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Abstract

Any two-distance space is uniquely up to a multiple represented by a distance $d_{G,t}$ for a graph G such that $d_{G,t}(ij)$ is equal to 1 or t depending on (ij) is an edge or nonedge of G. For a cone C_n^A of n-point distance spaces, we set $t^A(G) := \max\{t : d_{G,t} \in C_n^A\}$. We consider the cut cone $Cut_n = C_n^C$, the hypermetric cone $Hyp_n = C_n^H$, and the cone of negative type $Neg_n = C_n^N$. The values of $t^N(G)$ (in other terms) are considered by many authors, and are determined by roots of some polynomials. We give bounds on $t^H(G)$, and consider some classes of graphs G with a given value of $t^H(G)$, especially for $t^H(G) = 2$ and $t^H(G) = \frac{3}{2}$. The graphs G with $t^H(G) = 2$ are exactly graphs having the hypermetric truncated distance d_G^*.

1 Introduction

A point set in an m-dimensional *Euclidean space* $\mathbf{R}^m$ is called a two-distance *set* if the pairwise distances between the points take only two distances. We distinguish a two-distance *space* as an abstract distance space with two distances.

Two-distance sets have an intrinsic interest. Upper bounds on their cardinality may depend on the specific metric space where they are embedded, on its dimension, and on actual distances. Two-distance sets attract by their simplicity and their relation to some combinatorial objects, for example, to spherical designs.

Besides, a use of metric considerations simplifies combinatorial problems. For example, it is very interesting to compare the proofs in [16] and [10] of the nonexistance of a 7-point two-distance set in $\mathbf{R}^3$. The metric proof of [16] is much simpler of the combinatorial proof of [10].

There are some optimization problems related to sets V endowed by a distance function d. For example, the Traveling Salesman Problem is the problem to find an order $V \to \{1, 2, ..., |V|\}$ such that the sum $\sum_{i=1}^n d(i, i+1)$ (with $n+1 \equiv 1$) is minimal. This problem is NP-complete even for two-distance sets, since it is reduced to finding a Hamiltonian cycle in a graph. We think that many other problems are equally hard for two-distance spaces as for general distance spaces.

Any two-distance is, up to a multiple, the following distance $d_{G,t}$ for a graph G and some nonnegative t:

$$d_{G,t}(ij) = \begin{cases} 1 & \text{if } (ij) \text{ is an edge of } G \\ t & \text{if } (ij) \text{ is a nonedge of } G \end{cases}$$

Let C_n^A be a cone of distance spaces (V,d) on $n = |V|$ points. As C_n^A, we consider three combinatorially significant cones, namely, the cut cone $C_n^C = Cut_n$, the cone of l_1-embeddable semimetrics; the hypermetric cone $C_n^H = Hyp_n$; and the cone $C_n^N = Neg_n$, the cone of distances of negative type (i.e. squared Euclidean distances). For a given graph G, we set

$$t_0^A(G) = \min\{t : d_{G,t} \in C_n^A\},\ t^A(G) = \max\{t : d_{G,t} \in C_n^A\}.$$

Since $d_{\overline{G},t} = t d_{G,\frac{1}{t}}$, we obtain

$$t_0^A(G)t^A(\overline{G}) = t_0^A(\overline{G})t^A(G) = 1. \tag{1}$$

So if we know values of $t^A(G)$ for all G, we know $t_0^A(G)$, too. Hence we are restricted ourselves below by studing values of $t^A(G)$ only. In what follows, $t_1^A(G)$ in expression of type $t_{0,1}^A(G)$ denotes $t^A(G)$.

There is the fourth combinatorially important cone $C_n^M = Met_n$, the metric cone. We could consider the problem of a determination of values $t_{0,1}^M(G)$. But this problem is trivially solvable. In fact, $t^M(G) = 2$ if G is not a disjoint sum of complete graphs, and $t^M(G) = \infty$ otherwise.

In principle, the problem of determining of $t^N(G)$ is solved by Einhorn and Schoenberg [16], and this resolution is well known. It relates to roots of a polynomial, namely of the determinant of the Gram matrix of a representation of the distance space $d_{G,t}$. Other polinomials related to $t^N(G)$ were used by Seidel [29], Neumeier [26], Maehara [24].

For a graph G, the distance $d_{G,2} = d_G^*$ is a metric, and it is called the *truncated* metric of the graph G. If G is connected and has diameter 2, then d_G^* is the path metric of the graph G. For a graph G of diameter 2, the equality $t^C(G) = 2$ ($t^H(G) = 2$) means that G is an l_1-graph (hypermetric, respectively). For example, $t^C(K_{m\times 2}) = t^H(K_{m\times 2}) = 2$, where $K_{m\times 2}$ is the Cocktail-party graph. Conversely, if a graph G of diameter 2 is not hypermetric (not l_1-embeddable), then $t^H(G) < 2$ ($t^C(G) < 2$, respectively).

In this paper, we apply known properties of hypermetrics and l_1-metrics to two-distance spaces.

In Section 2, general properties of distance spaces are considered. In particular, it is noted that the convex hull $P(d)$ of representing points of $d \in Neg_n$ is a polytope distinct from a simplex iff d belongs to the boundary of the cone Neg_n. But if d belongs to the boundary of Hyp_n, $P(d)$ can remain a simplex. The role of $P(d)$ plays here the Delaunay polytope $P_D(d)$. Some properties of $t^{H,N}(G)$ are described in Section 3. Taking in attention that any hypermetric space has a spherical representation, we give, in Section 4, known bounds on $t^Q(G)$ of such a representation using the smallest eigenvalue of the

Gram matrix Q. In Sections 5 and 6, we give lower bounds on $t^{H,N}(G)$. The bound on $t^N(G)$ was known early, but it was not related to $t^N(G)$. In Section 7, we recall a relation of two-distance spaces with equiangular lines. We find in Section 8 exact values of $t^H(G)$ for some classes of bipartite graphs, and describe the corresponding Delaunay polytopes $P_D(G)$. In Section 9, we give $t^H(G)$ and $P_D(G)$ for G with small number of vertices. We consider some examples of hypermetric two-distance spaces with $t^H(G) = 2$ and $t^H(G) = \frac{3}{2}$, respectively, in Sections 10 and 11.

2 Distance spaces

A finite distance space (V, d) is a finite set V and a matrix $D = \{d(ij)\}$ or a function d on the set V^2 of all pairs of V, such that

(1) D is nonnegative, i.e. $d(ij) \geq 0$ for all $(ij) \in V^2$,

(2) D is symmetric, $d(ij) = d(ji)$,

(3) D vanishes on diagonal, i.e. $d(ii) = 0$.

A finite distance space (V, d) with $|V| = n$ has $\binom{n}{2} = \frac{n(n-1)}{2}$ distances $d(ij)$. Hence d can be considered as a point of the nonnegative orthant $\mathbf{R}_+^{\binom{n}{2}}$ of the Euclidean space $\mathbf{R}^{\binom{n}{2}}$. Let E_n be the set of all $\binom{n}{2}$ unordered pairs (edges) of distinct points of V. Then the coordinates of $\mathbf{R}^{\binom{n}{2}}$ are indexed by $(ij) \in E_n$, i.e. $\mathbf{R}^{\binom{n}{2}} = \mathbf{R}^{E_n}$.

There are some subcones in $\mathbf{R}_+^{\binom{n}{2}}$ given by linear conditions on the distance d and tightly related to combinatorics. For example, if d satisfies all triangle inequalities $d(ij) + d(ik) - d(jk) \geq 0$, then d is a *semimetric*. All semimetrics form the metric cone Met_n. The following inequality

$$\sum_{1 \leq i < j \leq n} b_i b_j d(ij) \leq 0, \; b_i \in \mathbf{Z}, \; 1 \leq i \leq n, \tag{2}$$

where $\sum_{1 \leq i \leq n} b_i = 1$, is a generalization of the triangle inequality. It is called the *hypermetric* inequality. Since the hypermetric inequality is linear in d, all distances satisfying (2) for all $b_i \in \mathbf{Z}$ with $\sum b_i = 1$ fill out the hypermetric cone Hyp_n.

If d satisfies (2) for all $b \in \mathbf{Z}^n$ with $\sum b_i = 0$, then d is called a distance of *negative type*. These distances form the negative type cone Neg_n.

Remark 1. Since the condition $\sum_{i=1}^n b_i = 0$ and the inequalities (2) are homogeneous, $d \in Neg_n$ satisfies (2) for all rational b_i, and by continuity, for all $b \in \mathbf{R}^n$. $\square$

The cut cone Cut_n is the convex hull of *cut metrics* c_S for $S \subset V$. The cut metrics span the common extreme rays of Cut_n, Hyp_n and Met_n. For $S \subseteq V$, the *cut* $\delta(S) \subseteq E_n$ is the set of edges having exactly one vertex in the set S. Then c_S is the indicator vector of the cut $\delta(S)$, i.e. the vector of the space $\mathbf{R}^{E_n}$ such that

$$c_S(ij) = \begin{cases} 1 & \text{if } |\{ij\} \cap S| = 1, \\ 0 & \text{otherwise.} \end{cases} \tag{3}$$

Clearly, $c_{V-S} = c_S$.

There are important problems on isometrical embedding of a distance space (V, d) into an Euclidean space with Euclidean and squared Euclidean distances. If (V, d) is embedded into an m-dimensional Euclidean space, it can be considered as a set of $n = |V|$ points in $\mathbf{R}^m$. These points (and the vectors whose endpoints they are) are called a *representation* (of dimension m) of the distance space (V, d).

We are interested only in representations in Euclidean spaces with Euclidean and squared Euclidean distances. The famous result of Schoenberg (1935) says that a distance space (V, d) is of negative type if and only if it has a representation with d equal to squared Euclidean distance.

Similarly to $t^N_{0,1}(G)$, one can define the values $t^E_{0,1}(G)$, i.e. minimal and maximal values of t such that the distance space $(V, d_{G,t})$ has an Euclidean representation with distance equal to Euclidean distance. Since $(d_{G,t})^2 = d_{G,t^2}$, both these representations of a two-distance space are, in a sense, equivalent. Obviously, $(t^E_{0,1}(G))^2 = t^N_{0,1}(G)$.

An advantage of the representation of (V, d) in $\mathbf{R}^m$ with squared Euclidean distance is that

there is a one-to-one correspondence between linear dependences on the set of distances and linear dependences on the set of representing vectors.

Let vectors $v_i \in \mathbf{R}^m$, $i \in V$, represent a distance space (V, d), $d \in Neg_n$, such that

$$d(ij) = (v_i - v_j)^2. \tag{4}$$

For $d \in Neg_n$, let

$$\mathcal{B}(d) = \{b \in \mathbf{R}^n : \sum_{1 \le i < j \le n} b_i b_j d(ij) = 0, \ \sum_{i=1}^n b_i = 0\}.$$

Fact 1. *$\mathcal{B}(d)$ is a subspace of $\mathbf{R}^n$.*

Proof. If $d(ij) = (v_i - v_j)^2$, then the inequality (2) with $\sum_{i=1}^n b_i = 0$ takes the form $(\sum_{i=1}^n b_i v_i)^2 \ge 0$. For $b \in \mathcal{B}(d)$, we have the linear equality $\sum_{i=1}^n b_i v_i = 0$. Since the condition $\sum_{i=1}^n b_i = 0$ is linear, too, we obtain that

$$\mathcal{B}(d) = \{b \in \mathbf{R}^n : \sum_{i=1}^n b_i v_i = 0, \ \sum_{i=1}^n b_i = 0\}$$

is a subspace of the space $\mathbf{R}^n$. □

For $d \in Neg_n$, let $P(d)$ be the convex hull of endpoints of representing vectors v_i, $i \in V$. It is known that $P(d)$, up to translations and rotations, depends only on d.

Proposition 1 *Dimension of $P(d)$ is equal to $n - 1 - dim\mathcal{B}(d)$. In particular, if d is an inner point of Neg_n, then $P(d)$ is an $(n-1)$-dimensional simplex.*

Proof. Since $b_n = -\sum_{i=1}^{n-1} b_i$, we have

$$\sum_{i=1}^n b_i v_i = \sum_{i=1}^{n-1} b_i (v_i - v_n).$$

Without loss of generality, we can suppose that the $(n-1)$ vectors $v_i - v_n$ belongs to $\mathbf{R}^{n-1}$. If the point d lies in the interior of the cone Neg_n, then the sum $\sum_{i=1}^{n-1} b_i(v_i - v_n)$ is not zero for all $b \in \mathbf{R}^n$. In other words, the vectors $v_i - v_n$, $1 \le i \le n-1$, are linearly independent, and $P(d)$ is an $(n-1)$-dimensional simplex.

If d lies on the boundary of Neg_n, then the space $\mathcal{B}(d)$ is not empty, and there are linear dependencies between vectors $v_i - v_n$. Obviously, dimension of $P(d)$ is equal to $n - 1 - \dim\mathcal{B}(d)$. $\square$

Usually one proves assertions like Proposition 1 using some characteristic determinants that we consider below.

Let d be an inner point of Neg_n. Then $P(d)$ is an $(n-1)$-dimensional simplex. We take the vectors $v_i - v_n$, $1 \le i \le n-1$, as a basis of $\mathbf{R}^{n-1}$. Let $Gr_n(d)$ be the Gram matrix of these vectors, i.e. $(Gr_n(d))_{ij} = (v_i - v_n, v_j - v_n)$, $1 \le i, j \le n-1$. Using (4) it is easy to see that

$$(Gr_n(d))_{ij} = \frac{1}{2}(d_{in} + d_{jn} - d_{ij}).$$

Note that, using the Gram matrix, we can rewrite the inequality (2) $b^T D b \le 0$ as follows $b^T D b = -2b'^T Gr_n(d) b' \le 0$, i.e. $b'^T Gr_n(d) b' \ge 0$ for any $b' \in \mathbf{R}^{n-1}$.

Gram matrix is positive semidefinite. It is positive definite if the vectors $v_i - v_n$ are linearly independent. In other words, its minimal eigenvalue is positive. When the point d belongs to the boundary of Neg_n, then these vectors are linearly dependent and Gram matrix is positive semidefinite with zero as the smallest eigenvalue. In this case dimension of $P(d)$ is equal to $n - 1 - f$, where f is the multiplicity of the zero eigenvalue.

