is $\frac{k}{2}+4$. Hence k is even. Consider a configuration of the six new vertices on C_{12}. Let q be the maximal number of vertices of C_{12} lying between two consecutive new vertices. Then $q \leq 4$, since two consecutive new vertices and q vertices of C_{12} between them belong to a hexagon. It is easy to see that q takes only 3 values: 4, 3 and 2. Not very complicated enumeration of configurations shows the following.

If $q = 4$, we have a $5_{36}(D_{2d})$ with $k = 0$ (one of two 5_{36} with symmetry D_{2d}).

If $q = 3$, we obtain two nonisomorphic fullerenes 5_{44} with $k = 4$ (with symmetry D_2, D_{3d}).

If $q = 2$, we have a homogeneous configuration of 6 new vertices on C_{12}. In this case $k = 6$, and we obtain a fullerene 5_{48} (D_{6d}), (unique, except $(2\text{-}APrism_6^3)^*$, $5_{48}(D_{6d})$).

The partitions of the inner and outer domains into hexagons are similar.

Proposition 9.2 *There is a unique fullerene with one 3-cycle and one 9-cycle of pentagons, namely the fullerene $5_{38}(C_{3v})$.*

Proof. Let C_9 and C'_9 be the inner and the outer cycles of a ring of 9 adjacent squares. We have to set 9 new vertices on edges of C_9 and C'_9 such that the 9 squares of the ring are transformed into 9 pentagons. Let a and $a' = 9 - a$ be the numbers of the new vertices on C_9 and C'_9, respectively. Let the 3-cycle of pentagons lies in the inner region of C_9. Hence the outer region of C'_9 has no pentagons. We connect the a' vertices of C'_9 by lines such that the outer region of C'_9 is partitioned into hexagons. The Euler relation implies that $a' = 3$. Hence $a = 9 - 3 = 6$. The simple enumeration shows that the 3 new vertices of C'_9 lie uniformly, i.e. there are exactly 3 vertices of C'_9 between two consecutive new vertices. Besides, there is only one other vertex of degree 3. Hence the outer region of C'_9 is partitioned into 3 hexagons.

Consider the inner region of C_9. We have to connect the 6 vertices of degree 2 of the 3-cycle of pentagons with the 6 new vertices of the cycle C_9 by lines. It is easy to see that if there is one vertex of the 3-cycle and one new vertex of C_9 connected by an edge, then the partition of the inner region of C_9 into hexagons is unique. In this case each new vertex of C_9 is connected by an edge with a vertex of degree 2 of the 3-cycle. The 6 new vertices of C_9 lie uniformly, i.e. there are two and three vertices of C_9 between a new vertex and its left and right neighbours. We obtain the fullerene $5_{38}(C_{3v})$.

If there is no pair of vertices of the 3-cycle and C_9 connected by an edge, then the 3-cycle is circumscribed by a ring of 6 hexagons. The outer cycle C_{15}, including these 6 hexagons, contains 9 vertices of degree 2. It is not difficult to show that it is not possible to connect these 9 vertices of C_{15} with 6 vertices of C_9 and obtain a partition into hexagons.

Similarly,define HR_i (*hexagon i-regular*) be any fullerene such that any hexagon is adjacent to exactly i hexagons.Then all HR_0 are the 5_{24}, the 5_{26}, the $5_{28}(T_d)$ and no HR_6 exists. Examples of HR_1 are the $5_{28}(D_2)$, the $5_{32}(D_3)$; of HR_2 - the $5_{30}(D_{5h})$, the $5_{32}(D_{3d})$, the $5_{32}(D_2)$, the $5_{32}(D_{3h})$; of HR_3 - the $5_{36}(D_2)$, the $5_{60}(I_h)$; of HR_4 - the $5_{50}(D_{5d}) = (2 - APrism_5^3)^*$, the $5_{80}(I_h)$, the $5_{80}(D_{5h})$; of HR_5 - the $5_{140}(I)$. For $n \leq 36$ no other HR_i (than above ten) exists. The only possible (i, n) for preferable fullerenes HR_i are $(3, 60), (4, 80), (5, 140)$, since they have $p_6 = \frac{60}{6-i}$;there are only above two for $n = 80$.Those four HR_i and PR_0 fullerenes have (see Section 4.4 in [FoMa95] minimal *steric strain* among PR_0 fullerenes. HR_i have (see page 79 in [FoMa95] maximal steric strain among fullerenes.Any 5_n which is HR_i has exactly $30 - \frac{p_6(6-i)}{2}$ edges separating two pentagons; so $n \leq 20 + \frac{120}{6-i} \leq 140$. Examples of 5_n which are both PR_i and $HR_{i'}$ are (besides of preferable HR_i) the 5_{24}, the $5_{28}(T_d)$, the $5_{32}(D_{3h})$ for $(i, i') = (4, 0), (3, 0), (3, 2)$, respectively.