Gram matrix is not symmetric with respect to indices, the index n is special. But $Gr_n(d)$ is tightly related to the *Cayley-Menger matrix* $CM_n(d)$ of the order $n+1$, where

$$CM_n(d) = \begin{pmatrix} 0 & j_n^T \\ j_n & D \end{pmatrix},$$

and $j_n = (1, ..., 1)_T$ is the all-one column. It is known (see, for example, [6]) that

$$\det Gr_n(d) = \frac{(-1)^n}{2^{n-1}} \det CM_n(d).$$

It is well known (see, for example, [13]) that $Cut_n \subseteq Hyp_n \subseteq Neg_n$, and for $n \ge 7$ all these inclusions are strict. Obviously, $Hyp_n \subseteq Met_n$, but Met_n and Neg_n are not comparable. The cones Cut_n, Hyp_n and Met_n are polyhedral, while Neg_n is not polyhedral. Moreover, all extreme rays of Cut_n are extreme rays of Hyp_n and Met_n, and all facets of Met_n are facets of Cut_n and Hyp_n.

Note that the cones Hyp_n and Neg_n both are described by the inequalities (2) with $\sum_{i=1}^n b_i$ equal to 1 and 0, respectively. But it is sufficient a finite number of these inequalities for Hyp_n, while the description of Neg_n needs infinite number of inequalities (2).

Since a hypermetric space (V, d) is of negative type, it has a representation with d equal to squared Euclidean distance. Assouad (1980) proved that (V, d) is hypermetric if and only if the endpoints of the vectors of this representation lie on an empty sphere of the lattice affinely generated by these vectors. An n-dimensional lattice is an Abelian group of vectors of $\mathbf{R}^n$ integrally generated by n linearly independent vectors. An empty sphere of a lattice is a sphere such that no lattice point lies in the interior of the sphere. One considers usually an empty sphere such that the lattice points on the sphere affinely generate $\mathbf{R}^n$. In this case, the convex hull of all lattice points on an empty sphere is a *Delaunay polytope* P_D of the lattice. So $P(d)$ is the convex hull of some vertices of the Delaunay polytope P_D. Note that in this case P_D is uniquely determined by the distance d (see [13]). Hence we denote it by $P_D(d)$.

Since l_1-metric space (V,d) is hypermetric, the set V can be represented as a set of vertices of a Delaunay polytope $P_D(d)$. But the l_1-embeddability implies that $P_D(d)$ can be inscribed into a box. A box is a Delaunay polytope of a rectangular lattice generated by a set of mutually orthogonal vectors. In general, the dimension of the box, where $P_D(d)$ is inscribed, is greater than dimension of $P_D(d)$.

Let (V,d) be hypermetric. Then we can take the center of the corresponding Delaunay polytope $P_D(d)$ as origin. Substituting the representation (4) with $v_i^2 = r^2(d)$, where $r(d)$ is the radius of $P_D(d)$, into (2) with $\sum_{i=1}^n b_i = 1$, we obtain

$$(\sum_{i=1}^n b_i v_i)^2 \geq r^2(d).$$

This means that any affine integer combination $v(b) \equiv \sum_{i=1}^n b_i v_i$ of vectors v_i, being a point of the lattice affinely generated by these vectors, does not lie inside the empty sphere circumscribing $P_D(d)$.

Similarly to $\mathcal{B}(d)$ we can introduce the set

$$\mathcal{B}^H(d) = \{b \in \mathbf{R}^n : \sum_{1 \leq i < j \leq n} b_i b_j d(ij) = 0,\ \sum_{i=1}^n b_i = 1\}.$$

Obviously $\mathcal{B}^H(d) \neq \emptyset$ iff d lies on the boundary of Hyp_n. Substituting the expression (4) for d, we obtain

$$\mathcal{B}^H(d) = \{b \in \mathbf{R}^n : (\sum_{1 \leq i < j \leq n} b_i v_i)^2 = r^2(d),\ \sum_{i=1}^n b_i = 1\}.$$

In other words, if $b \in \mathcal{B}^H(d)$, then the point $v(b)$ is a vertex of $P_D(d)$. There are two possibilities: either $v(b) \in V$, or $v(b)$ is a new vertex of $P_D(d)$. The first case is possible only if there is an affine dependencies between vectors v_i, when these vectors span an affine space of dimension not greater than $n-2$. This means that d lies on the boundary of Neg_n, too. Since v_i generate $P_D(d)$, the dimension of $P_D(d)$ is not greater than $n-2$. Hence we have

Proposition 2 *If $dim P_D(d) \leq n-2$, then d lies on both the boundaries of Hyp_n and Neg_n.* $\square$

When a point d comes to the boundary of Hyp_n, then the simplex $P_D(d)$ is glued with some other simplexes of the L-partition of the lattice determined by the representation of d. If d comes to a facet F of Hyp_n determined by exactly one hypermetric equality, then the glued simplexes form a special Delaunay polytope called *repartitioning* polytope. It is studied in detail in [3].

Let the facet F is determined by a hypermetric equality with $b \in \mathbf{Z}^n$, $\sum_{i=1}^n b_i = 1$. Let

$$V_+ = \{i : b_i > 0\},\ V_- = \{i : b_i < 0\},\ V_0 = \{i : b_i = 0\}. \tag{5}$$

Let vectors v_i represent $d \in F$. Then the vectors v_i, $i \in V$, and the vector $v_0 = v(b) = \sum_{i=1}^n b_i v_i$ represent the repartitioning polytope. It is constructed as follows. Let S_- be the simplex spanned by endpoints of vecors v_i for $i \in V_- \cup \{0\}$. Similarly, let S_+ and S_0 be simplexes spanned by v_i for $i \in V_+$ and V_0, respectively. The simplexes S_+ and S_- span spaces that intersect in the point $v_c = \frac{1}{k}\sum_{i\in V_+} b_i v_i = \frac{1}{k}(v_0 + \sum_{i\in V_-} b_i v_i)$, where $k = \sum_{i\in V_+} b_i = 1 + \sum_{i\in V_-} |b_i|$. The point v_c is the common baricenter of both the simplexes S_+ and S_-. Let $V^{n'}_{n_+,n_-}$ be the convex hull of dimension n' of vertices of both the simplexes S_+ and S_-, where $n_+ = |V_+| - 1$ and $n_- = |V_-|$ are dimensions of S_+ and S_-. The vertices of the simplex S_0 do not belong to the space spanned by $V^{n'}_{n_+,n_-}$. The repartitioning polytope is the convex hull of vertices of $V^{n'}_{n_+,n_-}$ and S_0. It is denoted as $V^{n-2}_{n_+,n_-}$, where n is the number of its vertices and $n-2$ is its dimension. Note that the notation $V^m_{p,q}$ does not describe a concrete polytope, but a class of affinely equivalent repartitioning polytopes of dimension m with simplexes of dimensions p and q intersecting in the common baricenter.

If the points of a representation lie on a sphere, then the representation is called *spherical.* We saw that (V,d) has a spherical representation if d belongs to Hyp_n. Moreover, since every simplex can be inscribed into a sphere, the distance space has a spherical representation if d is an inner point of Neg_n. In this case the radius r of the circumscribing $P(d)$ sphere is given (see [6]) by

$$r^2(d) = -\frac{1}{2}\frac{\det D}{\det CM_n(d)}.$$

We see that, according to Proposition 1, the only case when $P(d)$ may be not inscribed into a sphere is the case when d belongs to the boundary of Neg_n. But there are cases when $P(d)$ is inscribed into a sphere although d lies on the boundary of Neg_n.

Call a distance space *regular* (of *strength 1*, in terms of [26]) if its distance matrix D has the all-one vector j_n as an eigenvector. This means that the sum of all elements of a row (or a column) of D does not depend on this row (and column). It is well known that if a regular distance space has a representation, then it has necessarily a spherical representation, too. In fact, we can take the origin of the representation space in the center of mass $\frac{1}{n}\sum_{i=1}^n v_i$ of the vectors v_i such that then $\sum_{i=1}^n v_i = 0$. Let α be the eigenvalue of D corresponding to the eigenvector j_n. Then taking in attention (4), we have

$$(Dj_n)_i = \sum_{j=1}^n (v_i - v_j)^2 = nv_i^2 + \sum_{i=1}^n v_j^2 = \alpha,$$

i.e. v_i^2 does not depend on i, that is all v_i have the same *norm* =squared length r^2.

Setting in the above equality $v_i^2 = v_j^2 = r^2$, we obtain the following value of the radius of the spherical representation of a regular distance space:

$$r^2(d) = \frac{1}{2n}\sum_{j=1}^{n} d(ij). \tag{6}$$

In particular, setting $d(ij) = a^2$, we obtain the following very useful formula for the squared radius of a regular $(n-1)$-dimensional simplex with length of edges a

$$r^2 = \frac{a^2}{2}\frac{n-1}{n}. \tag{7}$$

3 Two-distance spaces

For fixed G and varying t, the two-distances $d_{G,t}$ form a line in $\mathbf{R}^{\binom{n}{2}}$ going through the point $(1,1,...,1) = d_{G,1}$. For $t_k = t_k^A$, $k = 0,1$, the points d_{G,t_k} are the endpoints of a segment of this line lying in the cone C_n^A. The segments of the line $d_{G,t}$ lying in Cut_n, Hyp_n and Neg_n are contained each in other, respectively.

There are $2^{\binom{n}{2}} - 1$ lines $d_{G,t}$, since there are $2^{\binom{n}{2}} - 1$ graphs on n vertices distinct from K_n. In fact, we consider subsets E of the set E_n of all unordered pairs (=edges) of distinct points of V. The (labeled) graph $G = G(E)$ induced by the set E of its edges is a convenient denotation of the subset E. The set of all labelled n-point graphs is partitioned into classes of isomorphic graphs. Note that the cones C_n^A, $A = C, H, M, N$, are invariant under permutations of coordinates. Hence the values of $t_{0,1}^A$ are the same for all isomorphic graphs for all A. So it is natural to consider nonlabeled graphs only.

For the complete graph K_n, $d_{K_n,t}$ degenerates into the point $(1,1,...,1)$. For the empty graph $\overline{K_n}$, the line $d_{\overline{K_n},t}$ coincides with the central axis $(t,t,...,t)$ of the cones Cut_n, Hyp_n, Met_n and Neg_n. We see, that the graph G determines, in a sense, the angle between the line $d_{G,t}$ and the central axis. The points, where the line $d_{G,t}$ intersects the boundary of the cones Cut_n and Hyp_n, are given by $t = t_{0,1}^{C,H}(G)$, respectively.

Since the cut metric c_S takes only two values 0 and 1, it is the two-distance semimetric $d_{K(S,\overline{S}),0}$, where $\overline{S} = V - S$ and $K(S,\overline{S})$ is the complete bipartite graph with the partition $V = S \cup \overline{S}$.

Below we use the common notations K_n, P_n, C_n for the complete graph, the path and the cycle, each on n vertices. $K_{p,q}$ is the complete bipartite graph with parts of size p and q. The *direct product* $G_1 \times G_2$ of graphs G_1 and G_2 is the graph with the set of all pairs (v_1, v_2), $v_1 \in V_1$, $v_2 \in V_2$ as the set of vertices. Two vertices (v_1, v_2) and (u_1, u_2) are adjacent iff one of the coordinates coincide and the other coordinates are adjacent in the corresponding graph. Denote by $G_1 + G_2$ the disjoint union of graphs G_1 and G_2 such that kG is the disjoint union of k graphs isomorphic to G. The graph $\overline{K_1 + G}$ is called *suspension* of G and is denoted as ∇G.

Let $Q = (q_1, ..., q_m)$ be a sequence of positive integers. Denote by $K(Q)$ the disjoint sum $\sum_{i=1}^{m} K_{q_i}$. The graphs $K(Q)$ play a special role. The distance spaces $(V, d_{K(Q),t})$

are just all *ultrametrics* in the class of two-distance spaces. An ultrametric is a metric d satisfying the inequalities $d(ij) \leq \max\{d(i,k), d(j,k)\}$ for every $k \in V$, and any $i, j \in V$. It is proved in [24] that all Euclidean representations of ultrametrics are $(|V|-1)$-dimensional simplexes. This implies, and it is obvious for two-distance spaces $(V, d_{K(Q),t})$, that

$$t^N(K(Q)) = \infty. \tag{8}$$

The representation of $(V, d_{K(Q),t})$ for every t is the simplex being the convex hull of m unit simplices $P(d_{K_{q_i},t})$ spanning mutually orthogonal $(q_i - 1)$-dimensional spaces, $1 \leq i \leq m$. Moreover, it is easy to verify that the simplex $P(d_{K(Q),t})$ is a Delaunay polytope for all $t \geq 1$. Hence

$$t^C(K(Q)) = t^H(K(Q)) = \infty.$$

The following graphs are special cases of $\overline{K(Q)}$ which is the complete multipartite graph $K_{q_1,\dots,q_m}$. In particular, if $m = 2$, K_{q_1,q_2} is the complete bipartite graph. $\overline{K_1 + K_q} = \nabla \overline{K_q} = K_{1,q}$ is a star, a special case of $K_{p,q}$. If $q_i = q$, $1 \leq i \leq m$, then $K_{q,\dots,q} = K_{m \times q}$ is the complete m-partite graph. In particular, if $q = 2$, $K_{2,\dots,2} = K_{m\times 2}$ is the Cocktail-party graph.

The distance $d_{\overline{K(Q)},t}$ is a metric for all t such that $0 \leq t \leq 2$. Moreover, if $m = 2$, then $\overline{K(Q)} = K_{q,p}$ and $d_{K_{q,p},0}$ is a cut metric and spans an extreme ray of Cut_n. Otherwise, $d_{\overline{K(Q)},0}$ is a 0-extension of $d_{K_m,1}$, and lies on a face of Cut_n. In all these cases, $d_{\overline{K(Q)},0}$ is l_1-metric. Similarly, the metric $d_{\overline{K(Q)},2}$ spans an extreme ray of Met_n if $\overline{K(Q)}$ contains $K_{2,3}$ as an induced subgraph. In this case $d_{\overline{K(Q)},2}$ is not hypermetric.

If $G \neq \overline{K(Q)}$, then $d_{G,t}$ is not a metric for $t < \frac{1}{2}$.

The property $d_{\overline{G},t} = t d_{G,\frac{1}{t}}$ implies that $d_{K(Q),t}$ is a metric for $t \geq \frac{1}{2}$ but not hypermetric for $t = \frac{1}{2}$.

Obviously,

$$t_0^H(G) \leq t_0^C(G) \leq t^C(G) \leq t^H(G).$$

By the properties of $d_{G,t}$, and since l_1-metrics and hypermetrics are metrics, if $G \neq K(Q)$, then $t^{C,H}(G) \leq 2$.

(1) and (8) imply that $t_0^N(\overline{K(Q)}) = 0$.

Recall that a subgraph H of G with the set of vertices $V' \subseteq V$ is called *induced* subgraph if edges of H are all edges of G with both ends in V'. The following lemma is useful in what follows.

Lemma 1 *If H is an induced subgraph of G, then $t^A(G) \leq t^A(H)$ for $A = C, H, N$.*

Proof. Obviously, any representation of G provides a representation for every induced subgraph of G. Similarly, if $d_{G,t}$ is an l_1-metric or a hypermetric, then for every induced subgraph H of G the distance $d_{H,t}$ is an l_1-metric or a hypermetric, respectively. □

4 Vector representations of two-distance spaces

Obviously, $\det Gr_n(d_{G,t})$ is a polynomial in t. It was used in [16]. The authors of [20] use the polynomial $P_G(t) = \det CM_n(d_{G,t}) = -(-2)^{n-1}\det Gr_n(d_{G,t})$. Maehara [24] considers the polynomial $P'_G(s) = \det CM_n(d_{\overline{G},1-s}) = P_{\overline{G}}(1-s)$ and proves that $s^{n-1}P'_G(\frac{1}{s}) = \phi(G;-s) - (-1)^n\phi(\overline{G};s-1)$, where $\phi(G;s) = det(sI - A(G))$.

When $t < t^N(G)$, the Gram matrix is not singular and $P_G(t) \neq 0$. If $t = t^N(G)$, there is a dependency between representing vectors and $P_G(t) = 0$. Hence $t^N(G)$ is the smallest root of $P_G(t)$ with $t > 1$.

Unfortunately, $t^H(G)$ is not always given by a root of a known polynomial. But, for a hypermetric distance, we have a spherical representation that simplifies some considerations.

Using the adjacency matrix $A(G)$ of G, we can express the distance matrix $D(G,t)$ of the two-distance space $(V, d_{G,t})$ as follows

$$D(G,t) = A(G) + tA(\overline{G}) = A(G) + t(J - I - A(G)) = t(J - I) + (1-t)A(G), \tag{9}$$

where I is the identity matrix and J is the all-one matrix.

If t is sufficient near to 1, then $d_{G,t}$ is hypermetric, since $Hyp_n \subset Neg_n$. Hence there is a Delaunay polytope $P_D(G,t) = P_D(d_{G,t})$ such that the set of vertices of G is a subset of vertices of $P_D(G,t)$, and the distance is equal to the squared Euclidean distance. When $d_{G,t}$ is an interior point of Hyp_n, then $P(d_{G,t}) = P_D(G,t)$ is a simplex.

The polytope $P_D(G,1)$ is a regular $(n-1)$-dimensional simplex $P(d_{G,1})$ with the length of edges 1. When t encreases (or decreases) such that $d_{G,t}$ lies in the interior of the hypermetric cone Hyp_n, then $P_D(G,t)$ continues to be simplex, but not regular. When $d_{G,t}$ goes out on the boundary of Hyp_n, $P_D(G,t)$ ceases to be simplex. But there are two possibilities depending on whether there are affine dependencies between representing points or not: either the set V spans a basic $(n-1)$-dimensional simplex of $P_D(G,t)$ or $P_D(G,t)$ has dimension less than $n-1$ and V spans a subpolytope of $P_D(G,t)$. We shall see later that both these cases occur for bipartite G.

For $t = t^H(G)$, denote $P_D(G,t)$ as $P_D(G)$. Note that the distance space generated by all vertices of $P_D(G)$ is not, in general, a two-distance space. For example, the unit cube γ_n realizes all distances k, for $1 \leq k \leq n$. For the star $K_{1,n}$, we have $t^H(K_{1,n}) = 2$ and $P_D(K_{1,n}) = \gamma_n$.

Let the origin be in the center of $P_D(G,t)$, and let v_i represent the vertex i of G. Then

$$d_{G,t}(i,j) = (v_i - v_j)^2 \tag{10}$$

Let r be the radius of the sphere circumscribing P. Then (10) implies

$$v_iv_j = \begin{cases} r^2 - \frac{1}{2} & \text{if } (ij) \in E(G), \\ r^2 - \frac{1}{2}t & \text{if } (ij) \notin E(G), \\ r^2 & \text{if } i = j. \end{cases} \tag{11}$$

Recall that a regular distance space has a spherical representation with radius given by (6). Of course, a two-distance space $(V, d_{G,t})$ is regular if and only if the graph G is regular.

Let G be regular of valency q. Then $\sum_{j=1}^{n} d_{G,t}(ij) = q + t(n-1-q)$. Hence, by (6), for the squared radius of the spherical representation we have

$$r^2(G,t) = \frac{q + t(n-1-q)}{2n}. \tag{12}$$

Of course if a space (V, d) has a spherical representation with a radius r, then it has a spherical representation with any radius greater than r (may be of dimension one greater).

We note the following useful fact proved in [20].

Fact 2. *Let $G = G_1 + G_2$ or $G = \overline{G_1 + G_2}$ such that $V = V_1 \cup V_2$. Let the distance space $(V, d_{G,t})$ have a representation. Then this representation is as follows: for $i = 1, 2$, $(V_i, d_{G_i,t})$ has a spherical representation such that V_1 and V_2 span orthogonal spaces that intersect in at most one point.* $\square$

Let the vectors v_i, $1 \le i \le n$, give a representation of the two-distance space $(V, d_{G,t})$. The matrix $Q(G,t) \equiv (v_i v_j)_{1 \le i,j \le n}$ is the Gram matrix of the vectors v_i, $1 \le i \le n$. If this representation is spherical, then, according to (11),

$$Q(G,t) = r^2 J - \frac{t}{2}(J - I) + \frac{t-1}{2} A(G) = r^2 J - \frac{t+1}{4}(J - I) - \frac{t-1}{4} B(G), \tag{13}$$

where $B(G) = (J - I) - 2A(G)$ is the Seidel (∓ 1)-adjacency matrix of G. Comparing with (9), we obtain

$$Q(G,t) = r^2 J - \frac{1}{2} D(G,t).$$

As a Gram matrix, $Q(G,t)$ is positive semidefinite, and all its eigenvalues are nonnegative. Let $\lambda_0(G,t)$ be the minimal eigenvalue of $Q(G,t)$. We set

$$t_0^Q(G) = \min\{t : \lambda_0(G,t) \ge 0\},\ t^Q(G) = \max\{t : \lambda_0(G,t) \ge 0\}.$$

Obviously, $t_0^Q(G) \le t_0^H(G)$, and $t^Q(G) \ge t^H(G)$. These inequalities hold as equalities if the representation given by $Q(G,t)$ is a hypermetric representation, i.e. the set of endpoints of vectors v_i, $1 \le i \le n$, is a subset of vertices of a Delaunay polytope.

If G is a regular graph of valency q, then one can give explicit values of $t_{0,1}^Q(G,t)$. In this case, the all one vector j is an eigenvector of $A(G)$ with the eigenvalue q. It is easy to see that j is also an eigenvector of the matrix $Q(G,t)$ with the eigenvalue $(r^2 - \frac{t}{2})n + \frac{t}{2} + \frac{t-1}{2}q$. Note that this eigenvalue is nonnegative only if $r^2 \ge r^2(G,t)$, where $r^2(G,t) \equiv \frac{q+t(n-1-q)}{2n}$ is $r^2(d_{G,t})$ of (6). It is no wonder, since $r(G,t)$, the radius of the sphere with the center in the center of mass, is the radius of the minimal sphere circumscribing $P(G,t)$.

Let $\lambda(G)$ be the second largest eigenvalue of G and $-\mu(G)$ be the smallest eigenvalue of G. The smallest eigenvalue is negative, since all eigenvalues of $A(G)$ are real and the sum of all eigenvalues equals the trace of $A(G)$, i.e. it is zero. Then

$$\lambda_0(G,t) = \begin{cases} \frac{t}{2} - \lambda(G)\frac{1-t}{2} & \text{if } t \le 1 \\ \frac{t}{2} - \mu(G)\frac{t-1}{2} & \text{if } t \ge 1 \end{cases}$$

It is known that $\mu(G) > 1$ if $G \neq K(Q)$ (see, for example, [5], Corollary 3.5.4.). Hence we have

$$t_0^Q(G) = \frac{\lambda(G)}{1+\lambda(G)},\ t^Q(G) = \frac{\mu(G)}{\mu(G)-1} \tag{14}$$

Note that $\mu(G) = k+1$ for the bipartite graph $K_{k+1,k+1}$. Hence $t^Q(K_{k+1,k+1}) = \frac{k+1}{k}$.

The condition (14) is a sufficient condition that the vertices of G have a representation by points v_i of a sphere with distances (10). But (14) says nothing about what is the obtained distance space. Is it a metric, hypermetric or l_1-space?

Note that the smallest eigenvalue of $Q(G,t)$ is equal to 0 if and only if the vectors v_i are linearly dependent.

Usually one interests in two-distance sets in an Euclidean space of given dimension m of maximal possible size $n(m)$. Obviously, the two-distance set with $n(1) = 3$ collinear points is given by two points and their mid-point. A regular pentagon provides $n(2) = 5$ points of the maximal two-distance set in $\mathbf{R}^2$. It is proved in [16] that two-distance sets in $\mathbf{R}^3$ with $n(3) = 6$ points are represented by 6 polyhedrons.

A regular m-dimensional cross-polytope (it is a regular octahedron if $m = 3$) with $\binom{m+1}{2}$ vertices provide a lower bound on $n(m)$. For $m = 6$ and $m = 22$, two-distance sets are known with $\frac{1}{2}m(m+3)$ points given by maximal sets of equiangular lines.

The best general upper bound is due to Blokhuis [7] who has shown that $n(m) \leq \binom{m+2}{2} = \frac{m(m+3)}{2} + 1$. It was known that this bound is tight for $m = 1$ and is not tight for $m = 2,3$. Recently, P. Lisoněk [23] found a 8-dimensional two-distance set with $n(8) = \binom{8+2}{2} = 45$ points proving that Blokhuis's bound is tight for $m = 8$, too. This set consists of vertices of the Johnson polytope $PJ(9,2)$ and the regular simplex $PJ(9,1)$ which is inscribed into $PJ(9,2)$ such that the vertices of the simplex lie beyond the boundary of the Johnson polytope $PJ(9,2)$. This polytope realized the distance $d_{G,2}$, where $G = \tilde{T}(10)$ is obtained from the triangular graph $T(10)$ by switching a maximal clique (of size 9).

We can reformulate problem of a maximal two-distance set in a space of given dimension in terms of $t^N(G)$ as follows. Let $P(G)$ be $P(d_{G,t})$ for $t = t^N(G)$, and $m(G) = \dim P(G)$. Let $m(n) = \min_G\{m(G) : |V(G)| = n\}$

What is minimal dimension $m(n)$ of polytopes $P(G)$ with n vertices?

The function $m(n)$ is the reciprocal function to $n(m)$.

It is proved in [21] the following striking fact (Theorem 2 in [21]): if $n > 2m(G)+3$, then $t^N(G) = \frac{k+1}{k}$, the quotient of two neighboring positive integers such that $k < \frac{1}{2} + \sqrt{\frac{1}{2}m(G)}$. Neumaier [26] improved this result showing that it is true for $n \geq \max\{5, 2m(G)+1\}$. We reformulate this fact as follows:

Theorem A *If $m(G) \leq \frac{n-1}{2}$, then $t^N(G) = \frac{k+1}{k}$ for some positive integer $k < \frac{1}{2} + \sqrt{\frac{n-1}{4}}$.*

5 Bounds on $t^N(G)$

Schoenberg [28] considered the following problem: find a distance space (V, d) with minimal distance 1 such that (V, d) has an $(n-2)$-dimensional representation, $n = |V|$, and the *diameter* of d

$$t_n = \max_{(ij)} d(ij)$$

is minimal. He called these spaces *quasiregular*. Schoenberg gave bounds for diameter t_n and conjectured that quasiregular distance spaces are two-distance spaces.

Seidel [29] proved that, for any n, there exists a unique (up to labelings of points) quasiregular distance space. This implies the validity of Schoenberg's conjecture and that the bound given by Schoenberg is the exact value of t_n.

According to Proposition 1, d lies on the boundary of the cone Neg_n. The following nice interpretation of this problem given in [28] shows that t_n is, in fact, a lower bound on $t^N(G)$.

Consider a subset $B(t) \subset \mathbf{R}^{\binom{n}{2}}$ of all d such that $1 \leq d(ij) \leq t$ for all (ij). Obviously $B(t)$ is the convex hull of all points $d_{G,t}$, where G takes all $2^{\binom{n}{2}}$ values. $B(t)$ is $\binom{n}{2}$-dimensional cube whose edges are parallel to the coordinate axes of $\mathbf{R}^{\binom{n}{2}}$, and the vertex $d_{K_n,t}$ is the point (1,1,...,1). The vertices $d_{G,t}$ and $d_{\overline{G},t}$ are opposite vertices of the cube $B(t)$. Schoenberg showed that t_n is the value of $t > 1$, when a vertex of $B(t)$ touch for the first time the boundary of the cone Neg_n.

Let G_0 be the graph corresponding to the vertex of $B(t_n)$ touching the boundary of Neg_n. Obviously, $t_n = t^N(G_0) \leq t^N(G)$. In fact, if G is not isomorphic to G_0, we have here a strict inequality, and this fact Schoenberg proved also. The graph G_0 is isomorphic to one of the bipartite graphs $K_{k,k}$ or $K_{k,k+1}$ depending on $n = 2k$ or $n = 2k+1$.

The same bound was found also in [14], where *metric transforms* of a distance space were considered. In particular, a value of c_n was given, where c_n is the largest c such that the metric transform $d^c_{G,2} = d_{G,2^c}$ of the truncated metric $d_{G,2}$ has an Euclidean representation for all n-points graphs G. It was shown in [14] that, for the bipartite graphs $K_{k,k}$ and $K_{k,k+1}$ the value of c_n is exact. In our terms, we have $d_{G,2^{2c_n}} \in Neg_n$ for all G, and $d_{G_0,2^{2c_n}}$ belongs to the boundary of Neg_n, i.e. $t^N(G_0) = 2^{2c_n}$, or $c_n = \frac{1}{2} log_2 t^N(G_0)$.

We give once more proof of this fact using the inequalities (2) of negative type.