The fullerenes 5_n are the case:

1. "at least 20 vertices" of the medial polyhedra [Gold35],
2. "a=5" of simple bifaced polyhedra with $p = (p_a, p_6)$; see Sections 7,8 above,
3. "b=6" of simple bifaced polyhedra with $p = (p_5, p_b)$ (For general b we have $p_5 + p_b(6 - b) = 12$ and it has $2\frac{b-5}{b-6}p_5 - \frac{4b}{b-6}$ vertices; see remark on fulleroids in the end of Section 6.For $b = 3, 4$ it is the Durer's octahedron, $Prism_5$,respectively.)

3-valent polyhedra with $p = (p_4 = 2, p_5 = 8, p_6)$ with high symmetry were proposed in Chemistry (see [GaHe93]);some are chemically better (i.e. they have smaller number of pairs of adjacent pentagons) than fullerenes with the same number of vertices. 3-valent maps with $p = (p_5, p_6, p_7)$ have $p_5 = p_7 + 12(1 - g), p_6$ any.Many species of plancton (for example, famous Aulonia hexagona of E.Haeckel, 1887) have, as their rigid skeleton, such polyhedral maps. In the chemically important case $p_5 = p_7$, they are realizable on the torus (having the genus $g = 1$) and they have $4p_5 + 2p_6$ vertices. They are called *toroidal polyhexes* if $p_5 = p_7 = 0$ and *azulenoids* if $p_5 = p_7 > 0$. In many applications 5- and 7-gons come by joined pairs,i.e.(in chemical terms) by *azulene* units.Toroidal polyhexes with $p_6 = 3, 7, 8, 12$ are not 5-gonal, but for $p_6 = 4$, it realizes the cube H_3. Toroidal realizations of 3-valent maps with $p = (p_5, p_6, p_8)$ also used in Chemistry; they have $p_5 = 2p_8, p_6$ any and $3p_5 + 2p_6$ vertices. 3-valent maps with $p = (p_6, p_7, p_8)$ have $p_7 + 2p_8 = 12(g - 1), p_6$ any. They are realized on some minimal surface of negative curvature and are called *schwarzites*. [King96] gives 4 examples,with the genus $g = 3$, of chemical relevance having $p = (p_7 = 24)$ (Klein map $\{7^3\}$) on the D surface), $p = (p_6 = 56, p_7 = 24)$ (the leapfrog of the previous one,it is an analog of $5_{60}(I_h)$), $p = (p_6 = 80, p_8 = 12)$ and $p = (p_6 = 80, p_7 = 24)$. They have 56,168,192,216 vertices, respectively, and last two are realized on the the surface P.

10 Polyhedra $(3,4)_n$

Those are 4-valent bifaced polyhedra with parameters $(4; 3, 4; p_3, p_4)$; so $p_3 = 8$ and $n = 6 + p_4$. Besides 3_n, 4_n, 5_n it is the only case of bifaced polyhedra for which p_a is fixed for given (k, b). [Grün67], p.282, gives the existence of $(3,4)_n$ for any $n \geq 6$, except 7.

From §6 above we have ambo $(3,4)_n = (3,4)_{2n}$. Another operation, namely, inserting a ring of m 4-gons into some $(3,4)_n$, produces a $(3,4)_{n+m}$; let us call it *m-elongation*. For example, 2-$Prism_4^t = (3,4)_{4t+6} \to H_{2t+2}$ is, m times iterated, 4-elongation of 2-$Prism_4$. Already iterated 3-elongations of β_3, 4-elongations of $APrism_4$ and of 2-$Prism_4$ give $(3,4)_n$ for $n = 6 + 3m$, $8 + 4m$, $10 + 4m$ for any m.