Denote by $h_G(t, b)$ the left hand side of the inequality (2) with $d = d_{G,t}$. According to Remark 1, we consider rational b_i, and suppose that $\sum_{i=1}^n |b_i| = 2$. Let $V_+(b) = \{i \in V : b_i > 0\}$, $V_-(b) = \{i \in V : b_i < 0\}$, and $V(b) = V_+(b) \cup V_-(b) \subseteq V$. Note that $\sum_{i \in V_+(b)} b_i = \sum_{i \in V_-(b)} |b_i| = 1$. Let $G(b)$ be the subgraph of G induced on the set $V(b)$. Let $K(b)$ be the complete bipartite graph $K_{p,q}$ on the set $V(b)$ with the partition $(V_+(b), V_-(b))$, i.e. $p = |V_+(b)|$, $q = |V_-(b)|$. Let $E_b(G) = E(G(b)) \Delta E(K(b))$ be the symmetric difference between the sets of edges of the graphs $G(b)$ and $K(b)$.

Similarly as it was shown in [3], one can show that

$$h_G(b, t) = (t-1) - \frac{t}{2}\sum_{i=1}^n b_i^2 - (t-1) \sum_{(ij) \in E_b(G)} |b_i||b_j|. \qquad (15)$$

If we set

$$h_G(t) = \max_b h_G(b,t),$$

then

$$t^N(G) = \max\{t : h_G(t) \le 0\}.$$

Proposition 3 *We set*

$$f_n(k) = \begin{cases} \frac{k}{k-1} & \text{if } n = 2k, \\ \frac{2k(k+1)}{2k^2-1} & \text{if } n = 2k+1. \end{cases} \tag{16}$$

Then

$$f_n(k) \le t^N(G) \le 2 + \sqrt{3}$$

with an equality in the left hand side if $G = K_{k,k}$ *or* $G = K_{k,k+1}$.

Proof. We show that $h_G(b,t) \le 0$ for t equal to values given in the right hand side of (16) for all b with $\sum b_i = 0$. Obviously, maximum of $h_G(b,t)$ is achived for $b \in \mathbf{R}^n$ such that the second and the third terms of (15) are minimal in absolute value. The sum $\sum_{i=1}^n b_i^2$ with $\sum_{i \in V_+(b)} b_i = \sum_{i \in V_-(b)} |b_i| = 1$ takes its minimal value when b_i's are almost equal for all i.

If $n = 2k$, then the sum $\sum_{i=1}^n b_i^2$ takes the minimal value $\frac{2}{k}$ for $|b_i| = \frac{1}{k}$. Hence we have

$$h_G(b,t) \le (t-1) - \frac{t}{2}\frac{2}{k} = \frac{k-1}{k}(t - \frac{k}{k-1}).$$

This implies that $h_G(b, \frac{k}{k-1}) \le 0$ for all b if $n = 2k$.

Now, let $n = 2k+1$. Then the sum $\sum_{i=1}^n b_i^2$ takes the minimal value $\frac{2k+1}{k(k+1)}$ when $b_i = |V_+(b)|^{-1}$ for $i \in V_+(b)$, $b_i = -|V_-(b)|^{-1}$ for $i \in V_-(b)$, and either $|V_+(b)| = |V_-(b)| - 1 = k$ or $|V_+(b)| - 1 = |V_-(b)| = k$. In these cases

$$h_G(b,t) \le (t-1) - \frac{t}{2}\frac{(2k+1)}{k(k+1)} = \frac{2k^2-1}{2k(k+1)}(t - \frac{2k(k+1)}{2k^2-1}).$$

This implies that $h_G(b, \frac{2k(k+1)}{2k^2-1}) \le 0$ for all b if $n = 2k+1$.

Let $G = K_{k,k}$ with the partition $V = V_1 \cup V_2$ such that $|V_1| = |V_2| = k$. Then for b such that $b_i = \frac{1}{k}$ for $i \in V_1$, and $b_i = -\frac{1}{k}$ for $i \in V_2$, we have $h_G(b, \frac{k}{k-1}) = 0$.

If $G = K_{k,k+1}$ with the partition $V = V_1 \cup V_2$ such that $|V_1| = k$ and $|V_2| = k+1$, we set $b_i = \frac{1}{k}$ for $i \in V_1$ and $b_i = -\frac{1}{k+1}$ for $i \in V_2$, we have $h_G(b, \frac{2k(k+1)}{2k^2-1}) = 0$.

Hence the above bound is tight for $G = K_{k,k}$ and $G = K_{k,k+1}$.

One can reformulate Theorem 1 of [19] as the inequality

$$t^N(G) \le 2 + \sqrt{3}$$

valid for $n \ge 4$, i.e. as an upper bound on $t^N(G)$. This bound is implied by the following. If $n \ge 4$, then any representation of $(V, d_{G,t})$ contains a representation of a 4-point distance space $(V_4, d_{G,t})$. But examples of all 4-points distance spaces (see below) show that $t^N(G) \le t^N(P_3 + K_1) = 2 + \sqrt{3}$ if $t^N(G) \neq \infty$. $\square$

Note that the upper bound of Proposition 3 does not depend on the dimension $m(G)$ of the corresponding representation of G. One can find in [20] more exact upper bounds depending on $m(G)$. We reformulate Theorems 1, 2 and 3 of [20] as follows.

Theorem B *Let $m(G)$ be dimension of a representation of $d_{G,t}$ for $t = t^N(G)$. Then*

1. If $m(G) = n-2$, then $t^N(G) \le t^N(P_3 + K_{n-3}) = \frac{9(n-3)-1+\sqrt{33(n-3)^2+14(n-3)+1}}{4(n-3)}$;

2. If $\frac{2}{3}n \le m(G) \le n-3$, then $t^N(G) \le 2 + \sqrt{2}$;

3. If $\frac{n}{2} \le m(G) < \frac{2}{3}n$, then $t^N(G) \le \tau^2$. $\square$

Recall here Theorem A which says that $t^N(G) = \frac{k+1}{k}$ if $m(G) \le \frac{n-1}{2}$.

The polytope $P(G,t)$ for $G = K_{k,k}$ or $G = K_{k,k+1}$ and $t = t^N(G)$ has dimension $n-2$. For $G = K_{k,k}$, it is a special case of the *repartitioning polytope*, and is a Delaunay polytope.

Recall that $t^M(K_{k,k}) = t^M(K_{k,k+1}) = 2$, what is greater than $t^N(K_{k,k})$ and $t^N(K_{k,k+1})$. Since there are graphs with $t^N(G) > 2$, we have examples exhibiting incomparability of the cones Met_n and Neg_n.

6 Bounds on $t^H(G)$

Let b determine a $(2q+1)$-gonal hypermetric inequality, i.e. $\sum_{i=1}^n |b_i| = 2q+1$. Note that since $\sum_{i=1}^n |b_i| \equiv \sum_{i=1}^n b_i = 1 \pmod 2$, the first sum here is odd. Note that $\sum_{i\in V_+(b)} b_i = \sum_{i\in V_-(b)} |b_i| + 1 = q+1$.

It is shown in [3] that

$$h_G(b,t) = q^2(t - \frac{q+1}{q}) - \frac{t}{2}\sum_{i=1}^n |b_i|(|b_i|-1) - (t-1)\sum_{(ij)\in E_b(G)} |b_i||b_j|, \tag{17}$$

where $E_b(G)$ is defined in the previous section.

Proposition 4 *Let $n = 2k+1$ or $n = 2k+2$ and G be not a sum of complete graphs. Then*

$$\frac{k+1}{k} \le t^H(G) \le 2,$$

with an equality in the left hand side if $G = K_{k,k+1}$ or $G = K_{k+1,k+1}$, and an equality in the right hand side if the truncated distance d^*_G is hypermetric.

Proof. Let

$$h_G(q,t) = \max\{h_G(b,t) : \sum_{i=1}^n |b_i| = 2q+1\}.$$

We show that $h_G(q,t) \le 0$ for $t = \frac{k+1}{k}$ and all $q \ge 1$. Obviously, $\sum_{(ij)\in E_b(G)} |b_i||b_j| \ge 0$. Similarly $\sum_{i=1}^n |b_i|(|b_i|-1) \ge 0$, since $x(x-1) \ge 0$ for all integer x. Hence $\max_b h_G(b,t)$

is achieved for $b \in \mathbf{Z}^n$ such that the second and the third terms of (17) are minimal in absolute value.

Let $a = \lfloor \frac{2q+1}{n} \rfloor$. The sum $\sum_{i=1}^n |b_i|(|b_i|-1)$ with $\sum_{i=1}^n |b_i| = 2q+1$ takes its minimal value when $|b_i|$'s are almost equal for all i. This means that either $|b_i| = a$ or $|b_i| = a+1$. If $|b_i| = a$ for x values of i, then $|b_i| = a+1$ for $n-x$ values of i. We have $xa+(n-x)(a+1) = 2q+1$, i.e. $x = n(a+1)-(2q+1)$. In this case $\sum_{i=1}^n |b_i|(|b_i|-1) = a(2q+1-\frac{n}{2}(a+1))$, and therefore

$$h_G(q,t) \le q^2(t - \frac{q+1}{q}) - ta(2q+1-\frac{n}{2}(a+1)).$$

We have $2q+1 = an+s$, where $0 \le s \le n-1$. If $a=0$, then $2q+1 \le n-1 \le 2k+1$, i.e. $q \le k$. Since $\frac{k+1}{k} \le \frac{q+1}{q}$ for $q \le k$, we prove that

$$h_G(q, \frac{k+1}{k}) \le 0 \text{ for } 2q+1 < n.$$

Hence we suppose below that $a \ge 1$.

We have $q = \frac{1}{2}(an+s-1)$. Substituting q in the above inequality with a by this value, we obtain

$$h_G(q,t) \le t[(\frac{1}{4}(an+s-1)^2 - a(an+s-\frac{n}{2}(a+1)] - \frac{1}{4}((an+s)^2-1).$$

Setting $f(a,n,s) = a(n-2)(an+2s)+(s-1)^2$, one can rewrite the above inequality as follows

$$h_G(q,t) \le \frac{1}{4} f(a,n,s)\left[t - \frac{n}{n-2} - \frac{2(s-1)(n-1-s)}{(n-2)f(a,n,s)}\right].$$

Since $f(a,n,s) \ge 0$, the inequality $h_G(q,t) \le 0$ is valid if the expression in the square parantheses is not greater than zero. If $s > 0$, then $(s-1)(n-1-s) \ge 0$. In particular, if n is even, then $s \ne 0$. Hence, for $n = 2k+2$ even, the expression in the square parantheses takes the maximal value $t - \frac{n}{n-2} = t - \frac{k+1}{k}$ for $s=1$ or for $s = n-1$. For $n = 2k+1$ odd, the expression in the square parantheses takes the maximal value

$$t - \frac{n}{n-2} + \frac{2}{(n-2)(n-1)} = t - \frac{n+1}{n-1} = t - \frac{k+1}{k}$$

for $s=0$ and $a=1$ when $f(1,n,0) = (n-1)^2$. In both these cases we obtain that $h_G(q,t) \le 0$ for $t = \frac{k+1}{k}$ and all q.

It is not difficult to verify that for $G = K_{k,k+1}$ and $G = K_{k+1,k+1}$ this bound is tight.

Since a hypermetric is a metric, there is an obvious upper bound $t^H(G) \le 2$ if $G \ne K(Q)$. Obviously this bound is achived if the truncated distance d^*_G is hypermetric. □

We reformulate here Proposition 2 for a two-distance space. Recall that $P_D(G)$ is the Delaunay polytope of the distance space $(V, d_{G,t})$ for $t = t^H(G)$.

Proposition 5 *If $dim P_D(G) \le n-2$, then $t^N(G) = t^H(G)$.* □

Note that $P_D(K_{k+1,k+1})$ is a special case of the class of repartitioning polytopes $V_{k,k}^{2k}$ of dimension $2k = 2(k+1) - 2 = n - 2$. Hence Proposition 5 explains why we have $t^N(K_{k,k}) = t^H(K_{k,k})$ in both Propositions 3 and 4.

The repartitioning polytope $P_D(K_{k+1,k+1})$ is important for what follows. It is the convex hull of two regular k-dimensional simplexes S_1 and S_2, spanning orthogonal spaces and intersecting in the common center. The edges of both these simplexes S_i have norm $t^N(K_{k+1,k+1}) = t^H(K_{k+1,k+1}) = \frac{k+1}{k}$. We denote this polytope shortly as B_k. So, the $2k$-dimensional polytope B_k has $2k+2$ vertices and $(k+1)^2$ facets. Each facet is a simplex and is obtained by deleting by a vertex from both the simplexes S_1 and S_2. By construction the set of vertices of B_k form a two-distance space with distances 1 and $\frac{k+1}{k}$.

Note that the right hand side of Proposition 4 also has the form $\frac{k+1}{k}$ for $k = 1$. It is worth to recall Theorem A. Taking in attention Theorem A and Proposition 2, we give the following conjectures.

Conjecture 1. *In the case when the conditions of Theorem A hold, the distance d_{G,t^N} lies on the boundary of Hyp_n.*

7 Two-distance spaces and equiangular lines

Suppose that $4r^2 = 1 + t$ in the spherical representation (11) of the two-distance $d_{G,t}$. Then $v_i v_j = \pm\frac{t-1}{2}$ if $i \neq j$, i.e. the vectors v_i span equiangular lines. If G is regular of valency q, this is possible only if $\frac{1+t}{4} \geq r^2(G,t)$, i.e. if

$$t \begin{cases} \leq 1 + \frac{2}{n-2q-2} & \text{for } n \geq 2q+2, \\ \geq 1 - \frac{2}{2q+2-n} & \text{for } n \leq 2q+2. \end{cases} \tag{18}$$

The last inequality shows that if $q \geq \frac{n-2}{2}$, then regular G always has a spherical representation spanning equiangular lines.

If we replace a vector v_i by $-v_i$, we obtain again a two-distance $d_{G^{sw},t}$. Here the graph G^{sw} is obtained from G by *switching* of the vertex i. The edges and nonedges of G incident to the vertex i are interchanged in G^{sw}. The switching of G by a set of vertices is clear. Two graphs are called *switching equivalent* if one is a switching of other. Clearly, switching equivalent graphs determine isomorphic sets of eqiangular lines.

According to (13), if $r^2 = \frac{1+t}{4}$, we have

$$Q(G,t) = \frac{t+1}{4}I - \frac{t-1}{4}B(G).$$

This expression shows explicitly that $Q(G,t)$ is related to equiangular lines, since the Seidel matrix $B(G)$ takes (∓ 1)-values.