First examples of $(3,4)_n$ are: $\beta_3 = (3,4)_6 \to \frac{1}{2}H_4$, $APrism_4 = (3,4)_8 \to \frac{1}{2}H_5$, ambo $Prism_3$ =3-elongated $\beta_3 = (3,4)_9 \to \frac{1}{2}H_6$, 2-$Prism_4 = (3,4)_{10} \to \frac{1}{2}H_6$. The six polyhedra on Fig.7 are not 5-gonal: $(3,4)_{11}$ and 4 polyhedra $(3,4)_{12}$, (Cuboctahedron= ambo β_3= 4-elongated $APrism_4$, anticuboctahedron, another 2 times 3-elongation of β_3 and another $(3,4)_{12}$ and ambo $APrism_4 = (3,4)_{16}$. Now, Rhombicuboctahedron $(3,4)_{24} \to \frac{1}{2}H_{10}$ but its twisted version ("14th Archimedian solid") is not 5-gonal; they are 8-elongations of twisted ambo $APrism_4$ which is a $(3,4)_{16} \to \frac{1}{2}H_8$, and of ambo $APrism_4$ which is not 5-gonal. Exactly eleven $(3,4)_n$ are regular-faced: above 4, β_3, $APrism_4$, 2-$Prism_4$, Cuboctahedron, anticuboctahedron and 6-elongations of last two, which are non 5-gonal $(3,4)_{18}$.

Between the duals of above $(3,4)_n$, all embeddable are three zonohedra: $\beta_3^* = \gamma_3$, (Cuboctahedron)$^* \to H_4$ and (2-$Prism_4^t)^* = Prism_4^{t+1} \to H_{t+3}$. Apropos, (ambo $Prism_3 = (3,4)_9)^*$ is the smallest convex polyhedron with odd number of faces, all of which are quadrilaterals; it is not 5-gonal.

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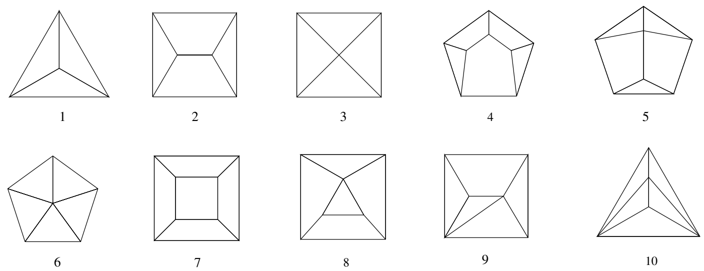

Figure 1: All polyhedra with at most six faces

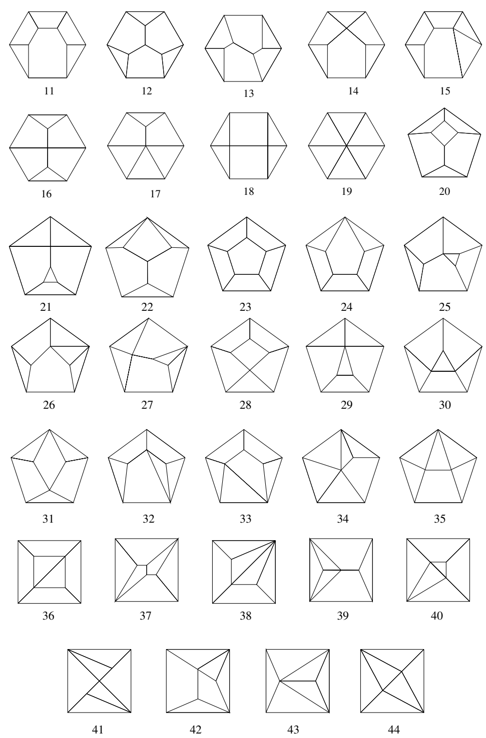

Figure 2: All polyhedra with seven faces

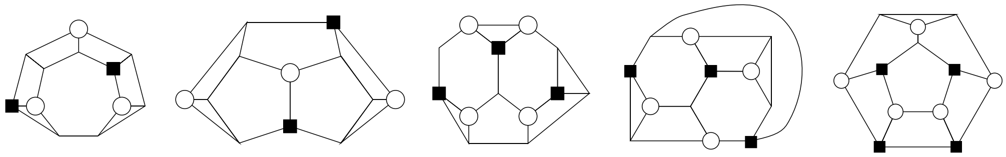

Figure 3: Some simple octahedra

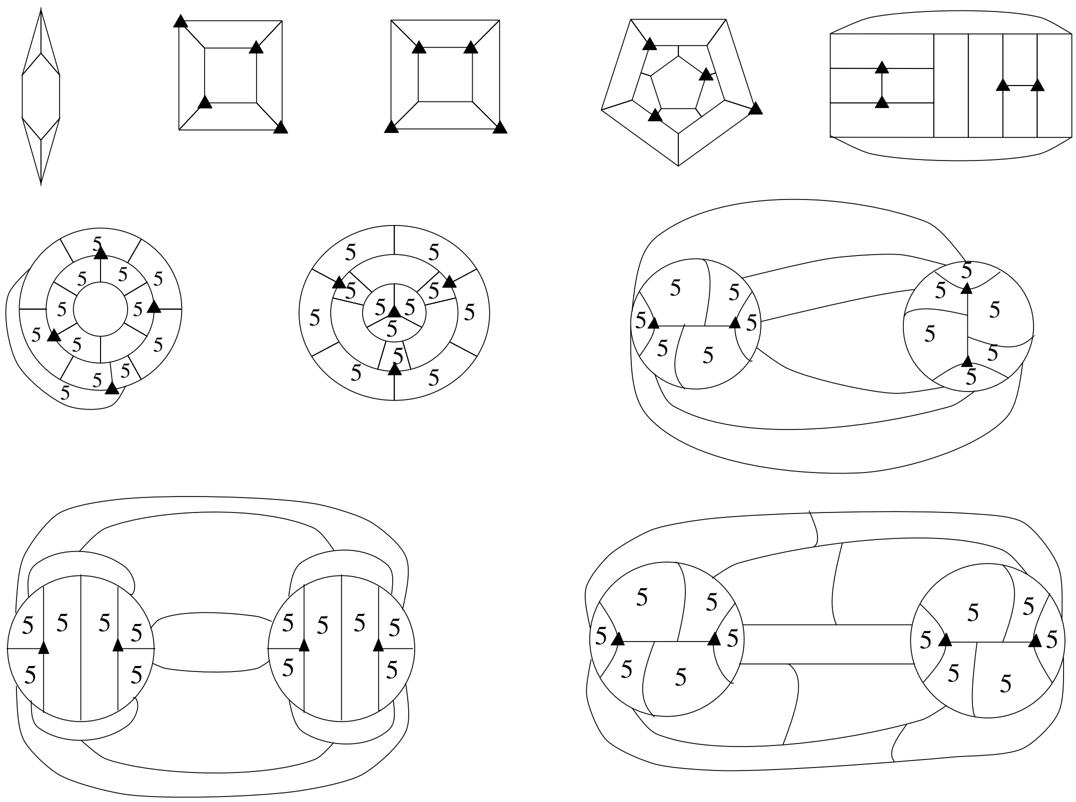

Figure 4: Some 3_n: "would-be" 3_8, both 3_{16}, both 3_{28}; 4-truncations of both 5_{28}, of the $5_{32}(D_2)$, of two 5_{40}