We can easy find the acute angle α between the corresponding equiangular lines. If $r^2 = \frac{1+t}{4}$ and the maximal inner product $v_i v_j = r^2 \cos\alpha$ is equal, according to $Q(G,t)$, to $\frac{t-1}{4}$, we have

$$\cos\alpha = \frac{t-1}{t+1}. \tag{19}$$

Note that, considering equiangular lines, one corresponds usually adjacency to v_i and v_j with negative inner product v_iv_j, i.e. one uses the complemented graph $\overline{G}$. Following to Neumaier [26], and using (13), we rewrite $Q(G,t)$ for $r^2 = \frac{1+t}{4}$ with $A(\overline{G})$:

$$Q(G,t) = \frac{t-1}{2}(\frac{1}{t-1}I - A(\overline{G}) + \frac{1}{2}J). \tag{20}$$

Hence if the largest eigenvalue $\lambda_{max}(\overline{G}) \leq \frac{1}{t-1}$, then $Q(G,t)$ is positive semidefinite, and we have

Proposition 6 *[26]. Let $\overline{G}$ be a graph which is switching equivalent to a graph H with $\lambda_{max}(H) \leq \frac{1}{t-1}$. Then $d_{\overline{G},t}$ is represented by equiangular lines with the angle $arcos\frac{t-1}{t+1}$.*

If G is regular, then eigenvalues of G and $\overline{G}$ are related as $\lambda(\overline{G}) = -(1+\lambda(G))$, $\lambda(G) \neq q(G)$, and $q(\overline{G}) = n-1-q(G)$. Hence the minimal eigenvalue $\lambda_{min}(G) = -\mu(G)$ corresponds to the second largest eigenvalue $\lambda_2(\overline{G})$ of $\overline{G}$.

There are two upper bounds on the number of equiangular lines in an m-dimensional space. The *absolute* bound does not depend on the angle between lines:

$$n \leq n_a(m) = \frac{m(m+1)}{2}.$$

The *special* bound is valid for $m < \frac{1}{\cos^2\alpha}$:

$$n \leq n_s(\alpha, m) = \frac{m(\cos^{-2}\alpha - 1)}{\cos^{-2}\alpha - m}.$$

In [22], the following analogue of Theorem A is proved.

Theorem C *If the number of equiangular lines in $\mathbf{R}^m$ is greater than $2m$, then $\frac{1}{cos\alpha}$ is an odd integer.* $\Box$

Denoting this integer by $2k+1$ we have $\cos\alpha = \frac{1}{2k+1}$. According to (19), we have in this case that

$$t = t(k) \equiv \frac{k+1}{k}.$$

For this value of $t = t(k)$, $r^2 = \frac{1+t}{4} = \frac{2k+1}{4k}$.

Take along each line two opposite vectors $\pm v$ of norm $v^2 = 2k+1$. Then $v_iv_j = \pm 1$ for $i \neq j$. It is convenient to denote the pair of opposite vectors $(v, -v)$ as (v, v^*). The set $\mathcal{V}$ of all these vectors is a special case of an odd system (see [12]). An *odd system* is a set $\mathcal{V}$ of vectors with odd inner products, and, in particular, having odd norms. Let

$$L(\mathcal{V}) = \{\sum_{v\in\mathcal{V}} b_v v : \sum_{v\in\mathcal{V}} b_v = 1 \pmod 2\}.$$

It is proved in [12] that $L(\mathcal{V})$ is a lattice. Let $\mathcal{U}_k$ be an m-dimensional odd system corresponding to a set of equiangular lines, i.e. $\mathcal{U}_k$ is a set of vectors of norm $2k+1$ with inner products ± 1. Let $P(\mathcal{U}_k)$ be the convex hull of all vectors of norm $2k+1$ of the lattice $L(\mathcal{U}_k)$.

$P(\mathcal{U}_k)$ is a symmetric m-dimensional polytope and turns out very often to be a symmetric Delaunay polytope of the lattice $L(\mathcal{U}_k)$. This implies that the corresponding two-distance space is hypermetric.

Let $Q(G,t)$ be the Gram matrix of $\mathcal{U}_k$. If G is not regular, then according to (20) $\lambda_{max}(\overline{G}) \le k$. If G is regular, then, by (14), $t^Q(G) = \frac{k+1}{k}$ if $\mu(G) = k+1$. This implies that $\lambda_2(\overline{G}) = k$. The cases $k=1$ and $k=2$ are especially interesting, since, for these k, conditions are known when $P(\mathcal{U}_k)$ is a Delaunay polytope. One knows many regular graphs with the second largest eigenvalue 2. We consider some of them in the last section.

For $\cos\alpha = \frac{1}{2k+1}$, the special bound takes the form

$$n_s(k,m) = \frac{4k(k+1)m}{(2k+1)^2 - m}. \tag{21}$$

Recall that this formula is valid only for $m < \frac{1}{\cos^2\alpha} = (2k+1)^2$.

Sets of equiangular lines, where the special bound is attained, are of a special interest. They are related to so-called *regular two-graphs*. Obviously, if the special bound is tight, then $n = n_s(k,m)$ is an integer. For a given k, there is a number of values of dimension m such that $n_s(k,m)$ is an integer.

There is a minimal value of m such that the integer $n_s(k,m) > m$. It is easy to find that the minimal value is equal to $m = 2k+1$ when

$$n_s(k, 2k+1) = 2(k+1).$$

Let $\mathcal{U}_k^0$ be the odd system corresponding to the set of $2(k+1)$ eqiangular lines. It consists of $2k+2$ vectors u_i, $1 \le i \le 2k+2$, of norm $2k+1$ with pairwise inner products -1 and of $2k+2$ its opposite u_i^*. It is easy to verify that $\sum_{i=1}^{2k+2} u_i = 0$. Note that, in any odd system of norm $2k+1$, every set of vectors with mutual inner products -1 has at most $2k+2$ vectors.

Let $\mathcal{V}_k^0 = \frac{1}{\sqrt{4k}}\mathcal{U}_k^0$, and $v_i = \frac{1}{\sqrt{4k}}u_i$. Let $V = \{1, 2, ..., 2k+1, 2k+2\}$, and $V = V_1 \cup V_2$, $|V_1| = p$, $|V_2| = q$, $p+q = 2k+2$, be a partition of V. Then the set of vectors $\{v_i : i \in V_1, v_j^* : j \in V_2\}$ represents the two-distance $d_{K_{p,q},t(k)}$, where $t(k) = \frac{k+1}{k}$. In the next section, we show that this representation is exact (i.e. with $t = t^H(G)$) if $p = k$, $q = k+2$ with $t^H(K_{k,k+2}) = t(k)$.

Baranovski [4] proves (in other terms) that $P(\mathcal{V}_k^0)$ coincides with the convex hull of $\mathcal{V}_k^0$, and $P(\mathcal{V}_k^0)$ is a symmetric Delaunay polytope of the lattice $L(\mathcal{V}_k^0)$ which is the Coxeter lattice A_{2k+1}^{k+1}. He denotes $P(\mathcal{V}_k^0)$ as $\mathcal{A}^{2k+1}$.

For a partition $V = V_1 \cup V_2$, $|V_1| = p$, $|V_2| = q$, let S^{p-1} and S^{q-1} be regular simplexes with edges of norm $t(k)$ such that these simplexes are the convex hulls of v_i, $i \in V_1$, and v_i^*, $i \in V_2$, respectivey. Let $S_{p,q}(t)$ be the convex hull of S^{p-1} and S^{q-1}. The distances between vertices of $S_{p,q}(t)$ from distinct simplexes are eqal to 1. Then $\mathcal{A}^{2k+1}$ is the convex hull of both $S_{p,q}(t)$ and its opposite with $p+q = 2k+2$. If $q = 0$, and $p = 2k+2$, we obtain that $\mathcal{A}^{2k+1}$ is the convex hull of the regular simplex S^{2k+1} and its opposite. So, all edges of $\mathcal{A}^{2k+1}$ have norm 1 or $t(k)$, and its diagonals have norm $1 + t(k)$. Each facet of $\mathcal{A}^{2k+1}$ is the $2k$-dimensional repartitioning polytope $P_D(K_{k+1,k+1}) = S_{k+1,k+1}(t(k))$, which we denoted in previous section as B_k. The polytope B_k is the convex hull of two regular k-dimensional simplexes spanning orthogonal spaces and intersecting in the common center. Note that $\mathcal{A}^3 = \gamma_3$, $B_1 = \gamma_2$, where γ_n is the unit n-dimensional cube.

8 Complete bipartite graphs

We saw that the complete bipartite graphs $K_{k,k+1}$ and $K_{k+1,k+1}$ provides the lower bound on $t^H(G)$ for G with the same number of vertices. We obtained that $t^H(K_{k,k+1}) = t^H(K_{k+1,k+1}) = t(k)$, and $P_D(K_{k,k+1}) = P_D(K_{k+1,k+1}) = B_k$. Now we consider the complete bipartite graphs $K_{p,q}$ for other values of p and q.

The case $p = 1$ is special. Recall that γ_q is the unit q-dimensional cube, which is the unique Delaunay polytope of the lattice $\mathbf{Z}^q$.

Proposition 7 *For $q \geq 2$, $t^H(K_{1,q}) = 2$ and $P_D(K_{1,q}) = \gamma_q$.*

Proof. The graph $K_{1,1}$ is the complete graph K_2 with $t^H(K_2) = \infty$. Hence we have to consider $q \geq 2$. By Proposition 4, $t^H(K_{1,q}) \leq 2$, for $q \geq 2$. The q mutually orthogonal edges of γ_q give a representation of $d_{K_{1,q},2}$. Obviously this representation generates the lattice $\mathbf{Z}^q$. $\square$

Let $V = V_1 \cup V_2$, where $|V_1| = p$ and $|V_2| = q$, are the set of vertices of $K_{p,q}$, and let $p \leq q$.

Recall that $h_G(k,t) = \max_b\{h_G(b,t) : \sum_{i=1}^n |b_i| = 2k+1\}$.

Lemma 2 *For $G = K_{p,q}$, $h_G(k,t) = k(k+1)((1 - f_{p,q}(k))t - 1)$, where*

$$f_{p,q}(k) = \frac{b_1(k - \frac{1}{2}(b_1+1)p) + b_2(k+1-\frac{1}{2}(b_2+1)q) + k}{k(k+1)},$$

and $b_1 = \lfloor \frac{k}{p} \rfloor$, $b_2 = \lfloor \frac{k+1}{q} \rfloor$.

Proof. Let b define a hypermetric inequality. Recall that $V_+(b) = \{i \in V : b_i > 0\}$, $V_-(b) = \{i \in V : b_i < 0\}$. Let b' be such that $b'_j = b_j$, $j \in V_2$, $b'_i = b_i$, $i \in V_1 - \{i_1, i_2\}$, $b'_{i_1} = b_{i_1} - \varepsilon$, $b'_{i_2} = b_{i_2} + \varepsilon$, ε is a positive integer. Let $\delta h_G(t) = h_G(b,t) - h_G(b',t)$. Then $\delta h_G(t) = t(\varepsilon(b_{i_2} - b_{i_1}) + \varepsilon^2)$. Obviously if $|b_{i_2} - b_{i_1}| > \varepsilon$, and signs of ε and $b_{i_2} - b_{i_1}$ are opposite, then $h_G(b',t) > h_G(b,t)$. We see that if there are two indexes i, i' both in V_1 or V_2 such that $|b_i - b_{i'}| > 1$, then there is a perturbation $b \to b'$ such that $h_G(b',t) > h_G(b,t)$. Hence, for b with maximal $h_G(b,t)$, b_i's of each part differs at most on 1. In particular, the coefficients of each part have the same sign.

If b defines a $(2k+1)$-gonal inequality, then $|\sum_{i\in V_2} b_j|$ is equal either to k or $k+1$. But comparing with (17), we see that if $|\sum_{i\in V_2} b_j| = k+1$, i.e. if $V_+(b) \subseteq V_2$, then $h_G(b,t)$ is not less than $h_G(b,t)$ for $V_+(b) \subseteq V_1$, since $q \geq p$.

Let b define a $(2k+1)$-gonal inequality. Then we saw that $\max_b h_G(b,t)$ for $G = K_{p,q}$ is achived for b_i such that $\sum_{i\in V_1} b_i = -k$, $\sum_{i\in V_2} b_i = k+1$, and $|b_i - b_{i'}| \leq 1$ for i, i' from the same part. Let $b_1 = \lfloor \frac{k}{p} \rfloor$, $b_2 = \lfloor \frac{k+1}{q} \rfloor$. Then $b_i = -b_1$ for $(b_1+1)p - k$ values of $i \in V_1$, and

$b_i = -(b_1+1)$ for other $k - b_1 p$ values of $i \in V_1$. Similarly, $b_j = b_2$ for $(b_2+1)q - (k+1)$ values of $j \in V_2$, and $b_j = b_2 + 1$ for other $k + 1 - b_2 q$ values of $j \in V_2$. In other words, the function $h_G(k,t)$ takes the form

$$h_G(k,t) = (k^2 - kb_1 - (k+1)b_2 + \frac{1}{2}pb_1(b_1+1) + \frac{1}{2}qb_2(b_2+1))t - k(k+1).$$

Introducing the function $f_{p,q}(k)$, we obtain the expression of $h_G(k,t)$ given in the formulation of this lemma. □

Proposition 8 *Let $G = K_{p,q}$ with $p \le q$, and $q = ps + r$, where $1 \le r \le p$. Then*

$$t^H(K_{p,q}) \le t(p,q) \equiv \frac{1}{1-\varphi(p,q)}, \text{ where } \varphi(p,q) \equiv \frac{(s+1)(q+r-2)}{2q(q-1)} = f_{p,q}(q-1),$$

with equality for $p = 1, q-2, q-1, q$, when

$$t^H(K_{p,q}) = t(p,q) = \begin{cases} 2 & \text{for } p = 1 \\ \frac{q}{q-1} & \text{for } p = q-1, q \\ \frac{q-1}{q-2} & \text{for } p = q-2 \end{cases}$$

Proof. Consider a $(2k+1)$-gonal inequality with $k = q-1$ and b such that $h_G(b,t) = h_G(k,t)$. Then $b_2 = 1$, and $b_1 = \lfloor \frac{k}{p} \rfloor = \lfloor \frac{q-1}{p} \rfloor = \lfloor \frac{ps+r-1}{p} \rfloor = s$, since $r \ge 1$. For these values of b_1 and b_2, $f_{p,q}(q-1) = \varphi(p,q)$.

Obviously, $h_{K_{p,q}}(q-1,t) = 0$ for $t = t(p,q)$, i.e. the distance $d_{G,t}$, for $G = K_{p,q}$ and $t = t(p,q)$ satisfies the above $(2q-1)$-gonal hypermetric equality. This implies that $t^H(K_{p,q}) \le t(p,q)$.

Note that if $p = 1$, then $s = q-1$ and $r = 1$, since $r \ge 1$. Hence $\varphi(1,q) = \frac{1}{2}$ and $t(1,q) = 2$, the result of Proposition 7. Similarly if $p = q-1$, then $s = r = 1$, and if $p = q$, then $s = 0$, $r = q$. Hence $\varphi(q-1,q) = \varphi(q,q) = \frac{1}{q}$, and $t(p,q) = \frac{q}{q-1} \equiv t(q-1) = t^H(K_{q-1,q}) = t^H(K_{q,q})$, by Proposition 4.

Now we consider $p = q-2$, when $s = 1$, $r = 2$, and $\varphi(q-2,q) = \frac{1}{q-1} = \frac{1}{p+1}$. Hence $t(p,p+2) = \frac{p+1}{p} = t(p)$. We know from Proposition 4 that $t^H(K_{p,p+1}) = t(p)$. Since $K_{p,p+1}$ is an induced subgraph of $K_{p,p+2}$, by Proposition 4 and Lemma 1, $t(p) = t^H(K_{p+1,p+1}) \le t^H(K_{p,p+2}) \le t^H(K_{p,p+1}) = t(p)$, i.e. $t^H(K_{p,p+2}) = t(p)$. □

Note, that if $b_j = 0$ for some $j \in V_2$, then the hypermetric inequality with this b is applied, in fact, to a graph $K_{p,q'}$ with $q' < q$. Since $K_{p,q'} \subseteq K_{p,q}$, $t^H(K_{p,q}) \le t^H(K_{p,q'})$, by Lemma 1. Hence, using induction on q, and that $\varphi(p,q) \le \varphi(p,q')$, we can suppose that $h_G(k,t) \le 0$ for $t = t(p,q)$, $G = K_{p,q}$, and for $k \le q-2$.

Conjecture 2. *$t^H(K_{p,q}) = t(p,q)$ where $t(p,q)$ is given in Proposition 8.*

For to prove this conjecture, we have to prove that $h_G(k,t) \le 0$ for $G = K_{p,q}$, $t = t(p,q)$ and all integers $k \ge q$. In other words, we have to prove that $\varphi(p,q) \le f_{p,q}(k)$ for all k. It is not difficult to verify that $\varphi(p,q) \le f_{p,q}(q)$. Unfortunately, the function $f_{p,q}(k)$ is not monotone on k and behaves very irregular when k encreases. We computed $f_{p,q}(k)$ for many values of p, q and k. For all these values the inequality $\varphi(p,q) \le f_{p,q}(k)$ holds.

The difficulty is such that, for given p and q, there is no unique expression of $f_{p,q}(k)$ for all k without the operation of taking integer part. But, for $q = ps$, we have

Proposition 9 $t^H(K_{p,ps}) = t(p,ps) = \frac{2p(ps-1)}{ps(2p-1)-(3p-2)}$.

Proof. If $q = ps$, then $r = p$. We set $k = bq + cp + a$, where $b \geq 1$, $cp + a < q$, i.e. $c \leq s-1$ and $a \leq p-1$. Now we have only two cases:

1) either $c \leq s-1$, $a \leq p-2$, or $c \leq s-2$, $a = p-1$, when $b_1 = bs + c$, $b_2 = b$, and

2) $c = s-1$, $a = p-1$, when $b_1 = bs + c$, $b_2 = b+1$.

It is easy to verify that, in the second case, when $k = (b+1)q - 1$,

$$f_{p,ps}((b+1)ps-1) - \varphi(p,ps) = \frac{b(s-1)}{2(ps-1)((b+1)ps-1)} > 0.$$

Tedious computations show that the inequality $f_{p,ps}(k) - \varphi(p,ps) \geq 0$ holds in the first case, too. $\square$

We describe the Delaunay polytopes $P_D(K_{p,ps})$ in Proposition 10 below.

For sufficient small t, the distance $d_{K_{p,q},t}$ is hypermetric, and $P_D(K_{p,q},t)$ is a simplex of the following form (cf. Fact 2 from Section 4). Let S^{p-1}, S^{q-1} be regular $(p-1)$- and $(q-1)$-dimensional simplexes with edges of norm t. Let S^{p-1} and S^{q-1} are imbedded in a $(p+q-1)$-dimensional space as follows. S^{p-1} and S^{q-1} span nonintersecting orthogonal spaces, the segment connecting their centers is orthogonal to both these spaces, and the distance between vertices of distinct simplexes is equal to 1. Then $P_D(K_{p,q},t)$ is the convex hull of S^{p-1} and S^{q-1}. We denote the $(p+q-1)$-dimensional simplex $P_D(K_{p,q},t)$ as $S_{p,q}(t)$. Obviously S^{p-1} and S^{q-1} are faces of $S_{p,q}(t)$. Moreover, any face of $S_{p,q}(t)$ is either $S^{p'-1}$, $S^{q'-1}$ or $S_{p',q'}(t)$ for some $p' \leq p$, $q' \leq q$. Besides $S_{p',q'}(t)$ is the Delaunay polytope $P_D(K_{p',q'},t)$.

Using (7), it can be shown that the squared radius of $S_{p,q}(t)$ is equal to

$$R^2_{p,q}(t) = \frac{pq - (p-1)(q-1)t^2}{4pq - 2(2pq - p - q)t}.$$

Note that if $R^2_{p,q}(t) = \frac{1}{2}$, then the centers of S^{p-1}, S^{q-1} and $S_{p,q}(t)$ coincide, and if $R^2_{p,q}(t) > \frac{1}{2}$, then the center of $S_{p,q}(t)$ lies beyond its boundary. For $t = 2$ and $q \geq 2$, $R^2_{1,q}(2) = \frac{q}{4}$ is the squared radius of the unit q-dimensional cube γ_q.

Take the center of the sphere circumscribing $S_{p,q}(t)$ as origin. Let $\mathcal{W}_{p,q}(t)$ be $p+q$ pairs of opposite vectors (w_i, w_i^*), $i \in V$, of norm $R^2_{p,q}(t)$ such that w_i, $i \in V$, represent vertices of the simplex $S_{p,q}(t)$. Denote by $\mathcal{D}_{p,q}(t)$ the convex hull of vectors of $\mathcal{W}_{p,q}(t)$. In other words, similar to $\mathcal{A}^{2k+1}$, $\mathcal{D}_{p,q}(t)$ is the convex hull of $S_{p,q}(t)$ and its opposite.

Conjecture 1 and the cases of $K_{2,2s}$ and $K_{k,k+2}$, considered below, imply

Conjecture 3 *The polytope $\mathcal{D}_{p,ps+2}(t)$ for $t = t(p,ps+2)$ is a Delaunay polytope.*

Recall that the Delaunay polytope B_k is defined in the previous section. For $q = ps$, and $p \geq 3$, let $D_{p,s}$ be the following polytope. We set a copy of the simplex S^{p-1} in the sphere circumscribing $S_{p,q}(t)$ such that its vertices tuch the sphere and the the space spanned by this copy is parallel to the space spanned by the original simplex S^{p-1}.

Set $\mathcal{D}_{2,2s}(t(2,2s)) = \mathcal{D}^{2s+1}_{2,s}$.

Proposition 10 *For $p \geq 1,\ s \geq 1,$*

$$P_D(K_{p,ps}) = \begin{cases} \gamma_s & \text{if } p=1,\ s\geq 2, \\ B_{p-1} & \text{if } p\geq 2,\ s=1, \\ \mathcal{D}^{2s+1}_{2,s} & \text{if } p=2,\ s\geq 2, \\ D_{p,s} & \text{if } p\geq 3,\ s\geq 2. \end{cases}$$

Proof. The case $p=1$ is considered in Proposition 7. It is shown in Proposition 8 that $t^H(K_{p,p}) = t(p,p)$ (the case $s=1$). In previous section we show that $P_D(K_{p,p}) = B_{p-1}$.

Recall that, for $G = K_{p,ps}$, $p \geq 2$, the distance $d_{G,t}$ satisfies the p equalities with $b_j = 1$ for all $j \in V_2$, $b_i = -s$, $i \in V_1 - \{i_0\}$, $b_{i_0} = -(s-1)$. This equalities, for $i \in V_1$, determine p vectors $v(b) = w'_i \equiv w_0 + w_i$, where $w_0 \equiv \sum_{j\in V_2} w_j - s\sum_{i\in V_1} w_i$, and $w_i \in \mathcal{W}_{p,ps}(t)$.

Using (11) for $G = K_{p,ps}$, $r^2 = R^2_{p,ps}(t)$ and $t = t(p,ps)$, one can show that $w_0 = 0$ only if $s = 1$ and $w_0 = -\sum_{i\in V_1} w_i$ if $p = 2$.

Hence if $s=1$, then $w'_i = w_i$, and the convex hull of $2p$ vectors w_i, $i \in V$, is B_{p-1}.

If $p=2$, $s\geq 2$, then $w'_i = w^*_i$ for $i \in V_1$. For $t = t(2,2s)$, the distance $d_{K_{2,2s},t}$ satisfies additionally $2s$ $(4s-1)$-gonal equalities with $b_i = -s$, $i \in V_1$, $b_j = 1$, $j \in V_2 - \{j_0\}$, $b_{j_0} = 0$. These equalities provide additionally $2s$ vectors $v(b) = w_0 - w_i = w^*_i$, $i \in V_2$, of the system $\mathcal{W}_{2,2s}(t(2,2s))$. This implies that $\mathcal{D}^{2s+1}$, the convex hull of $\mathcal{W}_{2,2s}(t(2,2s))$, is the Delaunay polytope $P_D(K_{2,2s})$.

If $p > 2$ and $s > 1$, then $w'_i \neq w_i, w^*_i$, but $w'_i - w'_{i'} = w_i - w_{i'}$ for $i, i' \in V_1$. Hence the convex hull of endpoints of the p vectors w'_i is a copy of S^{p-1} parallel to S^{p-1}, and the convex hull of $2p + ps$ vectors w_j, $j \in V_2$ and w_i, w'_i, $i \in V_1$, is the polytope $D_{p,s}$. $\square$

We give, in Proposition 11 below, an infinite sequence of pairs of bipartite graphs $(K_{p,p+1}, K_{p,p+2})$ such that t^H's of both the graphs of the sequence coincide and the Delaunay polytope of the first graph is a facet of the Delaunay polytope of the second graph.

Let $t_{p,s} = t(ps, ps+2) = \frac{2(ps+1)}{2ps-s+1}$. Then $t_{p,1} = \frac{p+1}{p} = t(p)$. For $q = ps+2$, we set $\mathcal{D}_{p,q}(t_{p,s}) = \mathcal{D}^{ps+p+1}_{p,s}$, $\mathcal{W}_{p,sp+2}(t_{p,s}) = \mathcal{W}_{p,s}$. Note that $\mathcal{D}^{2p+1}_{p,1} = \mathcal{A}^{2p+1}$,

Proposition 11 *If Conjecture 2 is true for $q = ps+2$, then it is true for $q = ps+1$, too, and for $p \geq 2$, $s \geq 1$, $P_D(K_{p,ps+1})$ belongs to the class $V^{ps+p}_{p,ps}$ and is a facet of $P_D(K_{p,ps+2}) = \mathcal{D}^{ps+p-1}$.*

Remark . Note that Conjecture 2 is true for $q = ps+2$ and either $p=1$ or $s=1$. In fact, the cases $p=1$ and $s=1$ are noted in Proposition 8. We saw that $P_D(K_{p,p+1})$ is a facet of $\mathcal{A}^{2p+1}$.

Proof. It is easy to verify that $t(p, ps+1) = t(p, ps+2) = t_{p,s}$. Suppose that $t^H(K_{p,ps+2}) = t_{p,s}$. Since $K_{p,ps+1}$ is an induced subgraph of $K_{p,ps+2}$, by Lemma 1, we have $t^H(K_{p,ps+1}) \geq t^H(K_{p,ps+2}) = t_{p,s}$. But, by Proposition 8, $t^H(K_{p,ps+1}) \leq t(p, ps+1) = t_{p,s}$. Hence $t^H(K_{p,ps+1}) = t(p, ps+1)$ if Conjecture 2 is true for $K_{p,ps+2}$.

Let $V = V_1 \cup V_2$ be the set of vertices of $K_{p,ps+2}$. Suppose that the set of vertices of $K_{p,ps+1}$ is the set $V' = V_1 \cup V'_2$, where V'_2 is V_2 without a vertex. Note that the distance $d_{K_{p,ps+1},t}$, for $t = t_{p,s}$, satisfies the $(2ps+1)$-gonal equality with $b_i = -s$, $i \in V_1$, $b_j = 1$, $j \in V'_2$.

Similarly, for $t = t_{p,s}$, the distance $d_{K_{p,ps+2},t}$ satisfies the following $p + ps + 2$ equalities:

1) $(2ps + 1)$-gonal equalities with $b_i = -s$, $i \in V_1$, $b_j = 1$, $j \in V_2 - \{j_0\}$, $b_{j_0} = 0$; we have $ps + 2$ such equalities for $j_0 \in V_2$;

2) $(2ps + 3)$-gonal equalities with $b_i = -s$, $i \in V_1 - \{i_0\}$, $b_{i_0} = -(s + 1)$ and $b_j = 1$, $j \in V_2$. We have p such equalities for $i_0 \in V_1$.