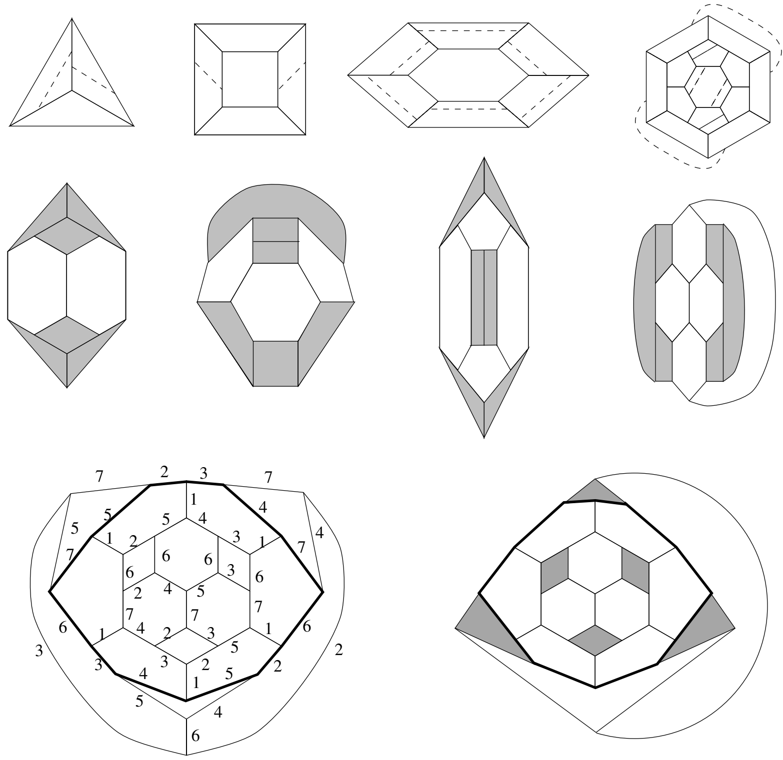

Figure 5: Some 4_n: γ_3, $Prism_6$, truncated β_3, a 4_{36} (as edge-truncations); unique 4_{14}, 4_{16}, 4_{18}; a 4_{20}, twisted Cham γ_3, Cham γ_3

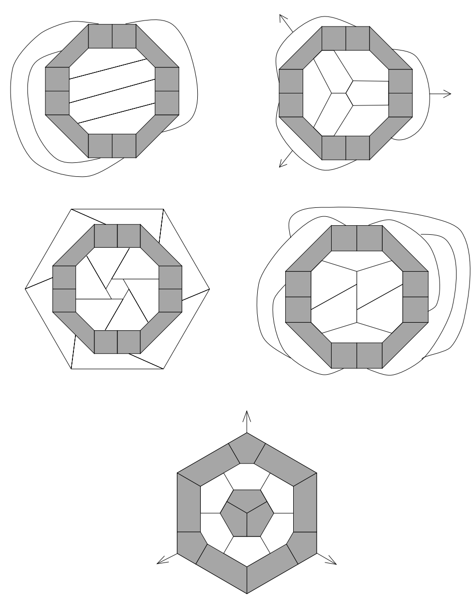

Figure 6: All pentagon-regular fullerenes with 12- or 9-belt of pentagons: a $5_{36}(D_{2d})$, a $5_{44}(D_{3d})$, a $5_{48}(D_{6d})$, a $5_{44}(D_2)$, the $5_{38}(C_{3v})$

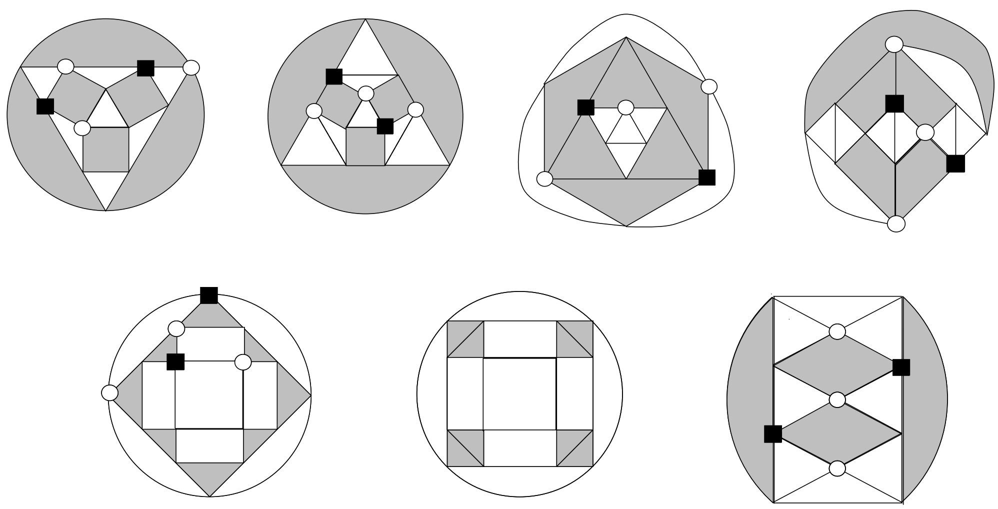

Figure 7: Some $(4,3)_n$: Cuboctahedron, its twist, two other $(3,4)_{12}$, ambo $APrism_4$, its twist, the $(3,4)_{11}$