Using (11) for $G = K_{p,ps+2}$, $t = t_{p,s}$ and $r^2 = R^2_{p,ps+2}$, one can show that $w_0 = \sum_{j \in V_2} w_j - s \sum_{i \in V_1} w_i = 0$ for $w_i \in \mathcal{W}_{p,s}$. Then the equalities 1) and 2) provide $sp + p + 2$ vectors $v(b) = w_0 - w_i = w_i^*$, $i \in V$, of the system $\mathcal{W}_{p,s}$. This implies that $\mathcal{D}^{ps+p+1}_{p,s}$, the convex hull of $\mathcal{W}_{p,s}$, is the Delaunay polytope $P_D(K_{p,ps+2})$. The convex hull of vectors w_i, $i \in V' = V - \{j_0\}$ and $w^*_{j_0}$ for $j_0 \in V_2$ and $t = t_{p,s}$ is $P_D(K_{p,ps+1})$. It is a facet of $\mathcal{D}^{ps+p+1}$ orthogonal to the vector $\beta \sum_{i \in V_1} w_i - \sum_{j \in V_2 - \{j_0\}} w_j$, where $\beta = \frac{(s-1)(ps+1)}{(s-1)p+2}$. $\square$

Corollary $t^H(K_{2,2s+1}) = t^H(K_{2,2s+2}) = t(2, 2(s + 1)) = \frac{2(2s+1)}{3s+1}$. *$P_D(K_{2,2s+1})$ belongs to the class $V^{2(s+1)}_{2,2s}$ and is a facet of $P_D(K_{2,2s+2}) = \mathcal{D}^{2s+1}$.*

Proof. We have $K_{2,2s+2} = K_{2,2(s+1)}$. By Proposition 10, Conjecture 2 holds for $K_{2,2s+2}$. Now we can apply Proposition 11. $\square$

We denote the facet $P_D(K_{p,ps+1})$ of $\mathcal{D}^{p(s+1)+1}_{p,s}$ by $F^{p(s+1)}_{p,s}$. Note that $F^{2p}_{p,1} = B_p$.

The cases of Proposition 7 and the above Corollary cover all complete bipartite graphs $K_{p.q}$ with $p + q \le 9$. In Table below, we write out the values of $t^H(K_{p,q})$ and $P(K_{p,q})$ for $3 \le p + q \le 9$, $1 \le p \le q$.

$p+q$	$K_{p,q}$	$t^H(K_{p,q})$	$P_D(K_{p,q})$
3	$K_{1,2} = P_3$	2	$\gamma_2 = B_1$
4	$K_{1,3}$	2	$\gamma_3 = \mathcal{A}^3$
4	$K_{2,2} = C_4$	2	$\gamma_2 = B_1$
5	$K_{1,4}$	2	γ_4
5	$K_{2,3}$	$\frac{3}{2}$	B_2
6	$K_{1,5}$	2	γ_5
6	$K_{2,4}$	$\frac{3}{2}$	$\mathcal{A}^5 = \mathcal{D}^5$
6	$K_{3,3}$	$\frac{3}{2}$	B_2
7	$K_{1,6}$	2	γ_6
7	$K_{2,5}$	$\frac{10}{7}$	$F^6_{2,2}$
7	$K_{3,4}$	$\frac{4}{3}$	B_3
8	$K_{1,7}$	2	γ_7
8	$K_{2,6}$	$\frac{10}{7}$	$\mathcal{D}^7_{2,2}$
8	$K_{3,5}$	$\frac{4}{3}$	$\mathcal{A}^7$
8	$K_{4,4}$	$\frac{4}{3}$	B_3
9	$K_{1,8}$	2	γ_8
9	$K_{2,7}$	$\frac{7}{5}$	$F^8_{2,3}$
9	$K_{3,6}$	$\frac{30}{23}$	$D_{3,2}$
9	$K_{4,5}$	$\frac{5}{4}$	B_4

9 Small two-distance sets with $n \le 7$

In [16], the numbers $g(n)$ of n-point graphs $G \neq K(Q)$, i.e. with $t^N(G) < \infty$, are given for $n = 4, 5, 6, 7$. In Table 1 below, we give these numbers together with numbers $h(n)$ of graphs having $t^H(G) = 2$, i.e. with the hypermetric distance d^*_G, and numbers $h^N(n)$ of graphs with $t^H(G) = t^N(G)$.

Table 1.

n	4	5	6	7
$g(n)$	6	27	145	1029
$h(n)$	6	23	95	?
$h^N(n)$	1	3		?

We use the Coxeter's notations of some polytopes. Let
α_n be an n-dimensional regular simplex, with 1-skeleton K_{n+1},
β_n be an n-dimensional cross-polytope with length of edges 1, with 1-skeleton $K_{n\times 2}$,
γ_n be an n-dimensional unite cube, $\gamma_n = \gamma_1^n$, with 1-skeleton K_2^n.

We denote by $PyrB$ the pyramid with the base B whose lateral edges have length greater than 1. Then $\dim PyrB = 1 + \dim B$. The 1-skeleton of $PyrB$ is $K_1 + G$, where G is the 1-skeleton of B. Besides, $Pyr^*(PyrB)$ denotes a polytope such that apex of the second pyramid is at distance 1 from the apex of the first pyramid.

It is known all combinatorial types of Delaunay polytopes in dimensions 2, 3 and 4. There are 2, 5 and 19 types in these dimensions, respectively (see [15]).

The 2-dimensional Delaunay polytope distinct from a simplex is a rectangle, of the combinatorial type γ_2. The combinatorial types of 3-dimensional Delaunay polytopes are the simplex α_3, the cross-polytope β_3, the prism $\alpha_2 \times \gamma_1$, the pyramid $Pyr\gamma_2$ and the cube γ_3.

Note that $\gamma_2 = P_D(K_{2,2})$ is the special case B_1 of the class $V^2_{1,1}$ of repartitioning polytopes. Similarly, $Pyr\gamma_2 = P_D(K_1 + K_{2,2})$ is the special case $PyrB_1$ of the class $V^3_{1,1}$.

In tables below, we give $t^H(G)$ and $P_D(G)$ of graphs G with $t^N(G) < \infty$. We use symbols of graphs from [16]. The symbol of a graph G is the triple $(n.r.s)$, where n and r are the numbers of vertices of G and of edges of $\overline{G}$, respectively, and s differs distinct graphs with the same n and r. The symbol $(n.r.s)'$ corresponds to the complement $\overline{G}$. (Graphs drawned in [16] have edges of length t, i.e. they are the complements of our graphs.) The D-simbol is either the *Delaunay symbol* denoting a Delaunay polytope of dimensions 3 and 4 in Tables IV and V or the symbol of Table VI taken from [15].

Table 2. 4-point graphs

symbol	G	$t^N(G)$	$t^H(G)$	$P_D(G)$
(4.1.1)	$K_4 - e = \nabla P_3$	3	2	β_3
(4.2.1)$'$	$\overline{P_3 + K_1}$	$2+\sqrt{3}$	2	$\alpha_2 \times \gamma_1$
(4.2.1)	$P_3 + K_1$	$2+\sqrt{3}$	2	$Pyr\gamma_2 = PyrB_1$
(4.2.2)	C_4	2	2	$\gamma_2 = B_1$
(4.3.1)	$K_{1,3}$	3	2	γ_3
(4.3.2)	P_4	τ^2	2	$\alpha_2 \times \gamma_1$

Table 3. 5-point graphs

N	symbol	G	$t^H(G)$	$P_D(G)$	D-symbol
G with $\dim P_D(G) = 3$ and $t^H(G) = t^N(G)$					
1	(5.2.2)	∇C_4	2	β_3	F_2^3
2	(5.4.6)	$\overline{P_5}$	2	$\alpha_2 \times \gamma_1$	F_1^3
3	$(5.4.5)'$	$K_1 + C_4$	2	$Pyr\gamma_2 = PyrB_1$	F^4
G with $P_D(G)$ of type $V_{2,2}^4$					
4	(5.4.4)	$K_{2,3}$	$\frac{3}{2}$	B_2	A
5	(5.3.2)	$\nabla^2(3K_1) =$ $= K_{2,3} + e'$	$\frac{5}{3}$	$\simeq V_{2,2}^4$	A
6	(5.3.4)	$K_{2,3} + e$	$\frac{5}{3}$	$\simeq V_{2,2}^4$	A
7	(5.5.3)	$K_{2,3} - e$	$\frac{5}{3}$	$\simeq V_{2,2}^4$	A
G with $P_D(G) = PJ(5,2)$					
8	(5.2.1)	$\nabla^2(K_2 + K_1)$	2	$PJ(5,2)$	F_4^4
9	(5.3.1)	$\nabla(K_3 + K_1)$	2	$PJ(5,2)$	F_4^4
10	(5.3.3)	∇P_4	2	$PJ(5,2)$	F_4^4
11	(5.4.3)	$\overline{K_{2,3} - V}$	2	$PJ(5,2)$	F_4^4
12	(5.5.4)	C_5	2	$PJ(5,2)$	F_4^4
G with $P_D(G)$ of type $PyrP$					
13	$(5.2.1)'$	$2K_1 + P_3$	2	$Pyr^2\gamma_2 = Pyr^2B_1$	F^8
14	$(5.3.1)'$	$K_1 + K_{1,3}$	2	$Pyr\gamma_3$	F_1^6
15	$(5.3.3)'$	$K_1 + P_4$	2	$Pyr(\alpha_2 \times \gamma_1)$	F_1^7
16	$(5.4.2)'$	$K_1 + \overline{K_1 + P_3}$	2	$Pyr(\alpha_2 \times \gamma_1)$	F_1^7
17	(5.5.1)	$K_1 + (K_4 - e)$	2	$Pyr\beta_3$	B
18	$(5.3.4)'$	$K_2 + P_3$	2	$Pyr^*(Pyr\gamma_2$	F^8
Other graphs					
19	(5.1.1)	$\nabla^3(2K_1) =$ $= \nabla(K_4 - e)$	2	β_4	C
20	$(5.4.1)'$	$K_{1,4}$	2	γ_4	F_6^3
21	(5.4.2)	$\nabla(K_1 + P_3)$	2	$\beta_3 \times \gamma_1$	F_5^4
22	$(5.4.3)'$	$K_{2,3} - V$	2	γ_4	F_6^3
23	(5.4.5)	$\nabla(2K_2)$	2	$\alpha_2 \times \alpha_2$	F_5^5
24	(5.4.6)	P_5	2	$\alpha_2 \times \alpha_2$	F_5^5
25	$(5.5.2) =$ $= (5.5.2)'$		2	$\alpha_2 \times \alpha_2$	F_5^5
26	$(5.5.1)'$	$\nabla(\overline{2K_1 + K_2})$	2	$\alpha_2 \times \gamma_2$	F_1^4
27	$(5.5.3)'$	$\overline{K_{2,3} - e}$	2	$\alpha_3 \times \gamma_1$	F_4^6

n=3. There is only one graph $P_3 = K_{1,2}$ on 3 vertices distict from $K(Q)$, i.e. with $t^N(G) < \infty$. We have $t^N(K_{1,2}) = 4$ and $t^H(K_{1,2}) = 2$ (see Propositions 3 and 4), $P_D(K_{1,2}) = \gamma_2 = B_1$.

n=4. The values of $t^N(G)$ are taken from [16]. They can be found also in [20].

Since $Cut_4 = Hyp_4 = Met_4$, we have $t^C(G) = t^H(G) = 2$ for all these six graphs G on 4 vertices with $t^N(G) < \infty$. Moreover, since $\lambda_{min}(C_4) = -2$, we have $t^H(C_4) = t^N(C_4) = 2$ and $P_D(C_4) = P(d_{C_4,2}) = \gamma_2$.

Besides these 6 four-point two-distance sets in $\mathbf{R}^2$ there is only one another two-distance set, the pentagon.

n=5. We have $Cut_5 = Hyp_5 \subset Met_5$. Hence $t^C(G) = t^H(G)$ for all 27 five-point graphs with $t^N(G) < \infty$. The values $t^N(G)$ can be found in [16]. There are 3 five-point graphs having 3-dimensional Delaunay polytopes $P_D(G)$, i.e. satisfying the conditions of Proposition 5. Hence for these graphs we have $t^H(G) = t^N(G) = 2$.

Among other 27-3=24 graphs there are additionally 20 graphs with $t^H(G) = 2 < t^N(G)$. The 4 graphs with $t^H(G) < 2$ are the 3 nonhypermetric five-point graphs (of diameter 2) and one graph with a nonhypermetric distance d^*_G. These graphs were found for the first time in [1]. One of these graphs is the complete bipartite graph $K_{2,3}$, the unique graph of Propositions 3 and 4, giving the minimal values of $t^N(G)$ and $t^H(G)$ for 5-point graphs. The edges e' and e in $K_{2,3} + e'$ and $K_{2,3} + e$ are added to distinct parts of the bipartite graph $K_{2,3}$. The Delaunay polytopes $P_D(G)$ of all these 4 graphs are 4-dimensional repartitioning polytopes. But only $P_D(K_{2,3}) = B_2$. $K_{2,3} - V$ is the graph $K_{2,3}$ without two edges incident to a vertex of the part of size 2.

Besides the graph $K_1 + C_4$, there are 5 graphs of the same type $G = K_1 + H$.

Note the graph $(5.5.4) = (5.5.4)' = C_5$, having $t^N(C_5) = \tau^2$ with a two-dimensional Euclidean representation. We have $t^H(C_5) = 2$ and $P_D(C_5)$ is the Johnson polytope $PJ(5,2)$ of dimension 4, whose 1-skeleton is the triangular graph $T(5)$.

The Johnson polytope $PJ(n,k)$ is the section of the cube γ_n by the hyperplane $\{x : xj_n = k\}$ orthogonal to the diagonal of γ_n spanned by the all-one n-dimensional vector j_n. We have $PJ(n, n-k) = PJ(n,k)$, $PJ(n,1) = \alpha_{n-1}$, $PJ(4,2) = \beta_3$.

n=6. In this case $Cut_6 = Hyp_6$. Hence $t^C(G) = t^H(G)$. There are 145 six-point graphs G of not the form $K(Q)$, i.e. with $t^N(G) < \infty$. It is noted in [1] that d^*_G is hypermetric, i.e. $t^H(G) = 2$, if either G does not contain one of the four 5-point graphs with $t^H(G) < 2$ (and there are 48 such 6-point graphs), or G is not one of the two 6-point graphs G_1 and G_2. Here $G_1 = K_{2,4} - 3e$, where all the 3 deleted edges are incident in $K_{2,4}$ to the same vertex of the part of size 2. The graph $G_2 = K_{2,4} - 2e + e'$, where the two deleted edges are incident in $K_{2,4}$ to the same vertex of the part of size 2, and the edge e' connects vertices of degree 2 of the part of size 4 in $K_{2,4} - 2e$.

The distance d^*_G for $G = G_1$ and $G = G_2$ does not satisfy the 7-gonal inequality with $b = (1,1,1,1,-1,-2)$. For this inequality and $G = G_1, G_2$, $h_G(t,b) = 5t - 9$. Hence $h_G(t,b) \leq 0$ if $t \leq \frac{9}{5}$. This implies $t^H(G) = \frac{9}{5} < 2$ for these two graphs.

Note that among 5 three-dimensional Delaunay polytopes there are two polytopes, β_3 and $\alpha_2 \times \gamma_1$, having 6 vertices with two distances between them. Obviously, for the corresponding graphs $(6.3.1)' = K_{3\times 2}$ and $(6.6.1)' = K_2 \times K_3$, we have $t^{H,N}(K_{3\times 2}) = t^{H,N}(K_2 \times K_3) = 2$. As it shown in [16], there are 4 another 6-point two-distance spaces $d_{G,t}$ with a 3-dimensional representations and with the same $t^N(G) = \tau^2$.

Among 19 four-dimensional Delaunay polytopes, there are two 6-vertex polytopes, $Pyr^2\gamma_2$ and B_2, giving two-distance sets d^*_G with $G = 2K_1 + C_4$ and $K_{3,3} = (6.6.3)'$, respectively. The 4 Delaunay polytopes β_3, $\alpha_2\times\gamma_1$, $Pyr^2\gamma_2$ and B_2 are examples, where the conditions of Proposition 5 hold. Hence $t^H(2K_1+C_4)=t^N(2K_1+C_4)=2$, and $t^H(K_{3,3})=t^N(K_{3,3})=\frac{3}{2}$.

n=7. We have $Cut_7\neq Hyp_7$, and there are 7-point graphs G with $t^C(G)<t^H(G)$. The first such graph was found by Avis [2]. Now, one knows 26 7-point graphs with this property. These graphs can be found in [11] and [13]. All these graphs lie on extreme rays of Hyp_7, and they are subgraphs of the Schläfli graph. The corresponding hypermetric distances are two-distances d^*_G. They have the common Delaunay polytope $P_D(G)=P_{Schl}$. The polytope P_{Schl} is a 6-dimensional asymmetric Delaunay polytope which is the convex hull of the representation of $Schl$.

For to find $t^C(G)$, one needs to find the facet of Cut_7, where $d_{G,t}$ lies. Obviously, this facet is not hypermetric.

10 Graphs with $t^H(G)=2$

Recall that $d_{G,2}=d^*_G$ is the truncated distance of the graph G. The equality $t^H(G)=2$ means that the distance d^*_G is hypermetric. The graphs G having hypermetrc d^*_G are studed in [1] and [11]. In particular, it is proved in [11] the following assertion.

Proposition 12 *If G is a connected regular graph, then d^*_G is hypermetric iff $\lambda_{min}(G)\geq -2$, where $\lambda_{min}(G)$ is the smallest eigenvalue of G.*

Recall that $\mu(G)=-\lambda_{min}(G)$ in (14). If $\mu(G)\leq 2$, then (14) shows that $t^Q(G)\geq 2$ with equality only if $\mu(G)=2$. Hence if $\lambda_{min}(G)>-2$, when $t^Q(G)>2$, the Gram matrix $Q(G,2)$ is not singular, and $P(d_{G,2})$ is a simplex. But if $\lambda_{min}(G)=-2$, then there are dependencies between representing vectors, and the conditions of Proposition 2 hold. Therefore if $\mu(G)=2$, we have $t^H(G)=t^N(G)=2$.

Note the class of strongly regular graphs with $\lambda_{min}(G)=-2$. These graphs were classified by Seidel (see [5], Theorem 3.12.4(i)). These graphs are

the triangular graph $T(n)$, $n\geq 5$,

the square $n\times n$ grid $K_n\times K_n$ (also called a lattice graph $L_2(n)$), $n\geq 3$,

the Cocktail party graph $K_{n\times 2}$, $n\geq 2$,

the Petersen Pe, the Clebsh Cle, the Schläfli $Schl$, the Shrikhande Shr, and three Chang Ch_i graphs.

If G is a connected regular graph with $\lambda_{min}(G)>-2$, then G is a complete graph or an odd cycle.

Since all these graphs (except $Schl$ and Ch_i) are l_1-graphs, we have that $t^C(G)=2$ if G is a strongly regular graph with $t^H(G)=2$, $G\neq Schl, Ch_i$, $1\leq i\leq 3$.

11 Graphs with $t^H(G)=\frac{3}{2}$

In this section we give examples of regular graphs G with $t^H(G)=\frac{3}{2}$ represented by odd systems of norm $2k+1$ related to equianglar lines. According to (19), the angle between these lines is equal to $arccos\frac{1}{5}$, i.e. $k = 2$.

Let $\mathcal{U}_2 = \{u_i, u_i^* : i \in V\}$ be the odd system of vectors of norm 5 related to the representation. It is proved in [12] that $P(\mathcal{U}_2)$ is a Delaunay polytope if the odd system $\mathcal{U}_2$ of norm 5 is not pillar. Recall that the cardinality of a maximal set of vectors of $\mathcal{U}_2$ with mutual inner products -1 is not greater than 6. Let $\{u_i : i \in C\}$ be such a set with $|C| \le 6$, and let $v \in \mathcal{U}_2$. Then the vector v partitions C into subsets $C_+ = \{i \in C : u_i v = 1\}$ and $C_- = \{i \in C : u_i v = -1\}$.

The odd system $\mathcal{U}_2$ is called *pillar* if this partition does not depend on the vector v. Let $\mathcal{U}_2^+$ be a subset of $\mathcal{U}_2$ containing from each pair (u_i, u_i^*) of opposite vectors exactly one vector. W.l.o.g., we denote the vector as u_i. Let G^+ be the graph with V as the set of vertices. Two vertices i, j of G^+ are adjacent iff $u_i u_j = -1$. If $\mathcal{U}_2$ is not pillar, then there are two vertices of G^+ having distinct neighborhoods in C, i.e. having distinct partitions of C.

Note that distance between endpoints of two non-opposite vectors (of norm 5) $u, u' \in \mathcal{U}_2$ is equal to 12 if $uu' = -1$ and to 8 if $uu' = 1$. Hence $\mathcal{U}_2^+$ represents the distance space $(8d_{G,t}, V)$, where $G = \overline{G^+}$ and $t = \frac{3}{2}$.

Let G' be a regular subgraph of G^+ with n vertices and valency q. It is represented by a subset of $\mathcal{U}_2^+$. Using (6), we find that the squared radius of the convex hull of endpoints of vectors of this representation is equal to $r^2 = 4 + \frac{2q-4}{n}$. Since $r^2 \le 5$, we have to have $n \ge 2q - 4$.

We saw in Section 7 that a regular graph G with $\lambda_2(\overline{G}) = 2$ has such a representation if its valency satisfies inequalities (18). For $t = \frac{3}{2}$, these inequalities take the form $n \le 2q+6$. For the graph $\overline{G}$ of valency $\overline{q} = n - q - 1$, the last inequality takes the form $\overline{q} \le \frac{n+4}{2}$.

Call a graph G with $\lambda_2(G) = 2$ *non-pillar* if it has the following property. There is a maximal clique C and two vertices $i, j \in V - C$ such that the neighborhoods of i and j in C are distinct, i.e. the partitions of C determined by i and j are distinct.

Proposition 13 *Let G be a regular non-pillar graph of valency $q \le \frac{n+4}{2}$ with $\lambda_2(G) = 2$ of multiplicity $f \ge 2$. Then $t^H(\overline{G}) = \frac{3}{2}$.*

Proof. We saw that $\overline{G}$ has a representation by an odd system $\mathcal{U}_2$ related to equiangular lines at angle $arccos\frac{1}{5}$ if valency q of G satisfies $q \le \frac{n+4}{2}$. The condition that G is non-pillar implies that the odd system $\mathcal{U}_2$ is not pillar. Hence $P(\mathcal{U}_2)$ is a Delaunay polytope. The dimension of this polytope is equal to $n - f \le n - 2$. By Proposition 2, $d_{\overline{G},\frac{3}{2}}$ lies on the boundary of Hyp_n. Hence this representation is exact, and $t^H(\overline{G}) = \frac{3}{2}$. $\square$

The most important case relates to equiangular lines corresponding to a regular two-graph. In table below, we give dimensions m for which one knows sets of $n_s(2, m) = \frac{24m}{25-m}$ (see (21)) equiangular lines corresponding to regular two-graphs. As usual, N is the number of known nonisomorphic two-graphs, and $\overline{N}$ denotes that this number is exact (cf. [12]).

m	5	10	13	15	21	22	23
$n_s(2, m)$	6	16	26	36	126	176	276
N	$\overline{1}$	$\overline{1}$	$\overline{4}$	227	1	1	$\overline{1}$

The minimal set of 6 lines is represented by the 5-dimensional odd system $\mathcal{V}_2^0 = \frac{1}{\sqrt{8}}\mathcal{U}_2^0$. The Delaunay polytope $P(\mathcal{V}_2^0) = \mathcal{A}^5$ is considered in the end of Section 7.

Now we consider in detail the case $m = 10$. Let $\mathcal{U}_2^1$ be the corresponding odd system. It is not pillar. The polytope $P(\mathcal{U}_2^1)$ coincides with the convex hull of $\mathcal{U}_2^1$ and is a Delaunay polytope. $\frac{1}{\sqrt{2}}P(\mathcal{U}_2^1)$ is the symmetrization of the cut polytope $PCut_5$ which is the convex hull of indicator vectors c_S of all cuts $\delta(S)$ of K_5 defined in Section 2.

The symmetrization of the cut polytope $PCut_5$ is the convex hull of all indicator vectors of cuts and their complements in the complete graph K_5. Let $V_5 = \{1, 2, ..., 5\}$ and $E_5 = \{ij : 1 \leq i < j \leq 5\}$ be the sets of vertices and edges of K_5, respectively.

Let j_{10} be the all-one vector. Denote by $c^*(S) = j_{10} - c(S)$ the indicator vector of the complement $\delta^*(S) = E_5 - \delta(S)$ of the cut $\delta(S)$. Since $c(V_5 - S) = c(S)$, we can use only $S \subseteq V_5$ with $|S| \leq 2$. We set $\mathcal{S} = \{S : S \subset V_5, |S| \leq 2\}$.

Denote by P_5 the convex hull of all vectors $c(S)$, $c^*(S)$, $S \in \mathcal{S}$. It is shown in [18] that P_5 is, up to a multiple, a Delaunay polytope of the 10-dimensional isodual lattice Q_{10} mentioned in [9].

All the vectors $c(S)$ and $c^*(S)$ are vertices of the 10-dimensional unit cube $\mathbf{B}^{10}$. In fact the set C_{10} of all 32 these (0,1)-vectors is the set of all codewords of a linear binary code with parameters $[n, k, d] = [10, 5, 4]$.

The set C_{10} with the Hamming distance d between its points is an l_1 metric space (C_{10}, d). The distance between points $a, b \in C_{10}$ is equal to the *norm* $(a-b)^2$ of the vector $a - b$. Since $|\delta(S)|$ take only two values 4 and 6 for $S \subseteq E_5$, $S \neq \emptyset$, E_5, we obtain that any subset of (C_{10}, d) without pairs of complemented vectors is a two-distance l_1-space with distances 4 and 6, i.e. it is $4d_{G,\frac{3}{2}}$ for some G.

The center of P_5 is $\frac{1}{2}j_{10}$, and the squared radius of P_5 is $\frac{5}{2}$. Hence the set of vectors $\sqrt{2}c(S)$, $\sqrt{2}c^*(S)$, $S \in \mathcal{S}$, form the odd system $\mathcal{U}_2^1$.

We consider graphs on subsets of $\mathcal{U}_2^1$ without opposite vectors such that two vertices are adjacent iff they are at distance 12. It is easy to verify that the corresponding graph on 16 vertices $\sqrt{2}c(S)$, $S \in \mathcal{S}$, is the strongly regular Clebsh graph Cle with parameters (16,10,6,6) and with $\lambda_2(Cle) = 2$. By construction, Cle represents the distance space $12d_{Cle,\frac{2}{3}} = 8d_{\overline{Cle},\frac{3}{2}}$. Using (12) with $t = \frac{2}{3}$, $q = 10$, $n = 16$, we find that $12r^2(Cle, \frac{2}{3}) = 5$, the norm of $\mathcal{U}_2^1$. Here Cle spans P_5.

The switching class of Cle contains another two strongly regular graphs, namely, the grid $L_2(4)$ and the Shrikhande graph Shr [5]. It is shown in [18] that $L_2(4)$ and Shr define facets of P_5 which are 9-dimensional Delaunay polytopes. Since $L_2(4)$ and Shr have $n = 16$ vertices, by Proposition 5, we have $t^H(\overline{L_2(4)}) = t^H(\overline{Shr}) = \frac{3}{2}$. The polytope P_5 has another facet defined by the Petersen graph Pe. Since dimension of the facet is 9 and $\overline{Pe} = T(5)$ has 10 vertices, we cannot apply Proposition 5. We have only $t^H(T(5)) \geq \frac{3}{2}$. In fact, we saw in the previous section, that $t^H(T(5)) = 2$.

In [5], 3 regular proper subgraphs of Shr are described. These graphs have 6, 10 and 12 vertices. The graphs on 10 and 12 vertices are denoted as G_{10} and G_{12}^1 in [18]. They define 8-dimensional faces of P_5. There is once more regular 12-vertex subgraph G_{12}^0 of Shr determining a 8-dimensional face of P_5 which is a line graph. Again, using

Proposition 5, we obtain that $t^H(\overline{G}) = \frac{3}{2}$ for $G = G_{10}, G_{12}^0, G_{12}^1$.

There are exactly 15 vertices of P_5 lying at the same distance from a vertex of P_5. These 15 vertices affinely generate a 9-dimensional hyperplane H and induce the triangular graph $T(6)$. The graph $T(6)$ is 1-skeleton of a Delaunay polytope which is an intersection of P_5 by the hyperplane H. By Proposition 5 we obtain that $t^H(\overline{T(6)}) = \frac{3}{2}$.

Recall that P_5 is inscribed in the unit cube γ_{10}. Since $t^C(G) \leq t^H(G)$ for any graph G, we have that $t^C(G) = t^H(G)$ for all above graphs, excluding $T(5)$.

Concluding this section we give some popular non-pillar strongly regular graphs G with $\lambda_2(G) = 2$. A survey of strongly regular graphs with $\lambda_2(G) = 2$ is given in [17].

1) The graphs $GQ(3,t)$ corresponding to generalized quadrangles with lines of size $s+1 = 4$, $t = 1, 3, 5, 9$. $GQ(3,1) = L_2(4)$.

2)The negative Latin square graphs $NL_2(m)$, $4 \leq m \leq 10$, $m \neq 7$. $NL_2(4) = Cle$, $NL_2(8) = GQ(3,5)$, $NL_2(10)$ is the Higman-Sims graph.

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